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## On the qualitative properties of differential equations of third order with retarded argument

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### Abstract

*By using the standard Liapunov-Krasovskii functional approach, in this paper, new stability, boundedness and ultimately boundedness criteria are established for a class of vector functional differential equations of third order with retarded argument.*

## 1. Introduction

Differential equations with retarded argument are used to describe many phenomena of physical interest. While ordinary differential equations without delay contain derivatives which depend on the solution at the present value of the independent variable ('time', differential equations with retarded argument contain in addition derivatives which depend on the solution at previous times. Differential equations with retarded argument arise in models throughout the sciences. Systems of differential equations with retarded argument now occupy a place of central importance in all areas of science and particularly in the biological sciences (e.g., population dynamics and epidemiology). Interest in such systems often arises when traditional point wise modeling assumptions are replaced by more realistic distributed assumptions, for example, when the birth rate of predators is affected by prior levels of predators or prey rather than by only the current levels in a predator-prey model. The manner in which the properties of systems of differential equations with retarded argument differ from those of systems of ordinary differential equations has been and remains an active area of research (see, e.g. Ahmad and Rama Mohana Rao [4], Burton [5], El'sgol'ts and Norkin [11], Hale [15], Hara [16], Krasovskii [17], Rauch [24], Smith [26] and references therein). On the other hand, differential equations of third order have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find some applications such as nonlinear oscillations (Afuwape et al. [3], Fridedrichs [14]), prototypical examples of complex dynamical systems in a high-dimensional phase space, displacement in a mechanical system, velocity, acceleration (Chlouverakis and Sprott [6], Eichhorn et al. [10], Linz [19]), the biological model and other models (Cronin- Scanlon [8]), electronic theory (Rauch [24]), the boundedness of some minimal chaotic attractors (Elhadj and Sprott [9]) and etc. Therefore, it is worth to investigate the qualitative behavior of differential equations of third order. At the same time, the theory of stability and boundedness of solutions are important branches of the qualitative theory of differential equations. In the last 60 years, there has been increasing interest in obtaining the sufficient conditions for the stability/instability/boundedness/ ultimately boundedness etc. of solutions of different classes of differential equations of third order such as linear and nonlinear ordinary, and functional differential equations; for a comprehensive treatment of subject, we refer the reader to the book by Reissig et al. [25] as a survey and the papers by Ademola and Arawomo [1],

Afuwape and Castellanos [2], Afuwape et. al. [3], Chukwu [7], Ezeilo ([12], [13]), Hara [16], Mehri and Shadman [20], Meng [21], Omeike and Afuwape [22], Qian [23], Rauch [24], Smith [26], Swick [27], Tejumola [28], Tunc [29-40], Tunc and Ates [41], Tunc and Ergören [42], Wall and Moe [43], Zhang and Yu [44] and references therein). It is well known that an important tool in the study of stability/instability/boundedness/ ultimately boundedness of solutions of the considerable number of ordinary/neutral/retarded differential equations of higher order is the Liapunov function/Liapunov-Krasovskii functional techniques. The Liapunov function technique goes back for ordinary differential equations as far as the approximation methods of Liapunov (1857-1918) (see, also, Liapunov [18]), which he developed in 1899, make it possible to define the stability of sets of ordinary differential equations and he created the modern theory of the stability of a dynamic system. Besides, in 1959, Krasovskii (1924-2012) carried out Liapunov-Krasovskii functional technique for a treatment of global behavior of solutions of a class of nonlinear differential equations with delay by suitable Liapunov-Krasovskii functionals. In his work, Krasovskii indicated the extension of Liapunov's method to a class of general differential equations with time delay in the literature. Especially, after the work of Krasovskii [17], stability/instability/boundedness/ convergence/ existence of periodic solutions, etc. of solutions of scalar neutral/retarded differential equations of higher order were investigated quite extensively by many authors (see the mentioned references and the references therein). It should be noted that, today, the same topics are also being investigated intensely in the literature and many good results are being obtained by the researchers. We would not like to give the details.

In 1968, Wall and Moe [43] considered the scalar differential equation of third order without delay

$$x'' + (1 + x'^2)x' + x' + x = 0.$$

The authors proved that the functional relationship to the original equations is evident by the direct way in which the desired Liapunov function was obtained from the original differential equations, and they gave sufficient conditions which guarantee globally asymptotic stability of this equation.

Later, in 2009, the result of Wall and Moe [43] was improved and extended by Tunc [35] for the stability and boundedness of the following

vector differential equation of third order without delay:

$$X'' + \Psi(X')X'' + BX' + cX = P(t),$$

when  $P(t) \equiv 0$  and  $P(t) = 0$ , respectively.

After that, in 2010, Omeike and Afuwape [22] studied the ultimate boundedness of solutions of Eq. (3). By this work, they did a good contribution to the topic. To the best of our knowledge from the literature, by now, no attention was given to the investigation of the stability/boundedness/ ultimately boundedness of solutions of the vector functional differential equations of third order with retardation. The basic reason for the absence of any work on this topic may be the difficulty of the construction or definition of appropriate Liapunov- Krasovskii functionals for differential equations of third order with retardation. Further, until now, construction or definition of suitable Liapunov functions/Liapunov- Krasovskii functionals for higher order nonlinear differential equations without and with retardation remains an as open problem in the literature. Therefore, it is worthwhile to do the discussion of the mentioned properties for those equations.

In this paper, we are concerned with the stability, boundedness and ultimately boundedness properties of the following type vector functional differential equations of third order with retarded argument,  $\tau_1 > 0$  :

$$(1.1) \quad X''' + \Psi(X')X'' + BX'(t - \tau_1) + cX(t - \tau_1) = P(t),$$

where  $\tau_1$  is the retarded argument;  $c$  is a positive constant;  $B$  is  $n \times n$ - an constant symmetric matrix;  $\Psi$  is an  $n \times n$ - continuous differentiable symmetric matrix function such that the Jacobian matrix  $J(\Psi(X')X' | X')$  exists and is symmetric and continuous, that is,

$$\begin{aligned} J(\psi(X')X' | X') &= \frac{\partial}{\partial x'_j} \left( \sum_{k=1}^n \psi_{ik} x'_k \right) \\ &= \psi(X') + \sum_{k=1}^n \frac{\partial \psi_{ik}}{\partial x'_j} x'_k, \quad (i, j = 1, 2, \dots, n), \end{aligned}$$

exists and is symmetric and continuous, where  $(x'_1, x'_2, \dots, x'_n)$  and  $(\psi_{ik})$  are components of  $X'$  and  $\Psi$ , respectively;  $P : \mathfrak{R}^+ \rightarrow \mathfrak{R}^n$  is a continuous function,  $\mathfrak{R}^+ = [0, \infty)$  and the primes in Eq.(1) indicate differentiation with respect to  $t$ ,  $t \geq t_0 \geq 0$ . For the third order functional differential equation with the retarded argument, let  $X = X_1$ ,  $X' = X_2$ ,  $X'' = X_3$ , then Eq. (1) can be written in the form

$$X'_1 = X_2,$$

$$\begin{aligned}
 X'_2 &= X_3, \\
 X'_3 &= -\Psi(X_2)X_3 - BX_2 + \int_{t-\tau_1}^t BX_3(s)ds \\
 (1.2) \quad &- cX_1 + c \int_{t-\tau_1}^t X_2(s)ds + P(t).
 \end{aligned}$$

Throughout this paper, we assume that the existence and the uniqueness of the solutions of Eq. (1) hold (*Èl'sgol'ts* and Norkin [11]). The motivation of this paper comes from the results established in Wall and Moe [43], Tunç [35], Omeike and Afuwape [22] and mentioned papers. The main purpose of this paper is to get some new stability/boundedness/ ultimately boundedness results of Eq. (1) by using the Liapunov- Krasovskii functional approach. This paper is the first attempt in the literature to obtain sufficient conditions which guarantee the stability/boundedness/ultimately boundedness of vector functional differential equations of third order with retarded argument, and it has a new contribution the topic in the literature. This case shows the novelty of this work. It should also be noted that the results to be established here may be useful for researchers working on the qualitative behaviors of solutions.

One tool to be used here is LaSalle's invariance principle. If we consider delay differential system

$$\dot{x} = F(x_t), \quad x_t = x(t + \theta), \quad -r \leq \theta \leq 0, \quad t \geq 0,$$

we take  $C = C([-r, 0], \mathfrak{R}^n)$  to be the space of continuous function from  $[-r, 0]$  into  $\mathfrak{R}^n$  and ask that  $F : C \rightarrow \mathfrak{R}^n$  be continuous. We say that  $V : C \rightarrow \mathfrak{R}$  is a Liapunov function on a set  $G \subset C$  relative to  $F$  if  $V$  is continuous on  $\bar{G}$ , the closure of  $G$ ,  $\dot{V}$  is defined on  $G$ , and  $\dot{V} \leq 0$  on  $G$ .

The following form of LaSalle's invariance principle can be found in Smith [26, Theorem 5.17]. Here,  $\omega$  denotes the omega limit set of a solution.

**Theorem A.** If  $V$  is a Liapunov function on  $G$  and  $x_t(\phi)$  is a bounded solution such that  $x_t(\phi) \in G$  for  $t \geq 0$ , then  $\omega(\phi) \neq \emptyset$  is contained in the largest invariant subset of  $E \equiv \{\psi \in \bar{G} : \dot{V}(\psi) = 0\}$ .

## 2. Main results

Before we state our main results, we give an algebraic result which will be required in the proofs.

**Lemma.** Let  $A$  be a real symmetric  $n \times n$ -matrix. Then for any  $X_1 \in R^n$

$$\delta_a \|X_1\|^2 \leq \langle AX_1, X_1 \rangle \leq \Delta_a \|X_1\|^2,$$

where  $\delta_a$  and  $\Delta_a$  are, respectively, the least and greatest eigenvalues of the matrix  $A$  (see [22]).

Let  $P(t) \equiv 0$ . The first main result of this paper is the following theorem.

**Theorem 1.** In addition to the basic assumptions imposed on  $\Psi(X_2)$ ,  $B$  and  $c$ , we assume that  $P(t) \equiv 0$  and there exist positive constants  $\alpha, \varepsilon, a_0, a_1, b_0$  and  $c$  such that the following conditions hold:

$n \times n$ - symmetric matrices  $B$  and  $\Psi$  commute with each other and

$$a_0 b_0 - c > 0, 1 - \alpha a_0 > 0, b_0 \leq \lambda_i(B) \leq b_1$$

and

$$a_0 + \varepsilon \leq \lambda_i(\Psi(X_2)) \leq a_1 \text{ for all } X_2 \in \mathfrak{R}^n.$$

if

$$\tau_1 < \min \left\{ \frac{\alpha a_0 b_0 c}{\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c}, \frac{k_1}{(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1}, \frac{k_2}{c + (2 + a_0 + \alpha a_0 b_0)b_1} \right\},$$

then, all solutions of Eq. (1) are bounded and the zero solution of Eq. (1) is globally asymptotically stable, where

$$k_1 = 2(a_0 b_0 - c) - \alpha a_0 b_0 [a_0 + c^{-1}(b_1 - b_0)^2] > 0$$

and

$$k_2 = 2\varepsilon [1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2] > 0.$$

**Proof.** To prove the theorem, we define a differentiable Liapunov- Krasovskii functional  $V(t) = V(X_1(t), X_2(t), X_3(t))$  by

$$\begin{aligned}
 2V &= a_0c\langle X_1, X_1 \rangle + a_0 \int_0^1 \langle \sigma \Psi(\sigma X_2) X_2, X_2 \rangle d\sigma + \alpha a_0 b_0^2 \langle X_1, X_1 \rangle \\
 &+ \langle B X_2, X_2 \rangle + \langle X_3, X_3 \rangle + 2\alpha a_0^2 b_0 \langle X_1, X_2 \rangle + 2\alpha a_0 b_0 \langle X_1, X_3 \rangle \\
 &+ 2a_0 \langle X_2, X_3 \rangle + 2c \langle X_1, X_2 \rangle - \alpha a_0 b_0 \langle X_2, X_2 \rangle \\
 (2.1) \quad &+ 2\lambda_1 \int_{-\tau_1}^0 \int_{t+s}^t \|X_2(\theta)\|^2 d\theta ds + 2\eta_1 \int_{-\tau_1}^0 \int_{t+s}^t \|X_3(\theta)\|^2 d\theta ds,
 \end{aligned}$$

where

$$(2.2) \quad 0 < \alpha < \min \left\{ \frac{1}{a_0}, \frac{a_0}{b_0}, \frac{a_0 b_0 - c}{a_0 b_0 [a_0 + c^{-1}(b_1 - b_0)^2]}, \frac{c}{a_0 b_0 (a_1 - a_0)} \right\},$$

$a_1 > a_0, b_1 \neq b_0$  and  $\lambda_1$  and  $\eta_1$  are certain positive constants to be determined later in the proof.

From (4), it follows that

$$\begin{aligned}
 2V &= a_0 b_0 \|a_0^{-\frac{1}{2}} X_2 + a_0^{-\frac{1}{2}} b_0^{-1} c X_1\|^2 + \|X_3 + a_0 X_2 + \alpha a_0 b_0 X_1\|^2 \\
 &+ a_0 \int_0^1 \langle \sigma \Psi(\sigma X_2) X_2, X_2 \rangle d\sigma - 2a_0^2 \langle X_2, X_2 \rangle + \langle (B - b_0 I) X_2, X_2 \rangle \\
 &+ \alpha a_0 b_0^2 (1 - \alpha a_0) \langle X_1, X_1 \rangle + c(a_0 - c b_0^{-1}) \langle X_1, X_1 \rangle + a_0(a_0 - \alpha b_0) \langle X_2, X_2 \rangle \\
 &+ 2\lambda_1 \int_{-\tau_1}^0 \int_{t+s}^t \|X_2(\theta)\|^2 d\theta ds + 2\eta_1 \int_{-\tau_1}^0 \int_{t+s}^t \|X_3(\theta)\|^2 d\theta ds.
 \end{aligned}$$

In view of the assumptions of Theorem 1, it is clear that

$$V(0, 0, 0) = 0,$$

$$\begin{aligned}
 &a_0 \int_0^1 \langle \sigma \Psi(\sigma X_2) X_2, X_2 \rangle d\sigma - 2a_0^2 \langle X_2, X_2 \rangle \\
 &= a_0 \int_0^1 \langle (\sigma \Psi(\sigma X_2) - a_0 I) X_2, X_2 \rangle d\sigma \geq \varepsilon a_0 \|X_2\|^2,
 \end{aligned}$$

$$\langle (B - b_0 I)X_2, X_2 \rangle \geq 0,$$

$$\alpha a_0 b_0^2 (1 - \alpha a_0) \langle X_1, X_1 \rangle = \mu_1 \|X_1\|^2,$$

$$\mu_1 = \alpha a_0 b_0^2 (1 - \alpha a_0) > 0,$$

$$c(a_0 - cb_0^{-1}) \langle X_1, X_1 \rangle = \mu_2 \|X_1\|^2,$$

$$\mu_2 = c(a_0 - cb_0^{-1}) > 0,$$

$$a_0(a_0 - \alpha b_0) \langle X_2, X_2 \rangle = \mu_3 \|X_2\|^2,$$

$$\mu_3 = a_0(a_0 - \alpha b_0) > 0.$$

By noting the above discussion, we get

$$\begin{aligned} V &\geq \frac{1}{2} a_0 b_0 \|a_0^{-\frac{1}{2}} X_2 + a_0^{-\frac{1}{2}} b_0^{-1} c X_1\|^2 + \frac{1}{2} \|X_3 + a_0 X_2 + \alpha a_0 b_0 X_1\|^2 \\ &+ \frac{1}{2} (\mu_1 + \mu_2) \|X_1\|^2 + \frac{1}{2} (a_0 \varepsilon + \mu_3) \|X_2\|^2 \\ &+ 2\lambda_1 \int_{-\tau_1}^0 \int_{t+s}^t \|X_2(\theta)\|^2 d\theta ds + 2\eta_1 \int_{-\tau_1}^0 \int_{t+s}^t \|X_3(\theta)\|^2 d\theta ds. \end{aligned}$$

Thus, one can obtain from the above estimate that there exist sufficiently small positive constants  $d_i$ , ( $i = 1, 2, 3$ ), such that

$$V \geq d_1 \|X_1\|^2 + d_2 \|X_2\|^2 + d_3 \|X_3\|^2.$$

Then, we conclude that Liapunov-Krasovskii functional  $V$  is positive definite. Let

$$d_4 = \min\{d_1, d_2, d_3\}.$$

Then, it is clear that

$$V \geq d_4 (\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2).$$

Let  $(X_1, X_2, X_3) = (X_1(t), X_2(t), X_3(t))$  be any solution of system (2). Differentiating this functional, (3), with respect to along system (2), we have

$$\begin{aligned}
\dot{V} &= -\alpha a_0 b_0 c \langle X_1, X_1 \rangle - a_0 \langle X_2, BX_2 \rangle + c \langle X_2, X_2 \rangle \\
&+ \alpha a_0^2 b_0 \langle X_2, X_2 \rangle - \alpha a_0 b_0 \langle X_1, \Psi(X_2)X_3 \rangle \\
&+ \alpha a_0^2 b_0 \langle X_1, X_3 \rangle - \langle \Psi(X_2)X_3, X_3 \rangle \\
&+ a_0 \langle X_3, X_3 \rangle - \alpha a_0 b_0 \langle X_1, BX_2 \rangle \\
&+ \alpha a_0 b_0^2 \langle X_1, X_2 \rangle + \langle X_3, \int_{t-\tau_1}^t BX_3(s)ds \rangle \\
&+ \langle X_3, c \int_{t-\tau_1}^t X_2(s)ds \rangle + \alpha a_0 b_0 \langle X_1, \int_{t-\tau_1}^t BX_3(s)ds \rangle \\
&+ \alpha a_0 b_0 c \langle X_1, \int_{t-\tau_1}^t X_2(s)ds \rangle + a_0 \langle X_2, \int_{t-\tau_1}^t BX_3(s)ds \rangle \\
&+ a_0 c \langle X_2, \int_{t-\tau_1}^t X_2(s)ds \rangle + \lambda_1 \tau_1 \langle X_2, X_2 \rangle + \eta_1 \tau_1 \langle X_3, X_3 \rangle \\
&- \lambda_1 \int_{t-\tau_1}^t \|X_2(s)\|^2 ds - \eta_1 \int_{t-\tau_1}^t \|X_3(s)\|^2 ds \\
&= -\frac{1}{2} \alpha a_0 b_0 c \langle X_1, X_1 \rangle - \langle (a_0 B - cI - \alpha a_0^2 b_0 I)X_2, X_2 \rangle \\
&- \langle (\Psi(X_2) - a_0 I)X_3, X_3 \rangle \\
&- \frac{1}{4} \alpha a_0 b_0 \left\| c^{\frac{1}{2}} X_1 + 2c^{-\frac{1}{2}} (\Psi(X_2) - a_0 I)X_3 \right\|^2 \\
&+ \frac{1}{4} \alpha a_0 b_0 \left\| 2c^{-\frac{1}{2}} (\Psi(X_2) - a_0 I)X_3 \right\|^2 \\
&- \frac{1}{4} \alpha a_0 b_0 \left\| c^{\frac{1}{2}} X_1 + 2c^{-\frac{1}{2}} (B - b_0 I)X_2 \right\|^2 \\
&+ \frac{1}{4} \alpha a_0 b_0 \left\| 2c^{-\frac{1}{2}} (B - b_0 I)X_2 \right\|^2 \\
&+ \langle X_3, \int_{t-\tau_1}^t BX_3(s)ds \rangle + \langle X_3, c \int_{t-\tau_1}^t X_2(s)ds \rangle \\
&+ \alpha a_0 b_0 \langle X_1, \int_{t-\tau_1}^t BX_3(s)ds \rangle + \alpha a_0 b_0 c \langle X_1, \int_{t-\tau_1}^t X_2(s)ds \rangle \\
&+ a_0 \langle X_2, \int_{t-\tau_1}^t BX_3(s)ds \rangle + a_0 c \langle X_2, \int_{t-\tau_1}^t X_2(s)ds \rangle \\
&+ \lambda_1 \tau_1 \langle X_2, X_2 \rangle + \eta_1 \tau_1 \langle X_3, X_3 \rangle
\end{aligned}$$

$$-\lambda_1 \int_{t-\tau_1}^t \|X_2(s)\|^2 ds - \eta_1 \int_{t-\tau_1}^t \|X_3(s)\|^2 ds.$$

Using the assumptions of Theorem 1 and elementary estimates, we obtain

$$\begin{aligned} \langle X_3, \int_{t-\tau_1}^t BX_3(s) ds \rangle &\leq \|X_3\| \left\| \int_{t-\tau_1}^t BX_3(s) ds \right\| \\ &\leq b_1 \|X_3\| \int_{t-\tau_1}^t \|X_3(s)\| ds \\ &\leq \frac{1}{2} b_1 \int_{t-\tau_1}^t \{ \|X_3(t)\|^2 + \|X_3(s)\|^2 \} ds \\ &= \frac{1}{2} b_1 \tau_1 \|X_3\|^2 + \frac{1}{2} b_1 \int_{t-\tau_1}^t \|X_3(s)\|^2 ds, \\ \langle X_3, c \int_{t-\tau_1}^t X_2(s) ds \rangle &\leq c \|X_3\| \left\| \int_{t-\tau_1}^t X_2(s) ds \right\| \\ &\leq c \|X_3\| \int_{t-\tau_1}^t \|X_2(s)\| ds \\ &\leq \frac{1}{2} c \int_{t-\tau_1}^t \{ \|X_3(t)\|^2 + \|X_2(s)\|^2 \} ds \\ &= \frac{1}{2} c \tau_1 \|X_3\|^2 + \frac{1}{2} c \int_{t-\tau_1}^t \|X_2(s)\|^2 ds, \\ \alpha a_0 b_0 \langle X_1, \int_{t-\tau_1}^t BX_3(s) ds \rangle &\leq \alpha a_0 b_0 b_1 \|X_1\| \left\| \int_{t-\tau_1}^t X_3(s) ds \right\| \\ &\leq \alpha a_0 b_0 b_1 \|X_1\| \int_{t-\tau_1}^t \|X_3(s)\| ds \\ &\leq \frac{1}{2} \alpha a_0 b_0 b_1 \int_{t-\tau_1}^t \{ \|X_1(t)\|^2 + \|X_3(s)\|^2 \} ds \\ &= \frac{1}{2} \alpha a_0 b_0 b_1 \tau_1 \|X_1\|^2 + \frac{1}{2} \alpha a_0 b_0 b_1 \int_{t-\tau_1}^t \|X_3(s)\|^2 ds, \\ \alpha a_0 b_0 c \langle X_1, \int_{t-\tau_1}^t X_2(s) ds \rangle &\leq \alpha a_0 b_0 c \|X_1\| \left\| \int_{t-\tau_1}^t X_2(s) ds \right\| \\ &\leq \alpha a_0 b_0 c \|X_1\| \int_{t-\tau_1}^t \|X_2(s)\| ds \\ &\leq \frac{1}{2} \alpha a_0 b_0 c \int_{t-\tau_1}^t \{ \|X_1(t)\|^2 + \|X_2(s)\|^2 \} ds \\ &= \frac{1}{2} \alpha a_0 b_0 c \tau_1 \|X_1\|^2 + \frac{1}{2} \alpha a_0 b_0 c \int_{t-\tau_1}^t \|X_2(s)\|^2 ds, \end{aligned}$$

$$\begin{aligned}
 a_0 \langle X_2, \int_{t-\tau_1}^t BX_3(s)ds \rangle &\leq a_0 \|X_2\| \left\| \int_{t-\tau_1}^t BX_3(s)ds \right\| \\
 &\leq a_0 b_1 \|X_2\| \int_{t-\tau_1}^t \|X_3(s)\| ds \\
 &\leq \frac{1}{2} a_0 b_1 \int_{t-\tau_1}^t \{ \|X_2(t)\|^2 + \|X_3(s)\|^2 \} ds \\
 &= \frac{1}{2} a_0 b_1 \tau_1 \|X_2\|^2 + \frac{1}{2} a_0 b_1 \int_{t-\tau_1}^t \|X_3(s)\|^2 ds, \\
 a_0 c \langle X_2, \int_{t-\tau_1}^t X_2(s)ds \rangle &\leq a_0 c \|X_2\| \left\| \int_{t-\tau_1}^t X_2(s)ds \right\| \\
 &\leq a_0 c \|X_2\| \int_{t-\tau_1}^t \|X_2(s)\| ds \\
 &\leq \frac{1}{2} a_0 c \int_{t-\tau_1}^t \{ \|X_2(t)\|^2 + \|X_2(s)\|^2 \} ds \\
 &= \frac{1}{2} a_0 c \tau_1 \|X_2\|^2 + \frac{1}{2} a_0 c \int_{t-\tau_1}^t \|X_2(s)\|^2 ds.
 \end{aligned}$$

Hence, it follows that

$$\begin{aligned}
 \dot{V}(t) &\leq -\frac{1}{2} \alpha a_0 b_0 c \langle X_1, X_1 \rangle - \langle (a_0 B - cI - \alpha a_0^2 b_0 I) X_2, X_2 \rangle \\
 &\quad - \langle (\Psi(X_2) - a_0 I) X_3, X_3 \rangle \\
 &\quad - \frac{1}{4} \alpha a_0 b_0 \|c^{\frac{1}{2}} X_1 + 2c^{-\frac{1}{2}} (\Psi(X_2) - a_0 I) X_3\|^2 \\
 &\quad + \frac{1}{4} \alpha a_0 b_0 \|2c^{-\frac{1}{2}} (\Psi(X_2) - a_0 I) X_3\|^2 \\
 &\quad - \frac{1}{4} \alpha a_0 b_0 \|c^{\frac{1}{2}} X_1 + 2c^{-\frac{1}{2}} (B - b_0 I) X_2\|^2 \\
 &\quad + \frac{1}{4} \alpha a_0 b_0 \|2c^{\frac{-1}{2}} (B - b_0 I) X_2\|^2 \\
 &\quad + \frac{1}{2} \alpha a_0 b_0 b_1 \tau_1 \|X_1\|^2 + \frac{1}{2} \alpha a_0 b_0 c \tau_1 \|X_1\|^2 \\
 &\quad + \frac{1}{2} a_0 b_1 \tau_1 \|X_2\|^2 + \frac{1}{2} a_0 c \tau_1 \|X_2\|^2 \\
 &\quad + \frac{1}{2} b_1 \tau_1 \|X_3\|^2 + \frac{1}{2} c \tau_1 \|X_3\|^2 \\
 &\quad + \lambda_1 \tau_1 \langle X_2, X_2 \rangle + \eta_1 \tau_1 \langle X_3, X_3 \rangle
 \end{aligned}$$

$$-\{\lambda_1 - \frac{1}{2}(a_0 + \alpha a_0 b_0 + 1)c\} \int_{t-\tau_1}^t \|X_2(s)\|^2 ds$$

$$-\{\eta_1 - (1 + a_0 + \frac{1}{2}\alpha a_0 b_0)b_1\} \int_{t-\tau_1}^t \|X_3(s)\|^2 ds.$$

Let

$$\lambda_1 = \frac{1}{2}(a_0 + \alpha a_0 b_0 + 1)c$$

and

$$\eta_1 = (1 + a_0 + \frac{1}{2}\alpha a_0 b_0)b_1.$$

Then

$$\begin{aligned} \dot{V}(t) \leq & -\frac{1}{2}\alpha a_0 b_0 c \langle X_1, X_1 \rangle \\ & -\langle (a_0 B - cI - \alpha a_0^2 b_0 I) X_2, X_2 \rangle \\ & -\langle (\Psi(X_2) - a_0 I) X_3, X_3 \rangle \\ & + \frac{1}{4}\alpha a_0 b_0 \|2c^{-\frac{1}{2}}(\Psi(X_2) - a_0 I) X_3\|^2 \\ & + \frac{1}{4}\alpha a_0 b_0 \|2c^{-\frac{1}{2}}(B - b_0 I) X_2\|^2 \\ & + \frac{1}{2}(\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c)\tau_1 \|X_1\|^2 + \frac{1}{2}(a_0 b_1 + a_0 c)\tau_1 \|X_2\|^2 \\ & + \frac{1}{2}(a_0 + \alpha a_0 b_0 + 1)c\tau_1 \|X_2\|^2 + \frac{1}{2}(b_1 + c)\tau_1 \|X_3\|^2 \\ & + \frac{1}{2}(1 + a_0 + \alpha a_0 b_0)b_1\tau_1 \|X_3\|^2. \end{aligned}$$

In view of the estimates

$$\begin{aligned} & \frac{1}{4}\alpha a_0 b_0 \|2c^{-\frac{1}{2}}(B - b_0 I) X_2\|^2 \\ & = \alpha a_0 b_0 \langle c^{-1}(B - b_0 I) X_2, (B - b_0 I) X_2 \rangle \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{4}\alpha a_0 b_0 \|2c^{-\frac{1}{2}}(\Psi(X_2) - a_0 I) X_3\|^2 \\ & = \alpha a_0 b_0 \langle (c^{-1}(\Psi(X_2) - a_0 I)) X_3, (\Psi(X_2) - a_0 I) X_3 \rangle, \end{aligned}$$

it follows that

$$\begin{aligned}
\dot{V}(t) &\leq -\frac{1}{2}\alpha a_0 b_0 c \langle X_1, X_1 \rangle \\
&\quad - \langle (a_0 B - cI - \alpha a_0^2 b_0 I) X_2, X_2 \rangle \\
&\quad + \alpha a_0 b_0 \langle c^{-1}(B - b_0 I) X_2, (B - b_0 I) X_2 \rangle \\
&\quad - \langle (\Psi(X_2) - a_0 I) X_3, X_3 \rangle \\
&\quad + \alpha a_0 b_0 \langle c^{-1}(\Psi(X_2) - a_0 I) X_3, (\Psi(X_2) - a_0 I) X_3 \rangle \\
&\quad + \frac{1}{2}(\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c) \tau_1 \|X_1\|^2 + \frac{1}{2}(a_0 b_1 + a_0 c) \tau_1 \|X_2\|^2 \\
&\quad + \frac{1}{2}(a_0 + \alpha a_0 b_0 + 1) c \tau_1 \|X_2\|^2 + \frac{1}{2}(b_1 + c) \tau_1 \|X_3\|^2 \\
&\quad + \frac{1}{2}(1 + a_0 + \alpha a_0 b_0) b_1 \tau_1 \|X_3\|^2.
\end{aligned}$$

By noting Lemma and the assumptions of Theorem 1, it can be obtained that

$$\begin{aligned}
\dot{V}(t) &\leq -\frac{1}{2}\{\alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c) \tau_1\} \|X_1\|^2 \\
&\quad - \langle \{(a_0 B - cI) - \alpha a_0 b_0 [a_0 I + c^{-1}(B - b_0 I)^2]\} X_2, X_2 \rangle \\
&\quad + \frac{1}{2}(a_0 b_1 + a_0 c) \tau_1 \|X_2\|^2 \\
&\quad + \frac{1}{2}(a_0 + \alpha a_0 b_0 + 1) c \tau_1 \|X_2\|^2 \\
&\quad - \langle \{(\Psi(X_2) - a_0 I)[I - \alpha a_0 b_0 c^{-1}(\Psi(X_2) - a_0 I)]\} X_3, X_3 \rangle \\
&\quad + \frac{1}{2}(b_1 + c) \tau_1 \|X_3\|^2 \\
&\quad + \frac{1}{2}(1 + a_0 + \alpha a_0 b_0) b_1 \tau_1 \|X_3\|^2 \\
&\leq -\frac{1}{2}\{\alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c) \tau_1\} \|X_1\|^2 \\
&\quad - \langle \{(a_0 b_0 - c) - \alpha a_0 b_0 [a_0 + c^{-1}(b_1 - b_0)^2]\} X_2, X_2 \rangle \\
&\quad + \frac{1}{2}\{(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1\} \tau_1 \|X_2\|^2 \\
&\quad - \varepsilon [1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2] \|X_3\|^2
\end{aligned}$$

$$+\frac{1}{2}(2b_1 + c + a_0b_1 + \alpha a_0b_0b_1)\tau_1\|X_3\|^2.$$

Let

$$k_1 = 2(a_0b_0 - c) - \alpha a_0b_0[a_0 + c^{-1}(b_1 - b_0)^2] > 0$$

and

$$k_2 = 2\varepsilon[1 - \alpha a_0b_0c^{-1}(a_1 - a_0)^2] > 0.$$

Hence

$$\dot{V}(t) \leq -\frac{1}{2}\{\alpha a_0b_0c - (\alpha a_0b_0b_1 + \alpha a_0b_0c)\tau_1\}\|X_1\|^2$$

$$-\frac{1}{2}\{k_1 - [(2a_0 + \alpha a_0b_0 + 1)c + a_0b_1]\tau_1\}\|X_2\|^2$$

$$-\frac{1}{2}\{k_2 - (2b_1 + c + a_0b_1 + \alpha a_0b_0b_1)\tau_1\}\|X_3\|^2.$$

If

$$\tau_1 < \min \left\{ \frac{\alpha a_0b_0c}{\alpha a_0b_0b_1 + \alpha a_0b_0c}, \frac{k_1}{(2a_0 + \alpha a_0b_0 + 1)c + a_0b_1}, \frac{k_2}{c + (2 + a_0 + \alpha a_0b_0)b_1} \right\},$$

then, for some positive constants  $\rho_1, \rho_2$  and  $\rho_3$ , it follows that

$$\dot{V}(t) \leq -\rho_1\|X_1\|^2 - \rho_2\|X_2\|^2 - \rho_3\|X_3\|^2 \leq 0.$$

In addition, we can easily see that

$$V(X_1, X_2, X_3) \rightarrow \infty \text{ as } \|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \rightarrow \infty.$$

We will now apply LaSalle's invariance principle, so consider the set

$$E \equiv \{(X_1, X_2, X_3) : \dot{V}(X_1, X_2, X_3) = 0\}.$$

Observe that  $(X_1, X_2, X_3) \in E$  implies that  $X_1 = X_2 = X_3 = 0$ . Clearly, the largest invariant set contained in  $E$  is  $(0, 0, 0) \in E$ , and so the zero solution of system (2) is globally asymptotically stable. Hence, the zero solution of Eq. (1) is globally asymptotically stable.

This completes the proof of Theorem 1.

In the case  $P(t) \neq 0$ , the second result of this paper is the following theorem.

**Theorem 2.** We assume that all assumptions of Theorem 1 hold, except  $P(t) \equiv 0$ . In addition, we assume that there exists a non-negative and continuous function  $\theta = \theta(t)$  such that the following condition holds:

$$\|P(t)\| \leq \theta(t) \text{ for all } t \geq 0, \max \theta(t) < \infty \text{ and } \theta \in L^1(0, \infty),$$

where  $L^1(0, \infty)$  denotes the space of Lebesgue integrable functions.

If

$$\tau_1 < \min \left\{ \frac{\alpha a_0 b_0 c}{\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c}, \frac{k_1}{(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1}, \frac{k_2}{c + (2 + a_0 + \alpha a_0 b_0)b_1} \right\},$$

where

$$k_1 = 2(a_0 b_0 - c) - \alpha a_0 b_0 [a_0 + c^{-1}(b_1 - b_0)^2] > 0$$

and

$$k_2 = 2\varepsilon [1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2] > 0,$$

then there exists a constant  $D > 0$  such that any solution  $(X_1(t), X_2(t), X_3(t))$  of system (2) determined by

$$X_1(0) = X_{10}, X_2(0) = X_{20}, X_3(0) = X_{30}$$

satisfies

$$\|X_1(t)\| \leq D, \|X_2(t)\| \leq D, \|X_3(t)\| \leq D$$

for all  $t \in \mathfrak{R}^+$ .

**Proof.** In the case of  $P(t) \neq 0$  under the assumptions of Theorem 2, it can be easily seen that

$$\begin{aligned} \dot{V}(t) &\leq -\frac{1}{2} \{ \alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c) \tau_1 \} \|X_1\|^2 \\ &\quad - \frac{1}{2} \{ k_1 - [(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1] \tau_1 \} \|X_2\|^2 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\{k_2 - (2b_1 + c + a_0b_1 + \alpha a_0b_0b_1)\tau_1\}\|X_3\|^2 \\
& + \langle X_3, P(t) \rangle + \alpha a_0b_0 \langle X_1, P(t) \rangle + a_0 \langle X_2, P(t) \rangle \\
& \leq (\alpha a_0b_0\|X_1\| + a_0\|X_2\| + \|X_3\|)\|P(t)\| \\
& \leq \sigma(\|X_1\| + \|X_2\| + \|X_3\|)\|P(t)\| \\
& \leq \sigma(3 + \|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2)\theta(t),
\end{aligned}$$

where

$$\sigma = \max\{\alpha a_0b_0, a_0, 1\}.$$

On the other hand, it is easy to see that

$$\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \leq d_4^{-1}V.$$

In view of the above discussion, we get

$$\dot{V}(t) \leq 3\sigma\theta(t) + d_4^{-1}V(t)\theta(t).$$

Integrating both sides of the last estimate from 0 to  $t$ , ( $t \geq 0$ ), one can easily obtain

$$V(t) - V(0) \leq 3\sigma \int_0^t \theta(s)ds + d_4^{-1} \int_0^t V(s)\theta(s)ds.$$

Then

$$V(t) \leq d + d_4^{-1} \int_0^\infty V(s)\theta(s)ds,$$

where

$$d = V(0) + 3\sigma \int_0^\infty \theta(s)ds.$$

By using Gronwall-Bellman inequality (see Ahmad and Rama Mohana Rao [4]), we conclude that

$$V(t) \leq d \exp(d_4^{-1} \int_0^\infty \theta(s)ds).$$

Thus, all solutions of system (2) are bounded.

For the case  $P(t) \neq 0$ , the third and last main result is the following theorem.

**Theorem 3.** We assume that all assumptions of Theorem 1 hold, except  $P(t) \equiv 0$ . In addition, we assume that there exists a positive constant  $\delta_0$  such that the condition

$$\|P(t)\| \leq \delta_0, (t \geq 0),$$

holds.

If

$$\tau_1 < \min \left\{ \frac{\alpha a_0 b_0 c}{\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c}, \frac{k_1}{(2a_0 + \alpha a_0 b_0 + 1)c + a_0 b_1}, \frac{k_2}{c + (2 + a_0 + \alpha a_0 b_0)b_1} \right\},$$

where

$$k_1 = 2(a_0 b_0 - c) - \alpha a_0 b_0 [a_0 + c^{-1}(b_1 - b_0)^2] > 0$$

and

$$k_2 = 2\varepsilon [1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2] > 0,$$

then there exists a constant  $d > 0$  such that any solution  $(X_1(t), X_2(t), X_3(t))$  of system (2) determined by

$$X_1(0) = X_{10}, X_2(0) = X_{20}, X_3(0) = X_{30}$$

ultimately satisfies

$$\|X_1(t)\|^2 + \|X_2(t)\|^2 + \|X_3(t)\|^2 \leq d$$

for all  $t \in \mathfrak{R}^+$ .

**Proof.** When  $P(t) \neq 0$ , under the assumptions of Theorem 3, we can re-arrange the time derivative of the Liapunov- Krasovskii functional  $\dot{V}(t)$  as the following:

$$\dot{V}(t) \leq -\rho_1 \|X_1\|^2 - \rho_2 \|X_2\|^2 - \rho_3 \|X_3\|^2$$

$$\begin{aligned}
& +(\alpha a_0 b_0 \|X_1\|^2 + a_0 \|X_2\| + \|X_3\|) \|P(t)\| \\
& \leq -\rho_1 \|X_1\|^2 - \rho_2 \|X_2\|^2 - \rho_3 \|X_3\|^2 \\
& +(\alpha a_0 b_0 \delta_0 \|X_1\|^2 + a_0 \delta_0 \|X_2\| + \delta_0 \|X_3\|).
\end{aligned}$$

Let

$$\bar{d}_2 = \frac{1}{2} \max\{\rho_1, \rho_2, \rho_3\}$$

and

$$\bar{d}_3 = \max\{\alpha a_0 b_0 \delta_0, a_0 b_0 \delta_0, \delta_0\}.$$

Therefore,

$$\dot{V}(t) \leq -2\bar{d}_2 \{\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2\} + \bar{d}_3 (\|X_1\| + \|X_2\| + \|X_3\|).$$

In view of Schwarz's inequality, it can be written that

$$\dot{V}(t) \leq -2\bar{d}_2 \{\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2\} + \bar{d}_4 (\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2)^{\frac{1}{2}},$$

where  $\bar{d}_4 = \sqrt{3}\bar{d}_3$ .

If

$$(\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2)^{\frac{1}{2}} \geq \bar{d}_5 = \bar{d}_4 \bar{d}_2^{-1},$$

then we get

$$\dot{V}(t) \leq -\bar{d}_2 (\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2).$$

Hence, it follows that there exists a positive constants  $\bar{d}_6$  such that

$$\dot{V}(t) \leq -1$$

if  $\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \geq \bar{d}_6^2$ .

The remaining of the proof can be completed easily by following a similar was as shown in Meng [21]. Therefore, we would not like to give the details of the proof.

### 3. Conclusion

A class of vector functional differential equations of third order with a constant retardation has been considered. The stability/boundedness/ ultimately boundedness of solutions of these equations have been discussed by using the Liapunov- Krasovskii functional approach. The obtained results extend and improve some recent results in the literature.

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