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Generalized b-closed sets in ideal bitopological spaces

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Abstract

In this article we introduce the concept of generalized b-closed sets with respect to an ideal in bitopological spaces, which is the extension of the concepts of generalized b-closed sets.

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1. Introduction.

The concept of bitopological spaces (X, τ_1, τ_2) was introduced by Kelly [6]. The bitopological spaces are equipped with two arbitrary topologies τ_1 and τ_2. The concept of ideals has been applied in topological spaces and studied by Kuratowski [7], Vaidyanathasamy [17] and Jankovic and Hamlett [5] and others.

An ideal I on a non-empty set X is a collection of subsets of X which satisfies $(i) A \in I$ and $B \subset A$ implies $B \in I$ and $(ii) A \in I$ and $B \in I$ implies $A \cup B \in I$. The notion of ideal has been applied for investigations in different directions. In sequence spaces ideal convergence has recently been studied by Tripathy and Hazarika [9], Tripathy and Mahanta [10], Tripathy etal. [16] and many others.

If I is an ideal on X, then (X, τ_1, τ_2, I) is called an ideal bitopological space. Andrijevic [3] introduced the notion of b-open sets in topological spaces. Later on this notion has been extended to bitopological setting by Abo Khadra and Nasef [1], Al-Hawary and Al-Omari [2] and many others. Recently, Sarsak and Rajesh [8], Tripathy and Sarma ([12], [13], [14]) have done some works on bitopological spaces using this notion. During recent years many topologists were interested in the study of different types of generalized closed sets. Mean while Fukutake [4] introduced the concept of generalized closed sets in bitopological spaces. On the other hand Tripathy and Sarma [15] introduced the notion of generalized b-closed sets in bitopological spaces and studied their basic properties. Recently different properties of the mixed topological spaces have been investigated from fuzzy settings by Tripathy and Ray [11] and others.

In this paper we introduce generalized b-closed sets with respect to an ideal in bitopological spaces and have studied some of its basic properties.

2. Preliminaries.

Throughout the paper (X, τ_1, τ_2) denotes a bitopological space on which no separation axioms are assumed and (X, τ_1, τ_2, I) be an ideal bitopological space, where $i, j \in \{1, 2\}, i \neq j$. Let A be a subset of X.

We use the following notations.

$(i)A$ is open with respect to τ_i if and only if A is i-open in (X, τ_1, τ_2, I).

$(ii)A$ is closed with respect to τ_i if and only if A is i-closed in (X, τ_1, τ_2, I).

Now we list some known definitions and results those will be used throughout this article.

The following definitions and results are due to Al-Hawary and Al-Omari [2].

Definition 2.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(i) $(i,j)-b$-open if $A \subset \tau_i - int(\tau_j - cl(A)) \cup \tau_j - cl(\tau_i - int(A))$.

(ii) $(i,j)-b$-closed if $\tau_i - cl(\tau_j - int(A)) \cap \tau_j - int(\tau_i - cl(A)) \subset A$.
By (i,j) we mean the pair of topologies (τ_i, τ_j).

Definition 2.2. Let A be a subset of a bitopological space (X, τ_1, τ_2).

(i) The $(i,j)-b$-closure of A denoted by $(i,j)-bcl(A)$, is defined by the intersection of all $(i,j)-b$-closed sets containing A.

(ii) The $(i,j)-b$-interior of A denoted by $(i,j)-bint(A)$, is defined by the union of all $(i,j)-b$-open sets contained in A.

Lemma 2.1. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then

(i) $(i,j)-bint(A)$ is $(i,j)-b$-open.

(ii) $(i,j)-bcl(A)$ is $(i,j)-b$-closed.

(iii) A is $(i,j)-b$-open if and only if $A = (i,j)-bint(A)$.

(iv) A is $(i,j)-b$-closed if and only if $A = (i,j)-bcl(A)$.

Lemma 2.2. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of

X. Then

(*i*) $x \in (i,j) - bcl(A)$ if and only if for every $(i,j) - b$-open set U containing x, $U \cap A \neq \emptyset$.

(*ii*) $x \in (i,j) - bint(A)$ if and only if there exists an $(i,j) - b$-open set U such that $x \in U \subset A$.

(*iii*) If $A \subset B$, then $(i,j) - bint(A) \subset (i,j) - bint(B)$ and $(i,j) - bcl(A) \subset (i,j) - bcl(B)$.

The following result is due to Sarsak and Rajesh [8].

Lemma 2.3. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X. Then

(*i*) $X - (i,j) - bint(A) = (i,j) - bcl(X - A)$.

(*ii*) $X - (i,j) - bcl(A) = (i,j) - bint(X - A)$.

The following definition is due to Tripathy and Sarma [15].

Definition 2.3. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i,j)-generalized b-closed (in short, $(i,j) - gb$-closed) set if $(j,i) - bcl(A) \subset U$ whenever $A \subset U$ and U is τ_i-open in X.

3. $(i,j) - I$-generalized b-closed Sets

Definition 3.1. Let (X, τ_1, τ_2, I) be an ideal bitopological space. A subset A of X is said to be $(i,j) - I$-generalized b-closed (in short, $(i,j) - Igb$-closed) set if $(j,i) - bcl(A) \setminus B \in I$ whenever $A \subset B$ and B is τ_i-open in X, for $i,j = 1,2$ and $i \neq j$.

Theorem 3.1. Every $(i,j) - gb$-closed set is $(i,j) - Igb$-closed.

Proof. Easy, so omitted.

Remark 3.1. The converse of the above Theorem is not necessarily true.

This is clear from the following example.

Example 3.1. Let $X = \{a, b, c\}$, consider the topologies $\tau_1 = \{\emptyset, \{a\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Here $\{a\}$ is (1, 2)-Igb-closed set but not (1, 2)-gb-closed since (2, 1)-$bcl(\{a\}) = X$ not a subset of $\{a\}$.

Theorem 3.2. Let (X, τ_1, τ_2, I) be an ideal bitopological space. If A is $(i, j) - Igb$-closed and $A \subset B \subset (j, i)$-$bcl(A)$ in X, then B is $(i, j) - Igb$-closed in X, where $i, j = 1, 2$ and $i \neq j$.

Proof. Let $B \subset V$ and V is τ_i-open. Since $A \subset B \subset (j, i) - bcl(A)$, we have $A \subset V$. By hypothesis $(j, i) - bcl(A) \setminus V \in I$. Further $B \subset (j, i) - bcl(A)$ implies that $(j, i) - bcl(B) \setminus V \subset (j, i) - bcl(A) \setminus V \in I$. Thus $(j, i) - bcl(B) \setminus V \in I$. Consequently B is $(i, j) - Igb$-closed.

Theorem 3.3. Union of two $(i, j) - Igb$-closed sets in an ideal bitopological space (X, τ_1, τ_2, I) is also $(i, j) - Igb$-closed.

Proof. Let A and B be two $(i, j) - Igb$-closed sets with $A \cup B \subset V$, where V is τ_i-open. Clearly $A \subset V$ and $B \subset V$. Since A and B are $(i, j) - Igb$-closed, we have $(j, i) - bcl(A) \setminus V \in I$ and $(j, i) - bcl(B) \setminus V \in I$. Now $(j, i) - bcl(A \cup B) \setminus V = ((j, i) - bcl(A) \cup (j, i) - bcl(B)) \setminus V = ((j, i) - bcl(A) \setminus V) \cup ((j, i) - bcl(B) \setminus V) \in I$. Thus $(j, i) - bcl(A \cup B) \setminus V \in I$ and hence $A \cup B$ is $(i, j) - Igb$-closed set.

Remark 3.2. The intersection of two (i, j)-Igb-closed sets is not necessarily a $(i, j) - Igb$-closed set is clear from the following example.

Example 3.2. Let $X = \{a, b, c\}$, consider the topologies $\tau_1 = \{\emptyset, \{a\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $I = \{\emptyset\}$. Here $\{a, b\}$ and $\{a, c\}$ are (1, 2)-Igb-closed sets but $\{a, b\} \cap \{a, c\} = \{a\}$ is not (1, 2)-Igb-closed.

Theorem 3.4. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Suppose A is $(i, j) - Igb$-closed in X and $A \subset Y \subset X$. Then A is $(i, j) - Igb$-closed relative to the subspace Y of X and with respect to the ideal $I_Y = \{P \subset Y : P \in I\}$.

Proof. Let V be τ_i-open in X and $A \subset Y \cap V$. Therefore we have $A \subset V$.

Since A is $(i,j)-Igb$-closed, therefore we have $(j,i)-bcl(A)\setminus V \in I$. Further we see that $((j,i)-bcl(A)\cap Y)\setminus(Y\cap V) = ((j,i)\text{-}bcl(A)\setminus V)\cap Y \in I_Y$. Thus for $A \subset Y\cap V$ and V is τ_i-open, we have $((j,i)-bcl(A)\cap Y)\setminus(Y\cap V) \in I_Y$. Hence A is $(i,j)-Igb$-closed relative to the subspace $(Y, \tau_1|Y, \tau_2|Y)$.

Definition 3.2. Let (X, τ_1, τ_2, I) be an ideal bitopological space. A subset A of X is said to be $(i,j)-I$-generalized b-open(in short, $(i,j)-Igb$-open) set if $X \setminus A$ is $(i,j)-Igb$-closed, for $i,j = 1,2$ and $i \neq j$.

Theorem 3.5. Let (X, τ_1, τ_2, I) be an ideal bitopological space. A subset A of X is $(i,j)-Igb$-open in X if and only if $B \setminus P \subset (j,i)-bint(A)$ for some $P \in I$, whenever $B \subset A$ and B is τ_i-closed.

Proof. Let $B \subset A$ and B be τ_i-closed. Clearly $X \setminus A \subset X \setminus B$. Since A is $(i,j)-Igb$-open, therefore we have $X\setminus A$ is $(i,j)-Igb$-closed. By definition $(j,i)-bcl(X\setminus A)\setminus(X\setminus B) \in I$. This implies $(j,i)-bcl(X\setminus A) \subset (X\setminus B)\cup P$ for some $P \in I$. This gives that $X \setminus ((X \setminus B)\cup P) \subset X \setminus (j,i)-bcl(X \setminus A)$. Thus $B \setminus P \subset X \setminus (X \setminus (j,i)-bint(A))$ and hence $B \setminus P \subset (j,i)-bint(A)$.

Conversely suppose that $B \subset A$ and B is τ_i-closed. By hypothesis we have $B \setminus P \subset (j,i)-bint(A)$ where $P \in I$. This implies $B \setminus P \subset X \setminus (j,i)-bcl(X \setminus A)$. Thus $X \setminus (X \setminus (j,i)-bcl(X \setminus A)) \subset X \setminus (B \setminus P)$ and consequently we have $(j,i)-bcl(X \setminus A) \subset (X \setminus B) \cup P$. Hence $(j,i)-bcl(X \setminus A) \setminus (X \setminus B) \in I$ for some $P \in I$. This shows that $X \setminus A$ is $(i,j)-Igb$-closed and so A is $(i,j)-Igb$-open.

Theorem 3.6. Let (X, τ_1, τ_2, I) be an ideal bitopological space. If A is $(i,j)-Igb$-open in X and $(j,i)\text{-}bint(A) \subset B \subset A$, then B is $(i,j)-Igb$-open in X.

Proof. Assume that A be $(i,j)-Igb$-open. Then $X\setminus A$ is $(i,j)-Igb$-closed. Since $(j,i)-bint(A) \subset B \subset A$, we have $X\setminus A \subset X\setminus B \subset X\setminus(j,i)-bint(A) = (j,i)-bcl(X\setminus A)$. Then by Theorem 3.2 , we have $X\setminus B$ is $(i,j)-Igb$-closed and hence B is $(i,j)-Igb$-open.

Theorem 3.7. The intersection of two $(i,j)-Igb$-open sets in an ideal bitopological space (X, τ_1, τ_2, I) is also $(i,j)-Igb$-open.

Proof. Suppose A and B be two $(i,j)-Igb$-open sets in X. Then $X \setminus A$ and $X\setminus B$ are $(i,j)-Igb$-closed. Now we have $X\setminus(A\cap B) = (X\setminus A)\cup(X\setminus B)$

is $(i,j)-Igb$-closed, by Theorem 3.3. Hence $A\cap B$ is (i,j)-Igb-open.

Theorem 3.8. Let (X,τ_1,τ_2,I) be an ideal bitopological space. If A and B are two $(i,j)-Igb$-open sets in X such that $(j,i)-bcl(A)\cap B=\emptyset$ and $A\cap(j,i)-bcl(B)=\emptyset$, then $A\cup B$ is $(i,j)-Igb$-open.

Proof. Let A and B be two $(i,j)-Igb$-open sets in X such that $(j,i)-bcl(A)\cap B=\emptyset$ and $A\cap(j,i)-bcl(B)=\emptyset$. Suppose V is τ_i-closed and $V\subset A\cup B$. Clearly $V\subset A$ and $V\subset B$. Then $V\cap(j,i)-bcl(A)\subset A\cap(j,i)-bcl(A)=A$ and $V\cap(j,i)-bcl(B)\subset B\cap(j,i)-bcl(B)=B$. By hypothesis we have $(V\cap(j,i)-bcl(A))\backslash P\subset(j,i)-bint(A)$ and $(V\cap(j,i)-bcl(B))\backslash Q\subset(j,i)-bint(B)$ for some $P,Q\in I$. This implies $(V\cap(j,i)-bcl(A))\setminus(j,i)-bint(A)\in I$ and $(V\cap(j,i)-bcl(B))\setminus(j,i)-bint(B)\in I$. Then $((V\cap(j,i)$-$bcl(A))\setminus(j,i)-bint(A))\cup((V\cap(j,i)-bcl(B))\setminus(j,i)-bint(B))\in I$. Which implies $(V\cap((j,i)-bcl(A)\cup(j,i)-bcl(B)))\setminus((j,i)-bint(A)\cup(j,i)-bint(B))\in I$. Thus $(V\cap(j,i)-bcl(A\cup B))\setminus((j,i)-bint(A)\cup(j,i)-bint(B))\in I$. Further, $V=V\cap(A\cap B)\subset V\cap(j,i)-bcl(A\cup B)$, we have $V\setminus(j,i)-bint(A\cup B)\subset(V\cap(j,i)-bcl(A\cup B))\setminus(j,i)-bint(A\cup B)\subset(V\cap(j,i)-bcl(A\cup B))\setminus((j,i)-bint(A)\cup(j,i)-bint(B))\in I$. This shows that $V\setminus R\subset(j,i)-bint(A\cup B)$ for some $R\in I$. Hence $A\cup B$ is $(i,j)-Igb$-open.

Theorem 3.9. Let (X,τ_1,τ_2,I) be an ideal bitopological space. If A is $(i,j)-Igb$-open set relative to B such that $A\subset B\subset X$ and B is $(i,j)-Igb$-open relative to X, then A is $(i,j)-Igb$-open relative to X.

Proof. Let $U\subset A$ and U be τ_i-closed. Suppose A is $(i,j)-Igb$-open relative to B. Then we have $U\setminus P\subset(j,i)-bint_B(A)$ for some $P\in I_B$, where I_B denotes the ideal of the set B. Which implies that there exists a $(j,i)-b$-open set V_1 such that $U\setminus P\subset V_1\cap B\subset A$. Let $U\subset B$ and U is τ_i-closed. Suppose B is $(i,j)-Igb$-open relative to X. Then we have $U\setminus Q\subset(j,i)-bint(B)$ for some $Q\in I$. Which implies that there exists a $(j,i)-b$-open set V_2 such that $U\setminus Q\subset V_2\subset B$. Further $U\setminus(P\cup Q)=(U\setminus P)\cap(U\setminus Q)\subset(V_1\cap B)\cap V_2)\subset(V_1\cap B)\cap B=V_1\cap B\subset A$. This shows that $U\setminus(P\cup Q)\subset(j,i)$-$bint(A)$ for some $P\cup Q\in I$. Hence A is $(i,j)-Igb$-open relative to X.

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