

## On square sum graphs

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*Received : July 2012. Accepted : March 2013*

### Abstract

A  $(p, q)$ -graph  $G$  is said to be square sum, if there exists a bijection  $f : V(G) \rightarrow \{0, 1, 2, \dots, p-1\}$  such that the induced function  $f^* : E(G) \rightarrow N$  given by  $f^*(uv) = (f(u))^2 + (f(v))^2$  for every  $uv \in E(G)$  is injective. In this paper we initiate a study on square sum graphs and prove that trees, unicyclic graphs,  $mC_n$ ,  $m \geq 1$ , cycle with a chord, the graph obtained by joining two copies of cycle  $C_n$  by a path  $P_k$  and the graph defined by path union of  $k$  copies of  $C_n$ , when the path  $P_n \cong P_2$  are square sum.

**Key Words :** *Square sum graphs.*

## 1. Introduction

If the vertices of the graph are assigned values subject to certain conditions is known as graph labeling. A dynamic survey on graph labeling is regularly updated by Gallian [4]. Pairs of positive integers whose sum of squares are distinct, constitute an important part of number theory. For example, in number theory a positive integer  $n$  has a representation as a sum of two squares if  $n = a^2 + b^2$  for some  $a, b \in \mathbb{Z}$ . Exploration of the following problems were of great interest which inspired the curiosity of number theorists.

**Problem 1.1.** *What numbers can be written as sum of squares of two numbers? In general does there exists a number  $k$  so that all numbers can be written as sum of  $k$  squares?*

In additive number theory, Pierre de Fermat's theorem on sums of two squares states that an odd prime  $p$  is expressible as  $p = x^2 + y^2$  with  $x$  and  $y$  integers, if and only if  $p \cong (1 \pmod{4})$ . For example, the primes 5, 13, 17, 29, 37 and 41 are all congruent to 1 *mod* 4, and they can be expressed as sums of two squares in the following ways:  $5 = 1^2 + 2^2$ ;  $13 = 2^2 + 3^2$ ;  $17 = 1^2 + 4^2$ ;  $29 = 2^2 + 5^2$ ;  $37 = 1^2 + 6^2$ ;  $41 = 4^2 + 5^2$  etc. On the other hand, the primes 3, 7, 11, 19, 23 and 31 are all congruent to 3 *modulo* 4, and none of them can be expressed as the sum of two squares. Albert Girard was the first to make the observation (in 1632) and Fermat was first to claim a proof of it. Fermat announced this theorem in a letter to Marin Mersenne dated December 25, 1640; for this reason this theorem is sometimes called Fermat's Christmas Theorem. Since the Brahmagupta-Fibonacci identity implies that the product of two integers that can be written as the sum of two squares is itself expressible as the sum of two squares, by applying Fermat's theorem to the prime factorization of any positive integer  $n$ , we see that if all of  $n$ 's odd prime factors congruent to 3 *mod* 4 occur to an even exponent, it is expressible as a sum of two squares. The converse also holds.

These wide-angular history of sum of squares of numbers motivated the authors to study the particular graphs named square sum graphs.

Unless mentioned otherwise, by a *graph* we shall mean in this paper a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [3]. Square sum graphs are vertex labeled graphs with the labels from the set  $\{0, 1, 2, \dots, p-1\}$  such that the induced edge labels as the sum of the squares of the labels of the end vertices are all distinct. Not every graph is square sum. For

example, any complete graph  $K_n$ , where  $n \geq 6$  is not square sum [2]. We are interested to study different classes of graphs, which are square sum. In this paper, we shall show that trees, unicyclic graphs,  $mC_n$ ,  $m \geq 1$ , cycle with a chord, the graph obtained by joining two copies of cycles  $C_n$  by a path  $P_k$  and the graph defined by path union of  $k$  copies of  $C_n$ , when the path  $P_n \cong P_2$  are square sum.

## 2. Square sum graphs

Acharya and Germina defined a square sum labeling of a  $(p, q)$ -graph  $G$  as follows [1, 2].

**Definition 2.1.** A  $(p, q)$ -graph  $G$  is said to be square sum, if there exists a bijection  $f : V(G) \rightarrow \{0, 1, 2, \dots, p - 1\}$  such that the induced function  $f^* : E(G) \rightarrow N$  is given by  $f^*(uv) = (f(u))^2 + (f(v))^2$ , for every  $uv \in E(G)$  is injective.

Some square sum graphs are depicted in Figure 1.

**Theorem 2.2.**  $P_n$  is square sum for every  $n$ .



Figure 1: Example of square sum graphs

**Proof.** Let  $V(P_n) = \{u_1, u_2, u_3, \dots, u_n\}$ . Define a function  $f : V(P_n) \rightarrow \{0, 1, 2, \dots, n-1\}$  by  $f(u_i) = i-1$ ;  $1 \leq i \leq n$ . Clearly, the edge labels of  $P_n$  are distinct as it form a strictly increasing sequence of odd numbers.  $\square$

**Theorem 2.3.** *Trees are square sum.*

**Proof.** Let  $v_{0,0}$  be a vertex with the maximum degree in a tree  $T$ . Choose  $v_{0,0}$  as the root vertex. Let  $k$  denote the height of the tree  $T$ . Let  $n_1$  be the number of vertices adjacent to  $v_{0,0}$  and let these vertices be denoted by  $v_{1,1}, v_{1,2}, \dots, v_{1,n_1}$ . These vertices are at a distance 1 from the root vertex and are taken as the first level vertices. Let  $n_2$  be the number of vertices adjacent to first level vertices. Denote the vertices adjacent to  $v_{1,1}, v_{1,2}, \dots, v_{1,n_1}$  by  $v_{2,1}, v_{2,2}, \dots, v_{2,n_2}$  in such a manner that the vertices adjacent to  $v_{1,i}$  are labeled first in an increasing order when compared to that of  $v_{1,j}$ , where  $1 \leq i < j \leq n_1$ . These vertices are at a distance 2 from the root vertex and are taken as the second level vertices. Proceeding like this, let  $n_k$  be the number of vertices adjacent to the vertices of  $(k-1)^{th}$  level. The vertices in  $k^{th}$  level are denoted by  $v_{k,1}, v_{k,2}, \dots, v_{k,n_k}$ . Thus we have  $|V(T)| = \sum_{i=1}^k n_i + 1$ . Define  $f : V(T) \rightarrow \{0, 1, \dots, \sum_{i=1}^k n_i\}$  as follows.

$$\begin{aligned} f(v_{0,0}) &= 0 \\ f(v_{1,i}) &= f(v_{0,0}) + i; \quad 1 \leq i \leq n_1 \\ f(v_{2,i}) &= f(v_{0,0}) + (n_1 + i); \quad 1 \leq i \leq n_2 \\ f(v_{3,i}) &= f(v_{0,0}) + (n_1 + n_2 + i); \quad 1 \leq i \leq n_3 \\ &\vdots \\ f(v_{k,i}) &= f(v_{0,0}) + (n_1 + n_2 + \dots + n_{k-1} + i); \quad 1 \leq i \leq n_k. \end{aligned}$$

Clearly  $f$  is injective. Let  $e_1, e_2 \in E(T)$  be any two arbitrary edges of  $T$ . If  $e_1$  and  $e_2$  are two incident edges, then  $f^*(e_1) \neq f^*(e_2)$ . Let  $e_1 = u_1u_2$  and  $e_2 = u_3u_4$  be the edges such that  $e_1$  and  $e_2$  have no common vertex. Without loss of generality, assume that  $f(u_1)$  is least among  $\{f(u_1), f(u_2), f(u_3), f(u_4)\}$ . Since  $T$  is a tree, at least one of the vertices  $u_3, u_4$  is not adjacent to  $u_1$ . If  $u_4$  is not adjacent to  $u_1$  then  $f(u_4) > f(u_2)$  and if  $u_3$  is not adjacent to  $u_1$  then  $f(u_3) > f(u_2)$ . Hence  $f^*(e_1) \neq f^*(e_2)$ , showing that  $T$  is square sum.  $\square$

**Theorem 2.4.** *Forests are square sum.*

**Proof.** Let  $F = \bigcup_{i=1}^k T_i$  be a forest, where each  $T_i$  is a tree. Let  $|V(T_i)| =$

$n_i$ ;  $1 \leq i \leq k$ , so that  $T_1 \cup T_2 \cup \dots \cup T_k$  is of order  $p = \sum_{i=1}^k n_i$ .

Define  $f : V(\bigcup_{i=1}^k T_i) \rightarrow \{0, 1, \dots, p - 1\}$  as follows. Let  $v$  be a vertex with maximum degree in tree  $T_1$ . Choose  $v$  as the root vertex. Visit all the vertices of  $T_1$  using BFS algorithm and label the vertices with the consecutive numbers  $0, 1, 2, 3, \dots, n_1 - 1$  in ascending order, in the order in which they are visited in a one to one manner. By Theorem 2.3,  $T_1$  is square sum. Then label the vertices of  $T_2$  as follows. Let  $u$  be a vertex with maximum degree in tree  $T_2$ . Choose  $u$  as the root vertex. Visit the vertices of  $T_2$  using BFS algorithm and label the vertices with the consecutive numbers  $n_1, n_1 + 1, n_1 + 2, \dots, n_1 + n_2 - 1$  in ascending order, in the order in which they are visited in a one to one manner. Clearly  $T_1 \cup T_2$  is square sum. Now label the vertices of  $T_3$  in a similar way applying BFS algorithm, by choosing maximum degree vertex as the root vertex, with label  $n_1 + n_2$ . Proceeding in this way, we label the vertices of  $T_k$  as follows. Let  $w$  be a vertex with maximum degree in  $T_k$ . Choose  $w$  as the root vertex. Visit the vertices of  $T_k$  using BFS algorithm and label the vertices with consecutive numbers  $(n_1 + n_2 + \dots + n_{k-1}), (n_1 + n_2 + \dots + n_{k-1} + 1), \dots, (n_1 + n_2 + \dots + n_{k-1} + n_k - 1)$  in ascending order, in the order in which they are visited. Clearly, the induced edge label of  $T_i$ ;  $1 \leq i \leq k$ , are in strictly increasing order. Hence  $\bigcup_{i=1}^k T_i$  is square sum.  $\square$

**Theorem 2.5.** *Cycles are square sum.*

**Proof.** Let  $C_n$  be a cycle of length  $n$  and let  $C_n : u_1, u_2, \dots, u_n, u_1$ .  $E(C_n) = \{u_i u_{i+1}; 1 \leq i \leq n - 1\} \cup u_n u_1$ .

Define  $f : V(C_n) \rightarrow \{0, 1, 2, \dots, n - 1\}$  by

$$f(u_1) = 0 \text{ and}$$

when  $n$  is odd

$$f(u_i) = \begin{cases} 2i - 3 & \text{if } 2 \leq i \leq \lceil \frac{n}{2} \rceil, \\ 2(n + 1 - i) & \text{if } \lceil \frac{n}{2} \rceil < i \leq n. \end{cases}$$

when  $n$  is even

$$f(u_i) = \begin{cases} 2i - 3 & \text{if } 2 \leq i \leq \frac{n}{2} + 1, \\ 2(n + 1 - i) & \text{if } \frac{n}{2} + 1 < i \leq n. \end{cases}$$

Clearly  $f$  is injective. There are three classes of edges.

1) Edges connecting consecutive distinct odd numbers,  $\{u_i u_{i+1}; 2 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ .

2) Edges connecting consecutive distinct even numbers,  $\{u_i u_{i+1}; \lfloor \frac{n}{2} \rfloor + 1 < i \leq n - 1\} \cup u_n u_1$  and

3) Edges  $u_1 u_2$  and  $u_{\lfloor \frac{n}{2} \rfloor + 1} u_{\lfloor \frac{n}{2} \rfloor + 2}$ , having end vertices with the labels in which one of them is even and the other is odd, which receive odd edge labels.

Clearly the labelings of class 1 and class 2 are distinct. In class 3,  $f^*(u_1 u_2) = 1$  and  $f^*(u_{\lfloor \frac{n}{2} \rfloor + 1} u_{\lfloor \frac{n}{2} \rfloor + 2})$  is strictly greater than 1. Hence  $f^*(u_1 u_2) \neq f^*(u_{\lfloor \frac{n}{2} \rfloor + 1} u_{\lfloor \frac{n}{2} \rfloor + 2})$ . Since the labels of the edges connecting consecutive even numbers and consecutive odd numbers are distinct, the edge labelings of class 1 edges are distinct from edge labeling of class 2 edges. Since the edge labelings of class 3 edges are distinct odd numbers, they are distinct from edge labeling of class 1 edges and class 2 edges of distinct even numbers. Hence all the induced edge labels are distinct and  $f$  is square sum labeling of  $C_n$ .  $\square$

**Theorem 2.6.** *Unicyclic graphs are square sum.*

**Proof.** Let  $G$  be a unicyclic graph with unique cycle  $C_n : u_1, u_2, u_3, \dots, u_n, u_1$  and define the function  $f : V(G) \rightarrow \{0, 1, 2, \dots, n - 1\}$  as follows.

Case (1):  $G \cong C_n$ . Choose an arbitrary vertex say  $u_1$  of  $C_n$ . Starting from  $u_1$ , visit all the vertices of  $G$  using BFS algorithm and label the vertices  $u_i$ ;  $1 \leq i \leq n$  in the order in which they are visited. By Theorem 2.5,  $f$  is a square sum labeling of  $G$ .

Case (2): Let  $G \not\cong C_n$ . We prove the theorem in following two steps. Choose any vertex  $u_1$  of  $C_n$  such that  $\deg(u_1) \geq 3$ .

Subcase (i):  $n$  is even. Starting from  $u_1$ , visit all the vertices of  $G$  using BFS algorithm subject to the following restrictions. First visit all the neighbors of  $u_1$  not on the cycle so that  $u_n$  and  $u_2$  are visited consecutively. Without loss of generality, assume that  $u_n$  is visited first. In general, if  $0 \leq i \leq \frac{n}{2}$  while we visit the neighbors of  $u_{n-i}$ , first visit the neighbors that are not on cycle and for neighbors of  $u_{i+2}$ , first visit its neighbor on the cycle. Label the vertices of  $G$  with the consecutive numbers  $0, 1, 2, \dots, n-1$  in the order in which we visit them. Clearly  $f(V) = \{0, 1, 2, \dots, n-1\}$ . Since  $G$  is a unicyclic graph, BFS algorithm gives exactly one back edge  $u_k u_{k+1}$ , where  $k = \frac{n}{2}$ . Now let  $e_1$  and  $e_2$  be any two edges of  $G$ . If either  $e_1$  and  $e_2$  are adjacent or  $e_1, e_2 \neq u_k u_{k+1}$ , then obviously  $f^*(e_1) \neq f^*(e_2)$ .

Suppose  $e_1 = u_k u_{k+1}$ .  $e_1$  and  $e_2$  are non-adjacent and  $e_2 = v_1 v_2$  with  $f(v_1) < f(v_2)$ . If  $f(v_2) < f(u_k)$  then  $f(v_1) < f(v_2) < f(u_k) < f(u_{k+1})$ . If  $f(u_k) < f(v_2) < f(u_{k+1})$ , then  $f(v_1) < f(u_{k+2}) < f(u_k) < f(v_2) < f(u_{k+1})$ . if  $f(u_k), f(u_{k+1}) < f(v_2)$  then  $f(v_1) > f(u_k)$  so that  $f(u_k) < f(v_1) < f(u_{k+1}) < f(v_2)$  or  $f(u_k) < f(u_{k+1}) < f(v_1) < f(v_2)$ . In all cases we have  $f^*(e_1) \neq f^*(e_2)$ .

Subcase(ii):  $n$  is odd. Starting from  $u_1$ , visit all the neighbors of  $u_1$ , using BFS algorithm, such that we first visit  $u_n$ , then the neighbors of  $u_1$  that are not on the cycle and then  $u_2$ . Next visit  $u_{n-1}$  and then the other neighbors of  $u_n$  so that  $u_2$  and  $u_{n-1}$  are visited consecutively. Now proceeding as in Subcase (i) we obtain a square sum labeling of  $G$ .  $\square$

**Theorem 2.7.** For integers  $m$  and  $n$ , the graph  $mC_n$  is square sum for all  $m \geq 1$  and for all  $n \geq 3$ .

**Proof.** Let  $G$  denote  $m$  copies of  $C_n$ . When  $m = 1$  by Theorem 2.5,  $C_n$  is square sum. Let  $m \geq 2$ .

Let  $V(mC_n) = \{u_{i,j}; 1 \leq i \leq m, 1 \leq j \leq n\}$  and  $E(mC_n) = \{u_{i,j}u_{i,j+1}; 1 \leq i \leq m; 1 \leq j \leq n-1 \cup u_{i,n}u_{i,1}; 1 \leq i \leq m\}$

Define  $f : V(mC_n) \rightarrow \{0, 1, \dots, mn-1\}$  as follows.

$f(u_{i,1}) = n(i-1); 1 \leq i \leq m$

and when  $n$  is odd

$$f(u_{i,j}) = \begin{cases} f(u_{i,1}) + 2j - 3 & \text{if } 1 \leq i \leq m; 2 \leq j \leq \lceil \frac{n}{2} \rceil, \\ f(u_{i,1}) + 2(n+1-j) & \text{if } 1 \leq i \leq m; \lceil \frac{n}{2} \rceil < j \leq n. \end{cases}$$

when  $n$  is even

$$f(u_{i,j}) = \begin{cases} f(u_{i,1}) + 2j - 3 & \text{if } 1 \leq i \leq m; \quad 2 \leq j \leq \frac{n}{2} + 1, \\ f(u_{i,1}) + 2(n + 1 - j) & \text{if } 1 \leq i \leq m; \quad \frac{n}{2} + 1 < j \leq n. \end{cases}$$

Clearly  $f$  is injective. The induced edge labels of  $mC_n$ ;  $m \geq 1$  are strictly increasing and all the edge labels are distinct. Hence  $mC_n$  is square sum for all  $m \geq 1$ .

□

**Definition 2.8.** *Path union of graphs:* Let graphs  $G_1, G_2, G_3, \dots, G_n$ ,  $n \geq 2$  be all copies of a fixed graph  $G$ . Adding an edge between  $G_i$  to  $G_{i+1}$  for  $i = 1, 2, \dots, n-1$  is called path union of  $G$ .

**Theorem 2.9.** *The graph obtained by joining two copies of cycle  $C_n$  by a path  $P_k$  is square sum.*

**Proof.** Let  $C_{n_1}; 1 \leq i \leq 2$  be two copies of cycle  $C_n$ . Let  $G$  be the graph obtained by joining two copies of cycle  $C_n$  with path  $P_k$ . Let  $u_i; 1 \leq i \leq n$  be the vertices of  $C_{n_1}$  and  $v_i; 1 \leq i \leq n$  be the vertices of  $C_{n_2}$ . Let  $w_i; 1 \leq i \leq k$  be the vertices of path  $P_k$  with  $u_1 = w_1$  and  $v_1 = w_k$ . Here we note that  $|V(G)| = 2n + k - 2$ . Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2n + k - 3\}$  as follows. Start from  $u_1$  of  $C_{n_1}$  visit all the vertices of  $C_{n_1}$ , using BFS algorithm and label the vertices  $u_i; 1 \leq i \leq n$  as  $n-1, n-2, \dots, 2, 1, 0$  in descending order, the order in which they are visited. By Theorem 2.5,  $f$  is a square sum labeling of  $C_{n_1}$ . Define  $f : V(P_k) \rightarrow \{n-1, n, n+1, n+2, \dots, (n+k-3), (n+k-2)\}$  by  $f(w_i) = f(u_1) + i - 1; 1 \leq i \leq k$ . Clearly the edge labels of path  $P_k$  are strictly increasing sequence of odd numbers, and hence they are distinct. Starting from  $w_k = v_1$ , of  $C_{n_2}$  visit all the vertices of  $C_{n_2}$  using BFS algorithm, and label the vertices  $v_i; 1 \leq i \leq n$  as  $(n+k-2), (n+k-1), (n+k), \dots, (2n+k-3)$ , in the order in which they are visited. The edge labeling of path  $P_k$  are strictly greater than edge labels of  $C_{n_1}$ . Also edge labels of  $C_{n_2}$  are strictly greater than that of  $P_k$ . Hence induced edge labels of  $C_{n_i}; 1 \leq i \leq 2$  with path  $P_k$  are distinct and  $f$  is a square sum labeling of  $G$ . □

**Theorem 2.10.** *The path union of  $k$  copies of  $C_n$ , where  $k \geq 1$  when the path  $P_n \cong P_2$  are square sum.*

**Proof.** Let  $C_{n_i}; 1 \leq i \leq k$  be the  $k$  copies of  $C_n$ . Let  $G$  be a graph obtained by joining  $k$  copies of  $C_n$  by path  $P_2$ . Here  $|V(G)| = kn$ . Let



$C_{n_1} : u_1, u_2, \dots, u_n, u_1$  and  $C_{n_2} : v_1, v_2, \dots, v_n, v_1$  be the first two copies of  $C_n$ , where  $u_1v_1$  be the path  $P_2$  joining  $C_{n_1}$  and  $C_{n_2}$ . Define  $f : V(G) \rightarrow \{0, 1, 2, \dots, kn - 1\}$  as follows. Start from vertex  $u_1$  of  $C_{n_1}$ , visit all the vertices of  $C_{n_1}$  using BFS algorithm and label the vertices  $u_i$ ;  $1 \leq i \leq n$  of  $C_{n_1}$  as  $n - 1, n - 2, \dots, 1, 0$  in descending order, the order in which they are visited. By Theorem 2.5, this labeling is a square sum labeling of  $C_{n_1}$ . Starting from  $v_1$  visit all the vertices of  $C_{n_2}$  using BFS algorithm and label the vertices of  $C_{n_2}$  as  $n, n + 1, \dots, 2n - 1$  in ascending order, the order in which they are visited. Clearly label of edge  $u_1v_1$ , joining  $C_{n_1}$  and  $C_{n_2}$  is strictly greater than the edge labels of  $C_{n_1}$  and edge label of  $C_{n_2}$  is strictly greater than that of  $u_1v_1$ . Hence edge labels of  $C_{n_1}$ ,  $C_{n_2}$  and path  $P_2$  are distinct. Label the end vertex of path  $P_2$  joining  $C_{n_2}$  and  $C_{n_3}$  by  $2n - 1$  and  $2n$  respectively. Start from vertex with label  $2n$  on  $C_{n_3}$ , visit all the vertices using BFS algorithm and label the vertices in ascending order. Continue this process and label the vertices of  $C_{n_4}, \dots, C_{n_k}$  respectively using BFS algorithm and lastly, label the end vertices of path  $P_2$  joining  $C_{n_{k-1}}$  and  $C_{n_k}$  by  $(k - 1)n - 1$  and  $(k - 1)n$  respectively. Clearly all the edge labels of path are increasing sequence of odd numbers and are distinct from the edge labels of cycles. Hence  $f$  is square sum labeling of  $G$ .  $\square$

**Theorem 2.11.** Finite union  $\bigcup_{i=1}^k C_i$  of cycles  $C_i$  are square sum.

**Proof.** Let  $\bigcup_{i=1}^k C_i$  be finite union of cycles. Let  $C_1 : u_1, u_2, \dots, u_{n_1}, u_1$  be a cycle of length  $n_1$ ,  $C_2 : v_1, v_2, \dots, v_{n_2}, v_1$  be a cycle of length  $n_2$ , and  $C_k : w_1, w_2, \dots, w_{n_k}, w_1$  be a cycle of length  $n_k$ . Start from vertex  $u_1$  of  $C_1$ , visit all the vertices of  $C_1$  using BFS algorithm, label the vertices  $u_i$ ;  $1 \leq i \leq n_1$  of  $C_1$  as  $0, 1, 2, \dots, n_1 - 1$ , in the order in which they are visited. By Theorem 2.5,  $f$  is square sum labeling of  $C_1$ . Choose an arbitrary vertex  $v_1$  of  $C_2$ . Start from vertex  $v_1$  of  $C_2$ , visit all the vertices of  $C_2$  using BFS algorithm, label the vertices  $v_i$ ;  $1 \leq i \leq n_2$  of  $C_2$  as  $n_1, n_1 + 1, \dots, n_1 + n_2 - 1$ , in the order in which they are visited. Clearly  $C_1 \cup C_2$  is square sum. Now label the vertices of  $C_3$  in a similar way. Proceeding in this way we label the vertices of  $C_k$  as follows. Choose an arbitrary vertex  $w_1$  of  $C_k$ . Start from the vertex  $w_1$  of  $C_k$ , visit all the vertices of  $C_k$  using BFS algorithm, and label the vertices  $w_i$ ;  $1 \leq i \leq n_k$  of  $C_k$  as  $(n_1 + n_2 + \dots + n_{k-1}), (n_1 + n_2 + \dots + n_{k-1} + 1), \dots, (n_1 + n_2 + \dots + n_{k-1} + n_k - 1)$ , in the order in which they are visited. Since the induced

edge labels of  $\bigcup_{i=1}^k C_i$ ;  $1 \leq i \leq k$  are strictly increasing, all the edge labels are distinct. Hence  $\bigcup_{i=1}^k C_i$ ;  $1 \leq i \leq k$  is square sum.  $\square$

**Theorem 2.12.** *Cycle with a chord is square sum.*

**Proof.** By theorem 2.5,  $C_n : u_1, u_2, \dots, u_n, u_1$  is square sum. Join the vertices  $u_2$  and  $u_n$  by a chord. The edge label of  $u_2u_n$  is clearly distinct from all the induced edge labels of  $C_n$ . Hence cycle with a chord is square sum.  $\square$

Before concluding this section let us point out an interesting problem which will have a strong link with number theory. Consider an edge labeled graph with distinct labels for the edges. It would be worthwhile to find out suitable labeling for the vertices such that sum of the squares of the labels on the end vertices become the given edge labeling. This in turn is a reformulation of the Problem 1.1.

Scope for further investigation

One may define a particular case of square sum by labeling the vertices of the  $(p, q)$ -graph  $G$ , with  $\{0, 1, 2, \dots, p-1\}$  so as to have the induced edge labelings as distinct prime numbers. Another version of square sum graphs can be defined as labeling the vertices of the graph  $G$  using labels from  $\{0, 1, 2, \dots, q-1\}$  so as to have the induced edge labelings as distinct integers (prime numbers). Structural properties of graphs which admit such labelings seems to be interesting for further investigation.

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