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On I-statistically convergent sequence spaces defined by sequences of Orlicz functions using matrix transformation

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Abstract

Recently Savas and Das [12] introduced the notion of I-statistical convergence of sequences of real numbers. In this article we introduced the sequence spaces $W^{I(S)}(M, A, p)$, $W_0^{I(S)}(M, A, p)$ *and* $W_\infty^{I(S)}(M, A, p)$ *of real numbers defined by I-statistical convergence using sequences of Orlicz function. We study some basic topological and algebraic properties of these spaces. We investigate some inclusion relations involving these spaces.*

Key words : *Ideal, I-statistical convergence, Orlicz function, matrix transformation.*

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1. Introduction

The notion of statistical convergence was introduced by Fast [4] and Schoenberg [11], independently.Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory by Buck [1], Esi and Et [3]. Moreover, statistical convergence is closely related to the concept of convergence in probability.

$$i.e.\;\; X_A(k) = \begin{cases} 1 & \text{if} \quad k \in A \\ \\ 0 & \text{if} \quad k \in N \setminus A \end{cases} \quad \text{and} \quad d_n(A) = \frac{1}{n}\sum_{k=1}^{n} X_A(k)$$

The idea of statistical convergence depends on the density of subsets of the set N of natural numbers. Let N be the set of natural numbers. If $A \subseteq N$, then χ_A denotes the characteristic function of the set A

Then the number $\underline{d}(A) = \liminf d_n(A)$ and $\overline{d}(A) = \limsup d_n(A)$ are called the lower and upper asymptotic density of A respectively. If $\underline{d}(A)$ $=\overline{d}(A)=d(A)$ then $d(A)$ is called the asymptotic density of A. We see that asymptotic density is limit of frequencies of numbers in the set $\{0, 1, 2, \ldots\}$, therefore it is (when it exists) intuitively correct measure of size of subsets of integers. It is clear that any finite subset of N has natural density zero and $d(A^c)$ =1- $d(A)$. Asymptotic density is (in some context) appropriate way to describe whether a subset of natural numbers is small or large.

A sequence $x = (x_n)$ is said to be statistically convergent to a number $L \in R$ if for each $\varepsilon > 0$, $d(A(\varepsilon)) = 0$,where $A(\varepsilon)$= $\{n \in N : |x_n - L| \geq \varepsilon\}$.

In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Kostyrko et.al.[7] presented a new generalization of statistical convergence and called it I-convergence. They used the notion of an ideal I of subsets of the set N to define such a concept.

Let X be a non-empty set. Then a family of sets $I \subset 2^X$ is said to be an ideal if I is additive, i.e, $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I$, $B \subset A \Rightarrow B \in I$. A non-empty family of sets $F \subset 2^X$ is said to be a filter on X if and only if i)$\emptyset \notin F$ ii) for all $A, B \in F \Rightarrow A \cap B \in F$ iii)$A \in F$, $A \subset B \Rightarrow B \in F$. An ideal $I \subset 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal I is called admissible iff $I \supset \{\{x\} : x \in X\}$. A non-trivial ideal I is maximal if there does not exist any non-trivial ideal J$\neq I$, containing I as a subset. For each ideal I there is a filter $F(I)$ corresponding to I i.e $F(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N - K$.

A sequence $x = (x_n)$ is said to be I-convergent to a number $L \in R$ if for a given $\varepsilon > 0$,we have $A(\varepsilon)= \{n \in N : |x_n - L| \geq \varepsilon\} \in I$. The element L is called the I- limit of the sequence $x = (x_n)$.

Example 1.1: Let $I=I_f=\{A \subseteq N : A$ is finite$\}$. Then I_f is nontrivial admissible ideal of N and the corresponding convergence coincides with ordinary convergence. If $I=I_d=\{A \subseteq N : d(A) = 0\}$, where $d(A)$ denotes the asymtotic density of the set A. Then I_d is a non-trivial admissible ideal of N and the corresponding convergence coincide with statistical convergence. For more on I-convergence one may refer to [2,16,19,20,21,23].

An Orlicz function M :$[0, \infty) \to [0, \infty)$ is a continuous, convex, non-decreasing function defined for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function. An Orlicz function M is said to satisfy $\triangle_2$-condition for all values of u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$. Let M be an Orlicz function which satisfies $\triangle_2$-condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}tM(2)$ for some constant $K > 0$. Two Orlicz functions M_1 and M_2 are said to be equivalent if there exists positive constants α, β and x_0 such that $M_1(\alpha) \leq M_2(x) \leq M_1(\beta)$, for all $0 \leq x < x_0$.

Lindenstrass and Tzafriri [8] studied some Orlicz type sequence spaces defined as follows:

$$\ell_M = \{(x_k) \in w : \sum M(\frac{|x_k|}{\rho}) < \infty, for \rho > 0\}$$

The space ℓ_M with the norm

$$\|x\| = inf\{\rho > 0 : \sum M(\frac{|x_k|}{\rho}) \leq 1\},$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to ℓ_p which is an Orlicz sequence space with $M(t)=|t|^p$, for $1 \leq p < \infty$. Different classes of Orlicz sequence spaces were introduced and studied by Parasar and Choudhury [10], Esi and Et [3], Tripathy and Hazarika [17] and many others.

The notion of paranormed sequences was introduced by Nakano [9]. It was further investigated by Tripathy et. al.[14,15,22] and many others.

Definition 1.2: (Savas and Das[12]) A sequence $x = (x_k)$ is said to be I-statistically convergent to a number $L \in R$ if for each $\varepsilon > 0$, $\{n \in N : \frac{1}{n}|k \leq n : \|x_k - L\| \geq \varepsilon| \geq \delta\} \in I$.

The number L is called I-statistical limit of the sequence (x_k) and we write $I - st - lim x_k = L$.

Remark 1.3: (Savas and Das[12]) Let I=I_f=$\{A \subseteq N : A \text{ is finite}\}$. Then I_f is nontrivial admissible ideal of N and I-statistical convergence coincides with statistical convergence.

Definition 1.4: A sequence space E is said to be solid (or normal) if $(y_k) \in E$ whenever $(x_k) \in E$ and $|y_k| \leq |x_k|$ for all $k \in N$.

Lemma 1.5: (One may refer to Kamthan and Gupta[6]) A sequence space E is normal implies that it is monotone.

Lemma 1.6: If $I \subset 2^N$ is a maximal ideal then for each $A \subset N$, we have either $A \in I$ or $N - A \in I$.

The following well-known inequality will be used throughout the article.

Let $p = (p_k)$ be any sequence of positive real numbers with $0 \leq p_k \leq \sup p_k = G$ and $D = max\{1, 2^{G-1}\}$. Then $|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k})$ for all $k \in N$ and $a_k, b_k \in C$.

Also $|a_k|^{p_k} \leq max\{1, |a|^G\}$ for all $a \in C$.

2. Main Result

Let $M = (M_k)$ be a sequence of Orlicz functions and $A = (a_{ik})$ be an infinite matrix and $x = (x_k)$ be a sequence of real or complex numbers. We write Ax=$(A_k(x))$ if $A_k(x)$ =$\sum_k a_{ik}x_k$ converges for each i.

We define the following sequence spaces in this article:

$W^{I(S)}(M, A, p)$=$\{(x_k) \in w : \{n \in N : \frac{1}{n}|\{k \leq n : \sum_{k=1}^{n}[M_k(\frac{\|A_k(x)-L\|}{\rho})]^{p_k} \geq \varepsilon\}| \geq \delta\} \in I$ for some $\rho > 0$ and $L \in R\}$.

$W_0^{I(S)}(M, A, p)$=$\{(x_k) \in w : \{n \in N : \frac{1}{n}|\{k \leq n : \sum_{k=1}^{n}[M_k(\frac{\|A_k(x)\|}{\rho})]^{p_k} \geq \varepsilon\}| \geq \delta\} \in I$ for some $\rho > 0$ $\}$.

$W_\infty^{I(S)}(M, A, p)$=$\{(x_k) \in w : \{n \in N : \frac{1}{n}|\{k \leq n : \sum_{k=1}^{n}[M_k(\frac{\|A_k(x)\|}{\rho})]^{p_k} \geq$ M$\}| \geq \delta\} \in I$ for some $M > 0$ $\}$.

$W_\infty(M, A, p)$=$\{(x_k) \in w : \{n \in N : \sup \frac{1}{n} \sum_{k=1}^{n}[M_k(\frac{\|A_k(x)\|}{\rho})]^{p_k} < \infty$ $\}\}$.

From the above definition it is obvious that
$W_0^{I(S)}(M, A, p) \subset W^{I(S)}(M, A, p) \subset W_\infty^{I(S)}(M, A, p)$.

Theorem 2.1: The spaces $W_0^{I(S)}(M,A,p)$, $W^{I(S)}(M,A,p)$ and $W_\infty^{I(S)}(M,A,p)$ are linear space.

Proof: We prove the result for the space $W_0^{I(S)}(M,A,p)$. The other result can be established in similar way.

Let $x=(x_k)$ and $y=(y_k)$ be any two elements in $W_0^{I(S)}(M,A,p)$. Then there exists $\rho_1>0$ and $\rho_2>0$ such that
$A=\{n\in N:\frac{1}{n}|\{k\le n:\sum_{k=1}^{n}[M_k(\frac{\|A_k(x)\|}{\rho_1})]^{p_k}\ge\frac{\varepsilon}{2}\}|\ge\delta\}\in I$

and $B=\{n\in N:\frac{1}{n}|\{k\le n:\sum_{k=1}^{n}[M_k(\frac{\|A_k(x)\|}{\rho_2})]^{p_k}\ge\frac{\varepsilon}{2}\}|\ge\delta\}\in I$

Let a,b be any scalars. By the continuity of the sequence $M=(M_k)$ the following inequality holds:

$$\sum_{k=1}^{n}[M_k(\frac{\|A_k(ax+by)\|}{|a|\rho_1+|b|\rho_2})]^{p_k}\le DK\sum_{k=1}^{n}[M_k(\frac{\|A_k(x)\|}{\rho_1})]^{p_k}+DK\sum_{k=1}^{n}[M_k(\frac{\|A_k(y)\|}{\rho_2})]^{p_k}$$

$$\le D\sum_{k=1}^{n}[\frac{|a|}{|a|\rho_1+|b|\rho_2}M_k(\frac{\|A_k(x)\|}{\rho_1})]^{p_k}+D\sum_{k=1}^{n}[\frac{|b|}{|a|\rho_1+|b|\rho_2}M_k(\frac{\|A_k(x)\|}{\rho_1})]^{p_k}$$

where $K=max\{1,\frac{|a|}{|a|\rho_1+|b|\rho_2},\frac{|b|}{|a|\rho_1+|b|\rho_2}\}$.

From the above relation we get the following:
$\{n\in N:\frac{1}{n}|\{k\le n:\sum_{k=1}^{n}[M_k(\frac{\|A_k(ax+by)\|}{|a|\rho_1+|b|\rho_2})]^{p_k}\ge\frac{\varepsilon}{2}\}|\ge\delta\}\subseteq$

$\{n\in N:\frac{1}{n}|\{k\le n:\sum_{k=1}^{n}DK[M_k(\frac{\|A_k(x)\|}{\rho_1})]^{p_k}\ge\frac{\varepsilon}{2}\}|\ge\delta\}$

$\bigcup\{n\in N:\frac{1}{n}|\{k\le n:\sum_{k=1}^{n}DK[M_k(\frac{\|A_k(y)\|}{\rho_2})]^{p_k}\ge\frac{\varepsilon}{2}\}|\ge\delta\}$

This completes the proof.

Theorem 2.2: The space $W_\infty(M,A,p)$ is a paranormed spaces (not totally paranormed) with the paranorm g defined by:
$g(x)=\inf\{\rho^{\frac{p_k}{H}}:\sup_k M_k(\frac{\|A_k(x)\|}{\rho})\le1,\text{ for }\rho>0\}$, where $H=\max\{1,\sup_k p_k\}$.

Proof: It is obvious that $g(\theta)=0$ (where θ is the sequence of zeros), $g(-x)=g(x)$ and it can be easily shown that $g(x+y)\le g(x)+g(y)$.

Let $t_n\to L$, where $t_n,L\in C$ and let $g(x_n-x)\to0$, as $n\to\infty$. To prove that $g(t_nx_n-Lx)\to0$, as $n\to\infty$. We put

$$A=\{\rho_1>0:\sup_k[M_k(\frac{\|A_k(x)\|}{\rho_1})]^{p_k}\leq 1\}$$

and

$$B=\{\rho_2>0:\sup_k[M_k(\frac{\|A_k(x)\|}{\rho_2})]^{p_k}\leq 1\}$$

.

By the continuity of the sequence $M=(M_k)$, we observe that

$M_k(\frac{\|A_k(t_nx_n-Lx)\|}{|t_n-L|\rho_1+|L|\rho_2})\leq M_k(\frac{\|A_k(t_nx_n-Lx_n)\|}{|t_n-L|\rho_1+|L|\rho_2})+M_k(\frac{\|A_k(Lx_n-Lx)\|}{|t_n-L|\rho_1+|L|\rho_2})$

$\leq \frac{|t_n-L|\rho_1}{|t_n-L|\rho_1+|L|\rho_2}M_k(\frac{\|A_k(x_n)\|}{\rho_1})+\frac{|L|\rho_2}{|t_n-L|\rho_1+|L|\rho_2}M_k(\frac{\|A_k(x_n-x)\|}{\rho_2})$

From the above inequality it follows that

$$\sup_k[M_k(\frac{\|A_k(t_nx_n-Lx)\|}{|t_n-L|\rho_1+|L|\rho_2})]^{p_k}\leq 1$$

and hence

$$g(t_nx_n-Lx)=\inf\{(|t_n-L|\rho_1+|L|\rho_2)^{\frac{p_k}{H}}:\rho_1\in A,\rho_2\in B\}$$

$$\leq(|t_n-L|)^{\frac{p_k}{H}}\inf\{\rho_1^{\frac{p_k}{H}}:\rho_1\in A\}+(|L|)^{\frac{p_k}{H}}\inf\{\rho_2^{\frac{p_k}{H}}:\rho_2\in B\}$$

$$\leq\max\{|t_n-L|,(|t_n-L|)^{\frac{p_k}{H}}\}g(x_n)+\max\{|L|,(|L|)^{\frac{p_k}{H}}\}g(x_n-x)$$

As $g(x_n)\leq g(x)+g(x_n-x)$ for all $n\in N$, hence the right hand side of the above relation tends to zero as $n\to\infty$.

This completes the proof.

Proposition 2.3: Let $M=(M_k)$ and $N=(N_k)$ be sequences of Orlicz functions. Then the following hold:

(i) $W_0^{I(S)}(N,A,p) \subseteq W_0^{I(S)}(MoN,A,p)$, provided $p=(p_k)$ such that $G_0 = \inf p_k > 0$.

(ii)$W_0^{I(S)}(M,A,p) \bigcap W_0^{I(S)}(N,A,p) \subseteq W_0^{I(S)}(M+N,A,p)$.

Theorem 2.4: The spaces $W_0^{I(S)}(M,A,p)$ and $W^{I(S)}(M,A,p)$ are normal and monotone.

Proof: Let $x=(x_k) \in W_0^{I(S)}(M,A,p)$ and $y=(y_k)$ be such that $|y_k| \leq |x_k|$. Then for $\varepsilon > 0$,

$$\{n \in N : \tfrac{1}{n}|\{k \leq n : \textstyle\sum_{k=1}^{n}[M_k(\frac{\|A_k(x)\|}{\rho})]^{p_k} \geq \varepsilon\}| \geq \delta\}$$

$$\supseteq \{n \in N : \tfrac{1}{n}|\{k \leq n : \textstyle\sum_{k=1}^{n}[M_k(\frac{\|A_k(y)\|}{\rho})]^{p_k} \geq \varepsilon\}| \geq \delta\} \in I.$$

The result follows from the above relation. Thus the space $W_0^{I(S)}(M,A,p)$ is normal and hence monotone by lemma 1.5. Similarly for the other.

Proposition 2.5: Let $0 < p_k \leq q_k$ and $\frac{q_k}{p_k}$ be bounded. Then $W_0^{I(S)}(M,A,q)$ $\subseteq W_0^{I(S)}(M,A,p)$

Proposition 2.6: For any two sequence $p=(p_k)$ and $q=(q_k)$ of positive real numbers, the following hold:

$Z(M,A,p) \bigcap Z(M,A,q) \neq \emptyset$ for Z=$W^{I(S)}$, $W_0^{I(S)}$, $W_\infty^{I(S)}$.

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