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Edge Detour Monophonic Number of a Graph

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Abstract

For a connected graph G of order at least two, an edge detour monophonic set of G is a set S of vertices such that every edge of G lies on a detour monophonic path joining some pair of vertices in S. The edge detour monophonic number of G is the minimum cardinality of its edge detour monophonic sets and is denoted by $edm(G)$. We determine bounds for it and characterize graphs which realize these bounds. Also, certain general properties satisfied by an edge detour monophonic set are studied. It is shown that for positive integers a, b and c with $2 \leq a \leq b \leq c$, there exists a connected graph G such that $m(G) = a, m_1(G) = b$ and $edm(G) = c$, where $m(G)$ is the monophonic number and $m_1(G)$ is the edge monophonic number of G. Also, for any integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $dm(G) = a$ and $edm(G) = b$, where $dm(G)$ is the detour monophonic number of a graph G.

Key Words : *monophonic number, edge monophonic number, detour monophonic number, edge detour monophonic number.*

AMS Subject Classification : *05C12.*

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m, respectively. For basic graph theoretic terminology we refer to Harary [4]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x-y$ path in G. An $x-y$ path of length $d(x, y)$ is called an $x-y$ *geodesic*. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete. A vertex v in G is said to be a *semi-extreme vertex* of G if $\Delta\left(< N(v) >\right) = |N(v)| - 1$. That is, the induced subgraph of $N(v)$ has a full degree vertex in $N(v)$.

For the graph G given in Figure 1.1, v_2, v_3, v_4, v_5 and v_6 are the semi-extreme vertices. In any graph G, each extreme vertex is a semi-extreme vertex.

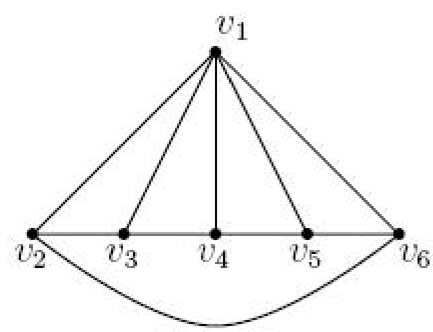

Figure 1.1: G

The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup\limits_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [1, 5] and further studied in [2].

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. A set S of vertices of a graph G is a *monophonic set* if each vertex v of G lies on an

$x - y$ monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G, denoted by $m(G)$. A longest $x-y$ monophonic path is called an $x-y$ *detour monophonic path.* A set S of vertices of a graph G is a *detour monophonic set* if each vertex v of G lies on an $x - y$ detour monophonic path for some $x, y \in S$. The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$. The detour monophonic number of a graph was introduced in [9] and further studied in [10].

An *edge monophonic set* of G is a set S of vertices such that every edge of G lies on a monophonic path joining some pair of vertices in S. The *edge monophonic number* of G is the minimum cardinality of its edge monophonic sets and is denoted by $m_1(G)$. An edge monophonic set of cardinality $m_1(G)$ is an m_1-set of G.

These concepts have interesting applications in Channel Assignment Problem in radio technologies, and the detour matrix of a connected graph is used to discuss the applications of the detour index and hyper-detour index to a class of graphs, which in turn, capture different aspects of certain molecular graphs associated to the molecules arising in special situations of molecular problems in theoretical Chemistry[3,6]. Also, there are useful applications of these concepts to security based communication network design. In the case of designing the channel for a communication network, although all the vertices are covered by the network when considering detour monophonic sets, some of the edges may be left out. This drawback is rectified in the case of edge detour monophonic sets so that considering edge detour monophonic sets is more advantageous to real life application of communication networks. This motivated us to introduce and investigate edge detour monophonic sets in a graph.

The following theorems will be used in the sequel.

Theorem 1.1. [8] Each extreme vertex of a graph G belongs to every monophonic set of G.

Theorem 1.2. [7] Each semi-extreme vertex of a graph G belongs to every edge monophonic set of G.

Theorem 1.3. [9] Each extreme vertex of a graph G belongs to every detour monophonic set of G.

Throughout this paper G denotes a connected graph with at least two vertices.

2. Edge detour monophonic number of a graph

Definition 2.1. Let G be a connected graph with at least two vertices. An *edge detour monophonic set* of G is a set S of vertices such that every edge of G lies on a detour monophonic path joining some pair of vertices in S. The *edge detour monophonic number* of G is the minimum cardinality of its edge detour monophonic sets and is denoted by $edm(G)$. An edge detour monophonic set of cardinality $edm(G)$ is an edm-set of G.

We observe that every edge detour monophonic set is also a detour monophonic set of G.

Example 2.2. For the graph G given in Figure 2.1, it is easily seen that no 3-element subset of vertices is an edge detour monophonic set. It is clear that $S_1 = \{z, v, w, x\}$ is an edge detour monophonic set of G so that $edm(G) = 4$. Also, $S_2 = \{z, v, w, u\}$, $S_3 = \{z, x, u, v\}$ and $S_4 = \{z, x, u, w\}$ are minimum edge detour monophonic sets of G.

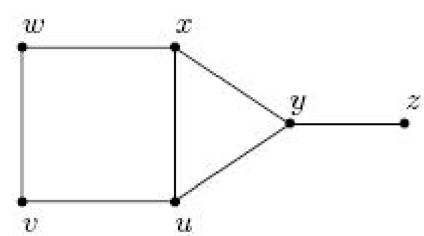

Figure 2.1: G

Note that for the graph G given in Figure 2.1, it is easily verified that $\{z, v\}$ is a minimum monophonic set of G and $\{z, v, w\}$ is a minimum detour monophonic set of G so that $m(G) = 2$ and $dm(G) = 3$. Thus the monophonic number, detour monophonic number and edge detour monophonic number of a graph are different.

Theorem 2.3. For any graph G of order n, $2 \leq edm(G) \leq n$.

Proof. An edge detour monophonic set needs at least two vertices and so $edm(G) \geq 2$. Clearly, the set of all vertices of G is an edge detour monophonic set of G so that $edm(G) \leq n$. □

The bounds in Theorem 2.3 are sharp. The even cycle C_n $(n \geq 4)$ has $edm(C_n) = 2$ and the complete graph K_n has $edm(K_n) = n$.

Theorem 2.4. Each semi-extreme vertex of a graph G belongs to every edge detour monophonic set of G. In particular, if the set S of all semi-extreme vertices of G is an edge detour monophonic set, then S is the unique minimum edge detour monophonic set of G.

Proof. Let S be the set of all semi-extreme vertices of G and let T be any edge detour monophonic set of G. Suppose that there exists a vertex $u \in S$ such that $u \notin T$. Since $\Delta(< N(u) >) = |N(u)| - 1$, there exists a $v \in N(u)$ such that $deg_{<N(u)>}(v) = |N(u)| - 1$. Since T is an edge detour monophonic set of G, the edge $e = uv$ lies on an $x - y$ detour monophonic path $P : x = x_0, x_1, \ldots, x_{i-1}, x_i = u, x_{i+1} = v, \ldots, x_n = y$ with $x, y \in T$. Since $u \notin T$, it is clear that u is an internal vertex of the path P. Since $deg_{<N(u)>}(v) = |N(u)| - 1$, we see that v is adjacent to x_{i-1}, which is a contradiction to the fact that P is an $x-y$ detour monophonic path. Hence S is contained in every edge detour monophonic set of G. □

Corollary 2.5. For any graph G with k semi-extreme vertices, $max\{2, k\} \leq edm(G) \leq n$.

Corollary 2.6. For the complete graph K_n $(n \geq 2)$, $edm(K_n) = n$.

Remark 2.7. The graph G given in Figure 2.2 is non-complete on 4 vertices with $edm(G) = 4$.

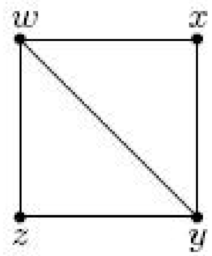

Figure 2.2: G

Theorem 2.8. Let G be a connected graph with cut-vertices and S an edge detour monophonic set of G. If v is a cut-vertex of G, then every component of $G-v$ contains an element of S.

Proof. Suppose that there is a component B of $G-v$ such that B contains no vertex of S. Let u be any vertex in B and let e be any edge incident with u, say $e = uw$. Since S is an edge detour monophonic set, there exist vertices $x, y \in S$ such that e lies on some $x-y$ detour monophonic path $P : x = u_0, u_1, \ldots, u, w, \ldots, u_t = y$ in G with $u \neq x, y$. Let P_1 be the $x-u$ subpath of P and P_2 be the $u-y$ subpath of P. Since v is a cutvertex of G, both P_1 and P_2 contain v, so that P is not a path, which is a contradiction. Thus every component of $G-v$ contains an element of S. □

Theorem 2.9. For any connected graph G, no cut-vertex of G belongs to any minimum edge detour monophonic set of G.

Proof. Let v be a cut-vertex of G and let S be a minimum edge detour monophonic set of G. Then by Theorem 2.8, every component of $G-v$ contains an element of S. Let U and W be two components of $G-v$ and let $u \in U$ and $w \in W$. Then v is an internal vertex of any $u-w$ detour monophonic path. Let $S'=S-\{v\}$. It is clear that every edge that lies on an $u-v$ detour monophonic path also lies on an $u-w$ detour monophonic path. Hence it follows that S' is an edge detour monophonic set of G, which is a contradiction to S a minimum edge detour monophonic set of G. □

Theorem 2.10. If T is a tree with k end vertices, then $edm(T) = k$.

Proof. This follows from Theorems 2.4 and 2.9. □

Theorem 2.11. For the cycle $C_n (n \geq 3)$,

$$\mathrm{edm}\,(Cn) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $C_n : v_1, v_2, v_3, ..., v_n, v_1$ be a cycle of order n. If n is even, then clearly $S = \{v_1, v_{\frac{n}{2}+1}\}$ is a minimum edge detour monophonic set of C_n and so $edm(C_n) = 2$. If n is odd, then clearly $S = \{v_1, v_2, v_3\}$ is a minimum edge detour monophonic set of C_n and so $edm(C_n) = 3$. □

Theorem 2.12. For the complete bipartite graph $G = K_{r,s}$ $(1 \leq r \leq s)$,
(i) $edm(G) = s$ if $r = 1$
(ii) $edm(G) = r$ if $r \geq 2$.

Proof. (i) This follows from Theorem 2.10.
(ii) Let $r \geq 2$ and let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be a bipartition of G. Let $S = U$. We prove that S is an *edm*-set of G. We observe that any $u - v$ detour monophonic path in G is of length at most 2. Any edge $u_i w_j$ $(1 \leq i \leq r, 1 \leq j \leq s)$ lies on the detour monophonic path $u_i w_j u_k$ for any $k \neq i$ so that S is an edge detour monophonic set of G. Let T be any set of vertices such that $|T| < |S|$. If $T \not U$, then there exists a vertex $u_i \in U$ such that $u_i \notin T$. Then any edge $u_i w_j$ $(1 \leq j \leq s)$, does not lie on a detour monophonic path joining a pair of vertices of T. Thus T is not an edge detour monophonic set of G. If $T \not W$, then the argument is similar. If $T \not S \cup W$ such that T contains at least one vertex from each of S and W, then since $|T| < |S|$, there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin T$ and $w_j \notin T$. It is clear that the edge $u_i w_j$ does not lie on a detour monophonic path joining any pair of vertices of T so that T is not an edge detour monophonic set of G. Hence S is an edge detour monophonic set with minimum cardinality so that $edm(G) = |S| = r$. □

A vertex v in a graph G is called an *independent vertex* if the subgraph induced by its neighbours contains no edges.

Theorem 2.13. Let G be a connected graph. Then $edm(G) = 2$ if and only if there exist two independent vertices u and v such that every edge of G lies on a $u - v$ detour monophonic path.

Proof. Let $edm(G) = 2$ and let $S = \{u, v\}$ be an edge detour monophonic set of G. If u and v are adjacent, then the graph is $G = K_2$, and the result is true. Suppose that u and v are non-adjacent in G. We prove that u and v are independent vertices. Suppose that u is not an independent vertex. Then there exists an edge xy such that $x, y \in N(u)$. It is clear that the edge xy does not lie on any $u - v$ detour monophonic path so that S is not an edge detour monophonic set, which is a contradiction. The converse is trivial. □

Theorem 2.14. Let G be a connected graph of order n. If G has more than one vertex of degree $n - 1$, then every edge detour monophonic set contains all vertices of degree $n - 1$.

Proof. Let G be a graph of order n with more than one vertex of degree $n - 1$. If u and v are two vertices of degree $n - 1$, then uv is an edge and it is not an edge of any detour monophonic path joining two vertices of G other than u and v. Hence it follows that both u and v belong to every edge detour monophonic set of G. □

Theorem 2.15. For any graph G of order n with at least two vertices of degree $n - 1$, $edm(G) = n$.

Proof. If all the vertices are of degree $n - 1$, then $G = K_n$ and so $edm(G) = n$. Otherwise, let $v_1, v_2, \ldots, v_k$ $(2 \leq k \leq n - 2)$ be the vertices of degree $n - 1$. Suppose that $edm(G) < n$. Let S be a edm-set of G such that $|S| < n$. By Theorem 2.14, S contains all the vertices $v_1, v_2, \ldots, v_k$. Let v be a vertex such that $v \notin S$. Then $deg(v) < n - 1$. Since any two of $v_1, v_2, \ldots, v_k$ are adjacent, the edge vv_i $(1 \leq i \leq k)$ does not lie on a detour monophonic path joining a pair of vertices v_j and v_l $(j \neq l)$. Similarly, since any v_j is adjacent to any vertex of S, which is different from $v_1, v_2, \ldots, v_k$, the edge vv_i $(1 \leq i \leq k)$ does not lie on a detour monophonic path joining a vertex v_j and a vertex of S, which is different from $v_1, v_2, \ldots, v_k$. Now, let u and w be two vertices of S different from $v_1, v_2, \ldots, v_k$. Since v_i is adjacent to both u and w, the edge vv_i does not lie on a detour monophonic path joining u and w. Thus we see that the edges vv_i $(1 \leq i \leq k)$ do not lie on any detour monophonic path joining a pair of vertices of S, which is a contradiction to S an edge deour monophonic set of G. Hence $edm(G) = n$. □

Remark 2.16. The converse of Theorem 2.15 is not true. For the graph G given in Figure 1.1, $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a minimum edge detour monophonic set of G so that $edm(G) = 6 = n$ and has exactly one vertex v_1 of degree $n - 1$.

Theorem 2.17. Let G be a graph of order $n \geq 3$. If G contains a cut-vertex of degree $n - 1$, then $edm(G) = n - 1$.

Proof. Let v be a cut-vertex of degree $n-1$. Clearly $S = V - \{v\}$ is an edge detour monophonic set of G and so $edm(G) \leq n - 1$. Now, we show that $edm(G) = n - 1$. Let T be any set of vertices with $|T| \leq n - 2$. Then there exist at least two vertices, say u and w, which are not in T. Since v is adjacent to all the remaining vertices of G, the edges vu and vw do not lie on any detour monophonic path joining any two vertices of T. Hence T is not an edge detour monophonic set of G and so $edm(G) = n - 1$. □

Remark 2.18. The converse of Theorem 2.17 is not true. For the graph G given in Figure 2.3, $S = V(G) - \{y\}$ is an edm-set and so $edm(G) = 4$. However, y is a cut-vertex of degree 3.

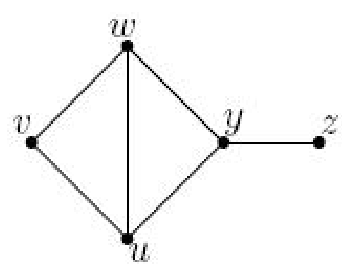

Figure 2.3: G

Problem 2.19. Characterize graphs G of order n for which $edm(G) = n - 1$.

3. Realisation Results

Theorem 3.1. For every pair k, n of integers with $2 \leq k \leq n$, there exists a connected graph G of order n with $edm(G) = k$.

Proof. For $k = n$, let $G = K_n$. Then by Corollary 2.6, we have $edm(G) = n$. Now, let $2 \leq k < n$. Let G be any tree of order n with k end vertices. Then by Theorem 2.10, $edm(G) = k$. □

Theorem 3.2. For any integers a, b and c with $2 \leq a \leq b \leq c$, there exists a connected graph G such that $m(G) = a, m_1(G) = b$ and $edm(G) = c$.

Proof. We consider four cases.

Case 1. For $a = b = c$, any tree with a end vertices has the desired property.

Case 2. $a < b = c$.

Subcase (i). $a = b-1$. Let $C : v_1, v_2, \ldots, v_6, v_1$ be a cycle of order 6. Let G be the graph obtained by adding a new vertices $u_1, u_2, \ldots, u_a$ to C and joining each $u_i (1 \leq i \leq a-1)$ to v_1, joining u_a to v_4, and joining the vertices v_3 and v_5. The graph G is shown in Figure 3.1. Let $S = \{u_1, u_2, \ldots, u_a\}$ be the set of all extreme vertices of G. By Theorems 1.1, 1.2 and 2.4, S is a subset of every monophonic set, edge monophonic set and edge detour monophonic set of G. Clearly, S is a monophonic set and so $m(G) = a$.

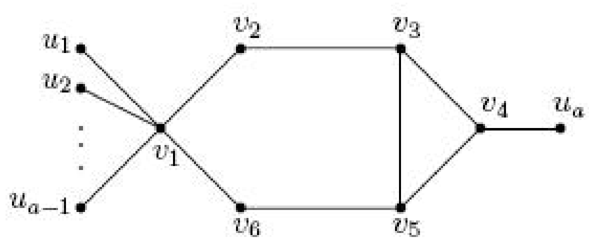

Figure 3.1: G

It is easily seen that S is not an edge monophonic set and an edge detour monophonic set of G, since the edge v_3v_5 does not lie any $x-y$ monophonic path or detour monophonic path, for some $x, y \in S$. Let $T = S \cup \{v_3\}$. Clearly T is an edge monophonic set and an edge detour monophonic set of G so that $m_1(G) = edm(G) = b = a + 1$.

Subcase (ii). $a \leq b - 2$. Let $P_{b-a+1} : u_1, u_2, \ldots, u_{b-a+1}$ be a path of order $b - a + 1$ and $P_2 : x, y$ be a path of order 2. Now, let H be the graph obtained by joining the vertices $u_i (1 \leq i \leq b-a+1)$ with y and also joining the vertices x and u_{b-a+1}. Let G be the graph obtained by adding $a-1$ new vertices $v_1, v_2, \ldots, v_{a-1}$ to H and joining each $v_i (1 \leq i \leq a-1)$ to x in H. The graph G is shown in Figure 3.2. Let $S = \{v_1, v_2, \ldots, v_{a-1}, u_1\}$ be the set of all extreme vertices of G. By Theorem 1.1, every monophonic set contains S. Clearly, S is a monophonic set of G and so $m(G) = a$. It is easily verified that S is not an edge monophonic set of G. Let $S' = S \cup \{u_2, \ldots, u_{b-a+1}\}$ be the set of all semi-extreme vertices of G. By Theroems 1.2 and 2.4, S' is a subset of every edge monophonic set and every edge detour monophonic set of G. Clearly, S' is an edge monophonic set and an edge detour monophonic set of G so that $m_1(G) = edm(G) = b$.

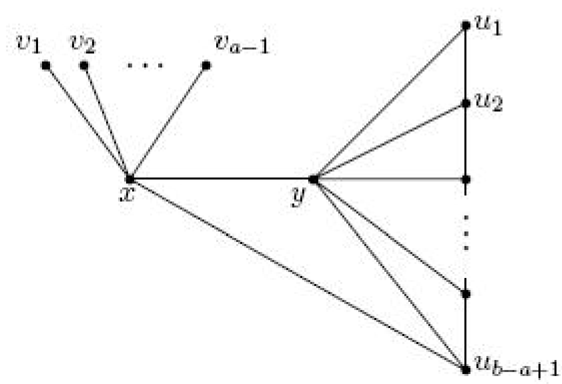

Figure 3.2: G

Case 3. $a = b < c$. Let $P_5 : v_1, v_2, \ldots, v_5$ be a path of order 5 and let $P_3 : x, y, z$ be a path of order 3. Let H be the graph obtained from P_5 and P_3 by joining the vertex x to v_2 and joining the vertex z to v_4. Let G be the graph obtained by adding $c - 2$ new vertices $u_1, u_2, \ldots, u_{a-1}, w_1, w_2, \ldots, w_{c-a-1}$ to H and joining each $w_i (1 \leq i \leq c - a - 1)$ to v_2, v_4; and also joining each $u_i (1 \leq i \leq a - 1)$ to v_1. The graph G is shown in Figure 3.3. Let $S = \{u_1, u_2, \ldots, u_{a-1}, v_5\}$ be the set of all extreme vertices of G. Then by Theorems 1.1, 1.2 and 2.4, S is contained in every monophonic set, edge monophonic set and edge detour monophonic set of G. It is easily seen that

S is a monophonic set and also an edge monophonic set of G so that $m(G) = m_1(G) = a$. It is easily verified that the edges $v_2w_i, w_iv_4 (1 \leq i \leq c-a-1)$ and v_2v_3, v_3v_4 are not internal edges of any $x-y$ detour monophonic path in G, for some $x, y \in S$. Let $T = S \cup \{v_3, w_1, w_2, \ldots, w_{c-a-1}\}$. Clearly, T is the unique edge detour monophonic set of G and so $edm(G) = c$.

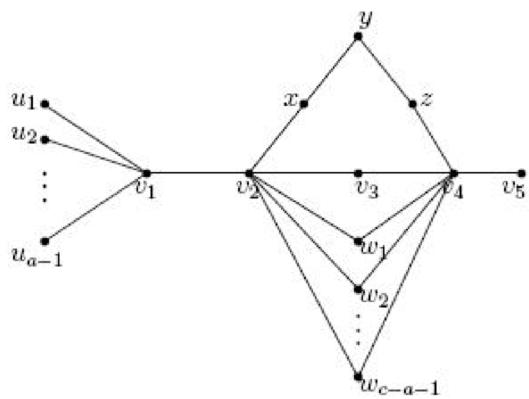

Figure 3.3: G

Case 4. $a < b < c$.

Subcase (i). $a = b-1$. Let $P_3 : x, y, z$ be a path of order 3, let $P_5 : v_1, v_2, \ldots, v_5$ be a path of order 5, and let $C : u_0, u_1, \ldots, u_5, u_0$ be a cycle of order 6. Let H be the graph obtained from P_3, P_5 and C by joining x to v_2; z to v_4; u_2 to u_4; and identifying the vertices v_5 and u_0. Let G be the graph obtained by adding $a+c-b-1$ new vertices $z_1, z_2, \ldots, z_a, w_1, w_2, \ldots, w_{c-b-1}$ to H and joining each $z_i (1 \leq i \leq a-1)$ to v_1, joining z_a to u_3, and also joining each $w_i (1 \leq i \leq c-b-1)$ to both v_2 and v_4. The graph G is shown in Figure 3.4. Let $S = \{z_1, z_2, \ldots, z_{a-1}, z_a\}$ be the set of all extreme vertices of G. Then by Theorems 1.1, 1.2 and 2.4, S is contained in every monophonic set, edge monophonic set and edge detour monophonic set of G. It is easily verified that S is a monophonic set of G and so $m(G) = a$. Clearly, S is not an edge monophonic set and an edge detour monophonic set of G. Let $T = S \cup \{u_2\}$. It is easily verified that T is an edge monophonic set and so $m_1(G) = a+1 = b$. Clearly, T is not an edge detour monophonic set of G. It is easily verified that the edges $v_2w_i, w_iv_4 (1 \leq i \leq c-b-1)$ and v_2v_3, v_3v_4 are not internal edges of any $x-y$ detour monophonic path in G with $x, y \in S$. Let $T' = T \cup \{v_3, w_1, w_2, \ldots, w_{c-b-1}\}$. Clearly, T' is an edge detour monophonic set of G and so $edm(G) = c$.

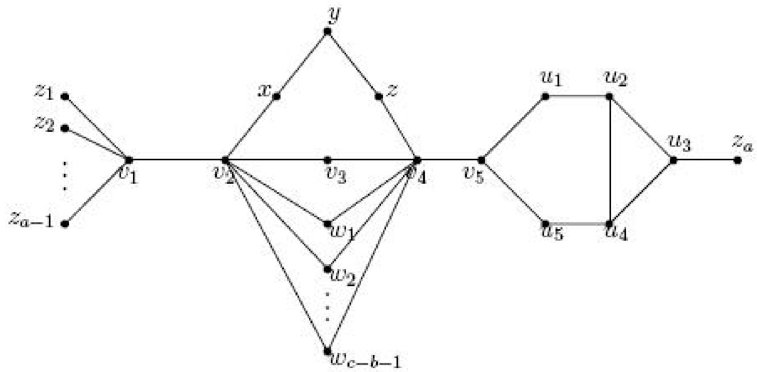

Figure 3.4: G

Subcase (ii). $a \le b-2$. Let $P_3 : x, y, z$ be a path of order 3, let $P_5 : v_1, v_2, \ldots, v_5$ be a path of order 5 and let $P_{b-a+1} : u_1, u_2, \ldots, u_{b-a+1}$ be a path of order $b-a+1$. Let H be the graph obtained from P_3, P_5 and P_{b-a+1} by joining x to v_2; z to v_4; v_4 to u_{b-a+1}; and each vertex $u_i (1 \le i \le b-a+1)$ to v_5. Let G be the graph obtained by adding $c + a - b - 2$ new vertices $z_1, z_2, \ldots, z_{a-1}, w_1, w_2, \ldots, w_{c-b-1}$ to H and joining each $z_i (1 \le i \le a-1)$ to v_1, and joining each $w_i (1 \le i \le c-b-1)$ to both the vertices v_2, v_4. The graph G is shown in Figure 3.5.

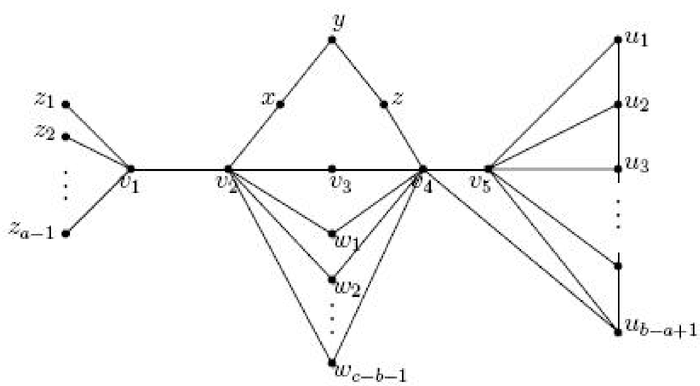

Figure 3.5:G

Let $S = \{z_1, z_2, \ldots, z_{a-1}, u_1\}$ be the set of all extreme vertices of G.

Then by Theorems 1.1, 1.2 and 2.4, S is contained in every monophonic set, edge monophonic set and edge detour monophonic set of G. It is easily verified that S is a monophonic set of G and so $m(G) = a$. Let $T = S \cup \{u_2, u_3, \ldots, u_{b-a+1}\}$ be the set of all semi-extreme vertices of G. Then by Theorems 1.2 and 2.4, T is contained in every edge monophonic set and edge detour monophonic set of G. It is easily verified that T is an edge monophonic set of G and so $m_1(G) = b$. Clearly, the edges $v_2w_i, w_iv_4 (1 \leq i \leq c-b-1)$ and v_2v_3, v_3v_4 are not internal edges of any $x-y$ detour monophonic path in G with $x, y \in S$. Let $T' = T \cup \{v_3, w_1, w_2, \ldots, w_{c-b-1}\}$. Clearly, T' is an edge detour monophonic set of G and so $edm(G) = c$. □

Theorem 3.3. For any integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $dm(G) = a$ and $edm(G) = b$.

Proof. We consider three cases.

Case 1. For $a = b$, any tree with a end vertices has the desired property.

Case 2. $a = b - 1$. Consider the graph G given in Figure 3.1. Let $S = \{u_1, u_2, \ldots, u_a\}$ be the set of all extreme vertices of G. By Theorems 1.3 and 2.4, S is contained in every detour monophonic set and every edge detour monophonic set of G. Clearly, S is a detour monophonic set of G and so $dm(G) = a$. It is easily seen that S is not an edge detour monophonic set of G. Let $T = S \cup \{v_3\}$. Then T is an edge detour monophonic set of G and so $edm(G) = b = a + 1$.

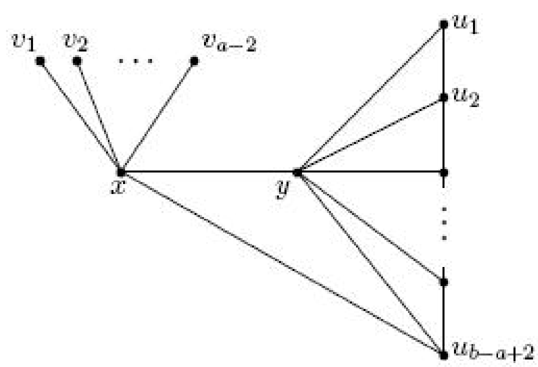

Figure 3.6: G

Case 3. $a \leq b - 2$. Let $P_{b-a+2} : u_1, u_2, \ldots, u_{b-a+2}$ be a path of order $b - a + 2$ and $P_2 : x, y$ be a path of order 2. Let H be the graph obtained by joining each vertex $u_i (1 \leq i \leq b-a+2)$ with y and also joining the vertices x and u_{b-a+2}. Let G be the graph obtained by addining $a-2$ new vertices $v_1, v_2, \ldots, v_{a-2}$ to H and joining each $v_i (1 \leq i \leq a-2)$ to x in H. The graph G is shown in Figure 3.6. Let $S = \{v_1, v_2, \ldots, v_{a-2}, u_1\}$ be the set of all extreme vertices of G. By Theorems 1.3 and 2.4, S is contained in every detour monophonic set and every edge detour monophonic set of G. Then S is not a detour monophonic set and an edge detour monophonic set of G. Let $S' = S \cup \{y\}$. It is easily verified that S' is a detour monophonic set of G and so $dm(G) = a$. Let $S'' = S \cup \{u_2, u_3, \ldots, u_{b-a+2}\}$ be the set of all semi-extreme vertices of G. Then by Theorem 2.4, every edge detour monophonic set contains S''. It is easily seen that S'' is an edge detour monophonic set of G and so $edm(G) = b$. □

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