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On Zweier I-convergent sequence spaces

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Abstract

In this article we introduce the Zweier I-convergent sequence spaces \mathcal{Z}^I , \mathcal{Z}_0^I and \mathcal{Z}_∞^I . We prove the decomposition theorem and study topological, algebraic properties and have established some inclusion relations of these spaces.

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1. Introduction

Let \mathbf{N} , \mathbf{R} and \mathbf{C} be the sets of all natural, real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbf{R} \text{ or } \mathbf{C}\},$$

the space of all real or complex sequences.

Let ℓ_∞ , c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively normed by

$$\|x\|_\infty = \sup_k |x_k|.$$

A sequence space λ with linear topology is called a K -space provided each of maps $p_i \rightarrow \mathbf{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbf{N}$.

A K -space λ is called an FK -space provided λ is a complete linear metric space.

An FK -space whose topology is normable is called a BK -space.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers (a_{nk}) , where $n, k \in \mathbf{N}$. Then we say that A defines a matrix mapping from λ to μ , and we denote it by writing $A : \lambda \rightarrow \mu$.

If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ , where

$$(1.1) \quad (Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbf{N}).$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \rightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1) converges for each $n \in \mathbf{N}$ and every $x \in \lambda$.

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have recently been employed by Altay, Başar and Mursaleen [1], Başar and Altay [2], Malkowsky [13], Ng and Lee [14], and Wang [21].

Şengönül[18] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i. e.,

$$y_i = px_i + (1 - p)x_{i-1}$$

where $x_{-1} = 0$, $1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1 - p, (i - 1 = k); (i, k \in \mathbf{N}), \\ 0, \text{otherwise.} \end{cases}$$

Following Başar and Altay[2], Şengönül[18] introduced the Zweier sequence spaces \mathcal{Z} and \mathcal{Z}_0 as follows :

$$\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$$

$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}.$$

Here we list below some of the results of Şengönül [18] which we will need as a reference in order to establish analogously some of the results of this article.

Theorem 1.1. The sets \mathcal{Z} and \mathcal{Z}_0 are linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm

$$\|x\|_{\mathcal{Z}} = \|x\|_{\mathcal{Z}_0} = \|Z^p x\|_c [\text{See (Theorem 2.1. [18])}].$$

Theorem 1.2. The sequence spaces \mathcal{Z} and \mathcal{Z}_0 are linearly isomorphic to the spaces c and c_0 respectively, i.e $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$ [See (Theorem 2.2.[18])]

Theorem 1.3. The inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly hold for $p \neq 1$. [See (Theorem 2.3. [18])].

Theorem 1.4. \mathcal{Z}_0 is solid.[See (Theorem 2.6.[18])].

Theorem 1.5. \mathcal{Z} is not a solid sequence space.[See (Theorem 3.6. [18])].

The concept of statistical convergence was first introduced by Fast [7] and also independently by Buck [3] and Schoenberg [17] for real and complex sequences.Further this concept was studied by Connor [4, 5], Connor,

Fridy and Kline [6] and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is said to be statistically convergent to L if for a given $\varepsilon > 0$

$$\lim_k \frac{1}{k} |\{i : |x_i - L| \geq \varepsilon, i \leq k\}| = 0.$$

The notion of I-convergence generalizes and unifies different notions of convergence including the notion of statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, Wilczyński [12]. Later on it was studied by Šalát, Tripathy, Ziman [15, 16]. Recently further it was studied by Tripathy [19, 20, 21, 22, 23, 24, 25, 26, 27], and V. A.Khan and Khalid Ebadullah [9-11].

Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. Then a family of sets $I \subseteq 2^X$ (2^X denoting the power set of X) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $\mathcal{L}(I) \subseteq 2^X$ is said to be filter on X if and only if $\emptyset \notin \mathcal{L}(I)$, for $A, B \in \mathcal{L}(I)$ we have $A \cap B \in \mathcal{L}(I)$ and for each $A \in \mathcal{L}(I)$ and $A \subseteq B$ implies $B \in \mathcal{L}(I)$.

An Ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$.

A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

For each ideal I , there is a filter $\mathcal{L}(I)$ corresponding to I . i.e

$$\mathcal{L}(I) = \{K \subseteq N : K^c \in I\}, \quad \text{where } K^c = N - K.$$

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\varepsilon > 0$.

$$\{k \in N : |x_k - L| \geq \varepsilon\} \in I.$$

In this case we write $I - \lim x_k = L$. The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbf{N} : |x_k - L| \geq \varepsilon\} \in I, \text{ for some } L \in \mathbf{C}\}.$$

Definition 1.7. A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$. In this case we write $I - \lim x_k = 0$.

Definition 1.8. A sequence $(x_k) \in \omega$ is said to be I-Cauchy if for every $\varepsilon > 0$ there exists a number $m = m(\varepsilon)$ such that

$$\{k \in N : |x_k - x_m| \geq \varepsilon\} \in I.$$

Definition 1.9. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{k \in N : |x_k| > M\} \in I.$$

Example 1.10. Take for I the class I_f of all finite subsets of \mathbf{N} . Then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X. (see [12]).

Definition 1.11. For $I = I_\delta$ and $A \subset \mathbf{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal, I_δ -convergence is said to be logarithmic statistical convergence(see[12]).

Definition 1.12. A map \hbar defined on a domain $D \subset X$ i.e $\hbar : D \subset X \rightarrow \mathbf{R}$ is said to satisfy Lipschitz condition if

$$|\hbar(x) - \hbar(y)| \leq K|x - y|,$$

where K is known as the Lipschitz constant.The class of K -Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$ (see[15,16]).

Definition 1.13. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \mathbf{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$ (See [15,16]).

Define a function $\hbar : F(I) \rightarrow \mathbf{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \rightarrow \mathbf{R}$ is a Lipschitz function. (see [15, 16]).

Definition 1.14. Let $(x_k), (y_k)$ be two sequences. We say that $(x_k) = (y_k)$ for almost all k relative to I (a.a.k.r.I), if

$$\{k \in \mathbf{N} : x_k \neq y_k\} \in I(\text{see}[19, 20]).$$

The following Lemmas will be used for establishing some results of this article :

Lemma 1.15. Let E be a sequence space. If E is solid then E is monotone. (see [8], page 53).

Lemma 1.16. If $I \subset 2^{\mathbf{N}}$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$. (see [19,20]).

2. Main Results

In this section we introduce the following classes of sequence spaces :

$$\mathcal{Z}^I = \{x = (x_k) \in \omega : \{k \in \mathbf{N} : I - \lim Z^p x = L, \text{ for some } L \in \mathbf{C}\}\};$$

$$\mathcal{Z}_0^I = \{x = (x_k) \in \omega : \{k \in \mathbf{N} : I - \lim Z^p x = 0\}\};$$

$$\mathcal{Z}_\infty^I = \{x = (x_k) \in \omega : \{k \in \mathbf{N} : \sup_k |Z^p x| < \infty\}\}.$$

We also denote by

$$m_{\mathcal{Z}}^I = \mathcal{Z}_\infty \cap \mathcal{Z}^I$$

and

$$m_{\mathcal{Z}_0}^I = \mathcal{Z}_\infty \cap \mathcal{Z}_0^I.$$

Throughout the article, for the sake of convenience now we will denote by

$$Z^p(x_k) = x^/, Z^p(y_k) = y^/, Z^p(z_k) = z^/ \text{ for } x, y, z \in \omega.$$

Theorem 2.1. The classes of sequences \mathcal{Z}^I , \mathcal{Z}_0^I , $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are linear spaces.

Proof. We shall prove the result for the space \mathcal{Z}^I .

The proof for the other spaces will follow similarly.

Let $(x_k), (y_k) \in \mathcal{Z}^I$ and let α, β be scalars. Then

$$I - \lim |x_k^/ - L_1| = 0, \text{ for some } L_1 \in \mathbf{C};$$

$$I - \lim |y'_k - L_2| = 0, \text{ for some } L_2 \in \mathbf{C};$$

That is for a given $\varepsilon > 0$, we have

$$(2.1) \quad \begin{aligned} A_1 &= \{k \in \mathbf{N} : |x'_k - L_1| > \frac{\varepsilon}{2}\} \in I, \\ A_2 &= \{k \in \mathbf{N} : |y'_k - L_2| > \frac{\varepsilon}{2}\} \in I. \end{aligned}$$

we have

$$\begin{aligned} |(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)| &\leq |\alpha|(|x'_k - L_1|) + |\beta|(|y'_k - L_2|) \\ &\leq |x'_k - L_1| + |y'_k - L_2| \end{aligned}$$

Now, by (1) and (2), $\{k \in \mathbf{N} : |(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)| > \varepsilon\} \subset A_1 \cup A_2$.

Therefore $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I$

Hence \mathcal{Z}^I is a linear space.

Theorem 2.2. The spaces $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are normed linear spaces, normed by

$$(2.2) \quad \|x'_k\|_* = \sup_k |Z^p(x)|,$$

where $x'_k = Z^p(x)$.

Proof: It is clear from Theorem 2.1 that $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are linear spaces.

It is easy to verify that (3) defines a norm on the spaces $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.

Theorem 2.3. A sequence $x = (x_k) \in m_{\mathcal{Z}}^I$ I-converges if and only if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbf{N}$ such that

$$(2.3) \quad \{k \in \mathbf{N} : |x'_k - x'_{N_\varepsilon}| < \varepsilon\} \in m_{\mathcal{Z}}^I$$

Proof. Suppose that $L = I - \lim x'$. Then

$$B_\varepsilon = \{k \in \mathbf{N} : |x'_k - L| < \frac{\varepsilon}{2}\} \in m_{\mathcal{Z}}^I \text{ for all } \varepsilon > 0$$

. Fix an $N_\epsilon \in B_\epsilon$. Then we have

$$|x'_{N_\epsilon} - x'_k| \leq |x'_{N_\epsilon} - L| + |L - x'_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$.

Hence $\{k \in \mathbf{N} : |x'_k - x'_{N_\epsilon}| < \epsilon\} \in m^I_{\mathcal{Z}}$.

Conversely, suppose that $\{k \in \mathbf{N} : |x'_k - x'_{N_\epsilon}| < \epsilon\} \in m^I_{\mathcal{Z}}$. That is $\{k \in \mathbf{N} : |x'_k - x'_{N_\epsilon}| < \epsilon\} \in m^I_{\mathcal{Z}}$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{k \in \mathbf{N} : x'_k \in [x'_{N_\epsilon} - \epsilon, x'_{N_\epsilon} + \epsilon]\} \in m^I_{\mathcal{Z}} \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x'_{N_\epsilon} - \epsilon, x'_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I_{\mathcal{Z}}$ as well as $C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbf{N} : x'_k \in J\} \in m^I_{\mathcal{Z}}$$

that is

$$diam J \leq diam J_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \dots \supseteq I_k \supseteq \dots$$

with the property that $diam I_k \leq \frac{1}{2} diam I_{k-1}$ for $(k=2,3,4,\dots)$ and

$$\{k \in \mathbf{N} : x'_k \in I_k\} \in m^I_{\mathcal{Z}} \text{ for } (k=1,2,3,4,\dots).$$

Then there exists a $\xi \in \cap I_k$ where $k \in \mathbf{N}$ such that $\xi' = I - \lim x'$, that is $L = I - \lim x'$.

Theorem 2.4. Let I be an admissible ideal. Then the following are equivalent.

- (a) $(x_k) \in \mathcal{Z}^I$;
- (b) there exists $(y_k) \in \mathcal{Z}$ such that $x_k = y_k$, for a.a.k.r.I;

(c) there exists $(y_k) \in \mathcal{Z}$ and $(z_k) \in \mathcal{Z}_0^I$ such that $x_k = y_k + z_k$ for all $k \in \mathbf{N}$ and $\{k \in \mathbf{N} : |y_k - L| \geq \epsilon\} \in I$;

(d) there exists a subset $K = \{k_1 < k_2, \dots\}$ of \mathbf{N} such that $K \in \mathcal{L}(I)$ and $\lim_{n \rightarrow \infty} |x_{k_n} - L| = 0$.

Proof.(a) implies (b). Let $(x_k) \in \mathcal{Z}^I$. Then there exists $L \in \mathbf{C}$ such that

$$\{k \in \mathbf{N} : |x'_k - L| \geq \epsilon\} \in I$$

.Let (m_t) be an increasing sequence with $m_t \in \mathbf{N}$ such that

$$\{k \leq m_t : |x'_k - L| \geq \frac{1}{t}\} \in I.$$

Define a sequence (y_k) by

$$y_k = x_k, \text{ for all } k \leq m_1.$$

For $m_t < k \leq m_{t+1}, t \in \mathbf{N}$.

$$y_k = \begin{cases} x_k, & \text{if } |x'_k - L| < t^{-1}, \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_k) \in \mathcal{Z}$ and form the following inclusion

$$\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : |x'_k - L| \geq \epsilon\} \in I.$$

We get $x_k = y_k$, for a.a.k.r.I.

(b) implies (c).For $(x_k) \in \mathcal{Z}^I$.

Then there exists $(y_k) \in \mathcal{Z}$ such that $x_k = y_k$, for a.a.k.r.I.

Let $K = \{k \in \mathbf{N} : x_k \neq y_k\}$, then $K \in I$.

Define a sequence (z_k) by

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $z_k \in \mathcal{Z}_0^I$ and $y_k \in \mathcal{Z}$.

(c) implies (d). Let $P_1 = \{k \in \mathbf{N} : |z_k| \geq \varepsilon\} \in I$ and

$$K = P_1^c = \{k_1 < k_2 < k_3 < \dots\} \in \mathcal{L}(I)$$

. Then we have $\lim_{n \rightarrow \infty} |x_{k_n} - L| = 0$.

(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < \dots\} \in \mathcal{L}(I)$ and $\lim_{n \rightarrow \infty} |x_{k_n} - L| = 0$.

Then for any $\epsilon > 0$, and by Lemma , we have

$$\{k \in \mathbf{N} : |x'_k - L| \geq \epsilon\} \subseteq K^c \cup \{k \in K : |x'_k - L| \geq \epsilon\}.$$

Thus $(x_k) \in \mathcal{Z}^I$.

Theorem 2.5. The inclusions $\mathcal{Z}_0^I \subset \mathcal{Z}^I \subset \mathcal{Z}_\infty^I$ are proper.

Proof: Let $(x_k) \in \mathcal{Z}^I$. Then there exists $L \in \mathbf{C}$ such that

$$I - \lim |x'_k - L| = 0$$

We have $|x'_k| \leq \frac{1}{2}|x'_k - L| + \frac{1}{2}|L|$.

Taking the supremum over k on both sides we get $(x_k) \in \mathcal{Z}_\infty^I$.

The inclusion $\mathcal{Z}_0^I \subset \mathcal{Z}^I$ is obvious.

Theorem 2.6. The function $\hbar : m_{\mathcal{Z}}^I \rightarrow \mathbf{R}$ is the Lipschitz function, where $m_{\mathcal{Z}}^I = \mathcal{Z}^I \cap \mathcal{Z}_\infty$, and hence uniformly continuous.

Proof: Let $x, y \in m_{\mathcal{Z}}^I$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbf{N} : |x'_k - \hbar(x')| \geq \|x' - y'\|_*\} \in I,$$

$$A_y = \{k \in \mathbf{N} : |y'_k - \hbar(y')| \geq \|x' - y'\|_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbf{N} : |x'_k - \hbar(x')| < \|x' - y'\|_*\} \in m_{\mathcal{Z}}^I,$$

$$B_y = \{k \in \mathbf{N} : |y'_k - \hbar(y')| < \|x' - y'\|_*\} \in m_{\mathcal{Z}}^I.$$

Hence also $B = B_x \cap B_y \in m_{\mathcal{Z}}^I$, so that $B \neq \phi$.

Now taking k in B ,

$$|\hbar(x') - \hbar(y')| \leq |\hbar(x') - x'_k| + |x'_k - y'_k| + |y'_k - \hbar(y')| \leq 3\|x' - y'\|_*$$

Thus \hbar is a Lipschitz function.

For $m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 2.7. If $x, y \in m_{\mathcal{Z}}^I$, then $(x.y) \in m_{\mathcal{Z}}^I$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof: For $\epsilon > 0$

$$B_x = \{k \in \mathbf{N} : |x' - \hbar(x')| < \epsilon\} \in m_{\mathcal{Z}}^I,$$

$$B_y = \{k \in \mathbf{N} : |y' - \hbar(y')| < \epsilon\} \in m_{\mathcal{Z}}^I.$$

Now,

$$\begin{aligned} |x'.y' - \hbar(x')\hbar(y')| &= |x'.y' - x'\hbar(y') + x'\hbar(y') - \hbar(x')\hbar(y')| \\ (2.4) \qquad \qquad \qquad &\leq |x'|\|y' - \hbar(y')\| + |\hbar(y')|\|x' - \hbar(x')\| \end{aligned}$$

As $m_{\mathcal{Z}}^I \subseteq \mathcal{Z}_{\infty}$, there exists an $M \in \mathbf{R}$ such that $|x'| < M$ and $|\hbar(y')| < M$.

Using eqn(5) we get

$$|x'.y' - \hbar(x')\hbar(y')| \leq M\epsilon + M\text{var}\epsilon = 2M\epsilon$$

For all $k \in B_x \cap B_y \in m_{\mathcal{Z}}^I$.

Hence $(x.y) \in m_{\mathcal{Z}}^I$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

For $m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 2.8. The spaces \mathcal{Z}_0^I and $m_{\mathcal{Z}_0}^I$ are solid and monotone .

Proof: We prove the result for the case \mathcal{Z}_0^I .

Let $(x_k) \in \mathcal{Z}_0^I$. Then

$$(2.5) \qquad \qquad \qquad I - \lim_k |x'_k| = 0$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbf{N}$. Then the result follows from (6) and the following inequality

$$|\alpha_k x'_k| \leq |\alpha_k| |x'_k| \leq |x'_k| \text{ for all } k \in \mathbf{N}.$$

That the space \mathcal{Z}_0^I is monotone follows from the Lemma 1.15.

For $m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 2.9. The spaces \mathcal{Z}^I and $m_{\mathcal{Z}}^I$ are neither monotone nor solid, if I is neither maximal nor $I = I_f$ in general .

Proof: Here we give a counter example.

Let $I = I_\delta$. Consider the K -step space X_K of X defined as follows,

Let $(x_k) \in X$ and let $(y_k) \in X_K$ be such that

$$(y'_k) = \begin{cases} (x'_k), & \text{if } k \text{ is odd,} \\ 1, & \text{otherwise.} \end{cases}$$

Consider the sequence (x'_k) defined by $(x'_k) = k^{-1}$ for all $k \in \mathbf{N}$.

Then $(x_k) \in \mathcal{Z}^I$ but its K -stepspace preimage does not belong to \mathcal{Z}^I . Thus \mathcal{Z}^I is not monotone. Hence \mathcal{Z}^I is not solid.

Theorem 2.10. The spaces \mathcal{Z}^I and \mathcal{Z}_0^I are sequence algebras.

Proof: We prove that \mathcal{Z}_0^I is a sequence algebra.

Let $(x_k), (y_k) \in \mathcal{Z}_0^I$. Then

$$I - \lim |x'_k| = 0$$

and

$$I - \lim |y'_k| = 0$$

Then we have

$$I - \lim |(x'_k \cdot y'_k)| = 0$$

Thus $(x_k \cdot y_k) \in \mathcal{Z}_0^I$

Hence \mathcal{Z}_0^I is a sequence algebra.

For the space \mathcal{Z}^I , the result can be proved similarly.

Theorem 2.11. The spaces \mathcal{Z}^I and \mathcal{Z}_0^I are not convergence free in general.

Proof: Here we give a counter example.

Let $I = I_f$. Consider the sequence (x'_k) and (y'_k) defined by

$$x'_k = \frac{1}{k} \quad \text{and} \quad y'_k = k \quad \text{for all } k \in \mathbf{N}$$

Then $(x_k) \in \mathcal{Z}^I$ and \mathcal{Z}_0^I , but $(y_k) \notin \mathcal{Z}^I$ and \mathcal{Z}_0^I .

Hence the spaces \mathcal{Z}^I and \mathcal{Z}_0^I are not convergence free.

Theorem 2.12. If I is not maximal and $I \neq I_f$, then the spaces \mathcal{Z}^I and \mathcal{Z}_0^I are not symmetric.

Proof: Let $A \in I$ be infinite.

If

$$x'_k = \begin{cases} 1, & \text{for } k \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then by lemma 1.16. $x_k \in \mathcal{Z}_0^I \subset \mathcal{Z}^I$. Let $K \subset \mathbf{N}$ be such that $K \notin I$ and $\mathbf{N} - K \notin I$. Let $\phi : K \rightarrow A$ and $\psi : \mathbf{N} - K \rightarrow \mathbf{N} - A$ be bijections, then the map $\pi : \mathbf{N} \rightarrow \mathbf{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & \text{otherwise.} \end{cases}$$

is a permutation on \mathbf{N} , but $x_{\pi(k)} \notin \mathcal{Z}^I$ and $x_{\pi(k)} \notin \mathcal{Z}_0^I$.

Hence \mathcal{Z}^I and \mathcal{Z}_0^I are not symmetric.

Theorem 2.13. The sequence spaces \mathcal{Z}^I and \mathcal{Z}_0^I are linearly isomorphic to the spaces c^I and c_0^I respectively, i.e $\mathcal{Z}^I \cong c^I$ and $\mathcal{Z}_0^I \cong c_0^I$.

Proof. We shall prove the result for the space \mathcal{Z}^I and c^I .

The proof for the other spaces will follow similarly.

We need to show that there exists a linear bijection between the spaces \mathcal{Z}^I and c^I . Define a map $T : \mathcal{Z}^I \longrightarrow c^I$ such that $x \rightarrow x' = Tx$

$$T(x_k) = px_k + (1-p)x_{k-1} = x'_k$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$.

Clearly T is linear.

Further, it is trivial that $x = 0 = (0, 0, 0, \dots)$, whenever $Tx = 0$ and hence injective.

Let $x'_k \in c^I$ and define the sequence $x = x_k$ by

$$x_k = M \sum_{i=0}^k (-1)^{k-i} N^{k-i} x'_i \quad (i \in \mathbf{N}),$$

where $M = \frac{1}{p}$ and $N = \frac{1-p}{p}$.

Then we have

$$\begin{aligned} & \lim_{k \rightarrow \infty} px_k + (1-p)x_{k-1} \\ &= p \lim_{k \rightarrow \infty} M \sum_{i=0}^k (-1)^{k-i} N^{k-i} x'_i + (1-p) \lim_{k \rightarrow \infty} M \sum_{i=0}^{k-1} (-1)^{k-i} N^{k-i} x'_i \\ &= \lim_{k \rightarrow \infty} x'_k \end{aligned}$$

which shows that $x \in \mathcal{Z}^I$.

Hence T is a linear bijection.

Also we have $\|x\|_* = \|Z^p x\|_c$.

Therefore,

$$\|x\|_* = \sup_{k \in \mathbf{N}} |px_k + (1-p)x_{k-1}|,$$

$$\begin{aligned}
&= \sup_{k \in \mathbf{N}} \left| pM \sum_{i=0}^k (-1)^{k-i} N^{k-i} x'_i + (1-p)M \sum_{i=0}^{k-1} (-1)^{k-i} N^{k-i} x'_i \right| \\
&= \sup_{k \in \mathbf{N}} |x'_k| = \|x'\|_{c^I}.
\end{aligned}$$

Hence $\mathcal{Z}^I \cong c^I$.

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