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Bounded linear operators for some new matrix transformations

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Abstract

In this paper, we define (σ,θ)-convergence and characterize (σ,θ)-conservative, (σ,θ)-regular, (σ,θ)-coercive matrices and we also determine the associated bounded linear operators for these matrix classes.

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1. Introduction and preliminaries

We shall write w for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let ϕ, ℓ_∞, c and c_0 denote the sets of all finite, bounded, convergent and null sequences respectively; and cs be the set of all convergent series. We write $\ell_p := \{x \in w : \sum_{k=0}^{\infty} \mid x_k \mid^p < \infty\}$ for $1 \leq p < \infty$. By e and $e^{(n)} (n \in \mathbf{N})$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, ...,$ and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0 (k \neq n)$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n-section.

Note that c_0, c, and ℓ_∞ are Banach spaces with the sup-norm $\|x\|_\infty = \sup_k |x_k|$, and $\ell_p (1 \leq p < \infty)$ are Banach spaces with the norm $\|x\|_p = (\sum |x_k|^p)^{1/p}$; while ϕ is not a Banach space with respect to any norm.

A sequence $(b^{(n)})_{n=0}^{\infty}$ in a linear metric space X is called *Schauder basis* if for every $x \in X$, there is a unique sequence $(\beta_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \beta_n b^{(n)}$.

Let X and Y be two sequence spaces and $A = (a_{nk})_{n;k=1}^{\infty}$ be an infinite matrix of real or complex numbers. We write $Ax = (A_n(x))$, $A_n(x) = \sum_k a_{nk} x_k$ provided that the series on the right converges for each n. If $x = (x_k) \in X$ implies that $Ax \in Y$, then we say that A defines a matrix transformation from X into Y and by (X, Y) we denote the class of such matrices.

Let σ be a one-to-one mapping from the set $\mathbf{N}$ of natural numbers into itself. A continuous linear functional φ on the space ℓ_∞ is said to be an *invariant mean* or a σ-*mean* if and only if (i) $\varphi(x) \geq 0$ if $x \geq 0$ (i.e. $x_k \geq 0$ for all k), (ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$, (iii) $\varphi(x) = \varphi((x_{\sigma(k)}))$ for all $x \in \ell_\infty$.

Throughout this paper we consider the mapping σ which has no finite orbits, that is, $\sigma^p(k) \neq k$ for all integer $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ denotes the *pth* iterate of σ at k. Note that, a σ-mean extends the limit functional on the space c in the sense that $\varphi(x) = \lim x$ for all $x \in c$, (cf [10]). Consequently, $c \subset V_\sigma$, the set of bounded sequences all of whose σ-means are equal.We say that a sequence $x = (x_k)$ is σ-*convergent* if and only if $x \in V_\sigma$.

$$V_\sigma = \{x \in \ell_\infty : \lim_{p \to \infty} t_{pn}(x) = L, \quad \text{uniformly in n}\}.$$

where $L = \sigma - lim\ x$, where

$t_{pn}(x) = \frac{1}{p+1} \sum_{m=0}^{p} x_{\sigma^m(n)}$,

Using the concept of Schaefer [17] defined and characterized the σ-conservative, σ- regular and σ- coercive matrices. If σ is translation then the σ- mean often called Banach Limit [2] and the set V_σ reduces to the set f of almost convergent sequence studied by Lorenz [9]. By a lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r := (k_{r-1} - k_r]$, and the ratio $k_r \Big/ k_{r-1}$ will be abbreviated by q_r (see Fredman et al[8]). Recently, Aydin[1] defined the concept of almost lacunary convergent as follow: A bounded sequence $x = (x_k)$ is said be almost lacunary convergent to the number ℓ if and only if

$\lim_r \frac{1}{h_r} \sum_{j \in I_r} x_{j+n} = \ell$, uniformly in n.

The idea of σ-convergence for double sequences was introduced in [4] and further studied recently in [3] and [15]. In [11]-[14] we study various classes of four dimensional matrices, e.g. σ-regular, σ-conservative, regularly σ-conservative, boundedly σ-conservative and σ-coercive matrices.

In this paper, we define (σ, θ)-convergence. We also generalize the above matrices by characterizing the (σ, θ)-conservative, (σ, θ)-regular and (σ, θ)-coercive matrices. Further, we also determine the associated bounded linear operators for these matrix classes. which is the generalized result of Mursaleen, M.A. Jarrah and S.Mouhiddin see ref [15]

2. (σ, θ) -Lacunary convergent sequences

We define the following:

Definition 2.1. [sir paper,2009]A bounded sequence $x = (x_k)$ of real numbers is said to be (σ, θ) -lacunary convergent to a number ℓ if and only if $\lim_r \frac{1}{h_r} \sum_{j \in I_r} x_{\sigma^j(n)} = \ell$, uniformly in n, and let $V_\sigma(\theta)$,denote the set of all such sequences, i.e where

$$V_\sigma(\theta) = \{x \in \ell_\infty : \lim_r \frac{1}{h_r} \sum_{j \in I_r} x_{\sigma^j(n)} = \ell, \text{ uniformly in } n\}$$

Note that for $\sigma(n) = n + 1$, σ- lacunary convergence is reduced to almost lacunary convergence. Results similar to that Aydin[1] can easily be proved for the space $V_\sigma(\theta)$,

Definition 2.2. A bounded sequence $x = (x_k)$ of real numbers is said to be σ -lacunary bounded if and only if $\sup_{r,n} |\frac{1}{h_r} \sum_{j \in I_r} x_{\sigma^j(n)}| < \infty$, and we let $V_\sigma^\infty(\theta)$, denot the set of all such sequences

$$V_\sigma^\infty(\theta) = \{x \in \ell_\infty : \sup_{r,n} |\tau_{r,n}(x)| < \infty\}.$$

Where

$\tau_{rn}(x) = \frac{1}{h_r}\sum_{j\in I_r} x_{\sigma^j(n)}$,

Note that $c \subset V_\sigma(\theta) \subset V_\sigma^\infty(\theta) \subset \ell_\infty$.

Definition 2.3. An infinite matrix $A = (a_{nk})$ is said to be (σ,θ)-*conservative* if and only if $Ax \in V_\sigma(\theta)$ for all $x = (x_k) \in c$ and we denote this by $A \in (c, V_\sigma(\theta))$.

Definition 2.4. We say that, infinite matrix $A = (a_{nk})$ is said to be (σ,θ)-*regular* if and only if it is $V_\sigma(\theta)$-conservative and $(\sigma,\theta)\text{-}\lim Ax = \lim x$ for all $x \in c$ and we denote this by $A \in (c, V_\sigma(\theta))_{reg}$.

Definition 2.5. A matrix $A = (a_{nk})$ is said to be (σ,θ)-*coercive* if and only if $Ax \in V_\sigma(\theta)$ for all $x = (x_k) \in \ell_\infty$ and we denote this by $A \in (\ell_\infty, V_\sigma(\theta))$.

Remark 2.6. If we take $h_r = r$ then $V_\sigma(\theta)$ is reduced to the space V_σ and (σ,θ)-conservative, (σ,θ)-regular, (σ,θ)-coercive matrices are respectively reduced to σ-conservative, σ-regular, σ-coercive matrices (cf [15]); and in addition if $\sigma(n) = n + 1$ then the space $V_\sigma(\theta)$ is reduced to the space f of almost convergent sequences (cf [9]) and these matrices are reduced to the almost conservative, almost regular (cf [7]) and almost coercive matrices respectively (cf [6]).

3. (σ,θ)-conservative matrices and bounded linear operators

In the following theorem we characterize (σ,θ)-conservative matrices and find the associated bounded linear operator.

Theorem 3.1. A matrix $A = (a_{nk})$ is (σ,θ)-conservative, i.e. $A \in (c, V_\sigma(\theta))$ if and only if it satisfies the condition

(i) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$;

(ii) $a_{(k)} = (a_{nk})_{n=1}^\infty \in V_\sigma(\theta)$, for each k;

(iii) $a = \left(\sum_k a_{nk}\right)_{n=1}^\infty \in V_\sigma(\theta)$.

In this case, the (σ,θ)-limit of Ax is $\lim x\left[u - \sum_k u_k\right] + \sum_k x_k u_k$, where $u = (\sigma,\theta)\text{-}\lim a$ and $u_k = (\sigma,\theta)\text{-}\lim a_k$, $k = 1, 2, \cdots$.

Proof. *Sufficiency.* Let the conditions hold. Let r be any non-negative integer and $x = (x_k) \in c$. For every positive integer n; write $\tau_{rn}(x) = \frac{1}{h_r}\sum_{k=1}^{\infty}\sum_{j\in I_r} a_{\sigma^j(n),k}x_k$ Then we have —$\tau_{rn}(x)| \leq \frac{1}{h_r}\sum_{k=1}^{\infty}\sum_{j\in I_r}|a_{\sigma^j(n),k}||x_k|$ $\leq \frac{\|x\|}{h_r}\sum_{k=1}^{\infty}\sum_{j\in I_r}|a_{\sigma^j(n),k}| \leq \|A\|\|x\|$. Since τ_{rn} is obviously linear on c, it follows that $\tau_{rn} \in c'$ and $\|\tau_{rn}\| \leq \|A\|$.

Now, $\tau_{rn}(e) = \frac{1}{h_r}\sum_{k=1}^{\infty}\sum_{j\in I_r} a_{\sigma^j(n),k} = \frac{1}{h_r}\sum_{j\in I_r}\sum_{k=1}^{\infty} a_{\sigma^j(n),k}$that is, $\lim_r \tau_{rn}(e)$ exists uniformly in n and $\lim_r \tau_{rn}(e) = u$ uniformly in n, the (σ,θ)-limit of a, since $a \in V_\sigma(\theta)$. Similarly, $\lim_r \tau_{rn}e^k = u_k$, the (σ,θ)-limit of $a_{(k)}$ for each k, uniformly in n. Since $\{e, e^1, e^2, \cdots\}$ is a fundamental set in c, and $\sup_r|\tau_{r,n}(x)|$ is finite for each $x \in c$, it follows that $\lim_r \tau_{rn}(x) = \tau_n(x)$, exists for all $x \in c$ (cf [5]). Furthermore, $\|\tau_n\| \leq \liminf_r \|\tau_{rn}\| \leq \|A\|$ for each n and $\tau_n \in c'$. Thus, each $x \in c$ has a unique representation x=$(\lim x)\left[e - \sum_k e_k\right] + \sum_k x_k e_k$ $\tau_n(x) = (\lim x)\left[t_n(e) - \sum_k t_n(e_k)\right] + \sum_k x_k t_n(e_k)$ $\tau_n(x) = (\lim x)\left[u - \sum_k u_k\right] + \sum_k x_k u_k$.By $L(x)$, we denote the right hand side of the above expression which is independent of n. Now, we have to show that $\lim_r \tau_{rn}(x) = L(x)$ uniformly in n. Put $F_{rn}(x) = \tau_{rn}(x) - L(x)$.Then $F_{rn} \in c'$, $\|F_{rn}\| \leq 2\|A\|$ for all r, n, $\lim_r F_{rn}(e) = 0$ uniformly in n, and $\lim_r F_{rn}(e^k) = 0$ uniformly in n for each k. Let K be an arbitrary positive integer. Then x=$(\lim x)e + \sum_{k=1}^{K}(x_k - \lim x)e^k + \sum_{k=K+1}^{\infty}(x_k - \lim x)e^k$.Now applying F_{rn} on both sides of the above equality, we have $F_{rn}(x) = (\lim x)F_{rn}(e) + \sum_{k=1}^{K}(x_k - \lim x)F_{rn}(e^k) + F_{rn}\left(\sum_{k=K+1}^{\infty}(x_k - \lim x)e^k\right)$.(3.1.1)Now, $\left|F_{rn}\left(\sum_{k=K+1}^{\infty}(x_k - \lim x)e^k\right)\right| \leq 2\|A\|\sum_{k\geq K+1}\{|x_k - \lim x|\}$, for all r, n. After choosing fixed K large enough, it is easy to see that the absolute value of each term on the right hand side of (3.1.1) can be made uniformly small for all sufficiently large r. Therefore, $\lim_r F_{rn}(x) = 0$ uniformly in n; so that $Ax \in V_\sigma(\theta)$ and the matrix A is (σ,θ)-conservative.

Necessity. Suppose that A is (σ,θ)-conservative. Then Ax=$(A_n(x))_{n=1}^{\infty} = \left(\sum_k a_{nk}x_k\right)_{n=1}^{\infty} \in V_\sigma(\theta)$, for all $x \in c$. Let $x = (x_k) = e^k$. Therefore (σ,θ)-$\lim_n \sum_k a_{nk}e^k = (\sigma,\theta)$-$\lim_n a_{nk} = a_{(k)}$.Hence (ii) holds. Now, let $x = e$. Then (σ,θ)-$\lim_n \sum_k a_{nk}e = (\sigma,\theta)$-$\lim_n \sum_k a_{nk} = a$, so that (iii) must hold. Since $Ax = (A_n(x)) \in V_\sigma(\theta) \subset \ell_\infty$. It follows that $\sup_n |A_n(x)| <$

∞, (A_n) is a sequence of bounded operators. Therefore, by Banach-Steinhaus theorem, $\sup_n |A_n| < \infty$, which implies $\sup_n \sum_k |a_{nk}| < \infty$ and hence $\|A\| = \sup_n \sum\limits_k |a_{nk}| < \infty$, i.e. (i).

This completes the proof of the theorem.

Now, we deduce the following.

Corollary 3.2. $A = (a_{nk})$ is (σ,θ)-regular if and only if the conditions (i), (ii) with (σ,θ)-limit zero for each k, and (iii) with (σ,θ)-limit 1 of Theorem 3.1 hold.

Proof. For $x \in c$, (σ,θ)-$\lim Ax = L(x)$, which reduces to $\lim x$, since $u = 1$ and $u_k = 0$ for each k. Hence A is (σ,θ)-regular.

Conversely, let A be (σ,θ)-regular. Then (σ,θ)-$\lim Ae = 1 = (\sigma,\theta)$-$\lim Aa$, (σ,θ)-$\lim Ae^k = 0 = (\sigma,\theta)$-$\lim A_{(k)}$ and $\|A\|$ is finite as condition (i) of Theorem 3.1.

This completes the proof of the Corollary 3.2.

4. (σ,θ)-coercive matrices

We use the following lemma in our next theorem.

Lemma 4.1. Let $B(n) = (b_{mk}(n))$, $n = 0, 1, 2, \cdots$ be a sequence of infinite matrices such that

(i) $\|B(n)\| < H < +\infty$ for all n; and

(ii) $\lim\limits_m b_{mk}(n) = 0$ for each k, uniformly in n.

Then $\lim\limits_m \sum\limits_k b_{mk}(n)x_k = 0$ uniformly in n for each $x \in \ell_\infty$(4.1.1) if and only if $\lim\limits_m \sum\limits_k |b_{mk}(n)| = 0$ uniformly in n.(4.1.2)

Theorem 4.2. A matrix $A = (a_{nk})$ is (σ,θ)-coercive, i.e. $A \in (\ell_\infty, V_\sigma(\theta))$ if and only if (i) and (ii) of Theorem 3.1 hold, and

(iii) $\lim\limits_r \sum\limits_{k=1}^{\infty} |\sum\limits_{j\in I_r} a_{\sigma^j(n),k} - u_k|$ uniformly in n.

In this case, the (σ,θ)-limit of Ax is $\sum_k x_k u_k \quad \forall x \in \ell_\infty,$ where $u_k = (\sigma,\theta)$-$\lim a_k$.

Proof. *Sufficiency.* Let the conditions hold. For any positive integer K $\sum\limits_{k=1}^{K} |u_k| = \sum\limits_{k=1}^{K} \lim_r \left|\sum\limits_{j\in I_r} a_{\sigma^j(n),k}\right| \Big/ h_r \; = \lim_r \sum\limits_{k=1}^{K} \left|\sum\limits_{j\in I_r} a_{\sigma^j(n),k}\right| \Big/ h_r \;\; \leq$

$\limsup_r \sum_{j\in I_r}\sum_{k=1}^{\infty}\left|a_{\sigma^j(n),k}\right| \Big/ h_r \le ||A||$. This shows that $\sum_{k=1}^{\infty}|u_k|$ converges, and that $\sum_{k=1}^{\infty} u_k x_k$ is defined for every $x=(x_k)\in \ell_\infty$.

Let $x=(x_k)$ be any arbitrary bounded sequence. For every positive integer r $\left\|\sum_{k=1}^{\infty}\left(\frac{1}{h_r}\sum_{j\in I_r} a_{\sigma^j(n),k}-u_k\right)x_k\right\| = \left\|\sum_{k=1}^{\infty}\left[\sum_{j\in I_r}[a_{\sigma^j(n),k}-u_k]\Big/ h_r\right]x_k\right\|$ $\le \sup_n\left[\left|\sum_{k=1}^{\infty}\left[\sum_{j\in I_r}[a_{\sigma^j(n),k}-u_k]\Big/ h_r\right]x_k\right|\right] \le \|x\|\sup_r\left[\sum_{k=1}^{\infty}\left|\sum_{j\in I_r}[a_{\sigma^j(n),k}-u_k]\right|\Big/ h_r\right]$.

Letting $r\to\infty$ and using condition (iii), we get

$$\frac{1}{h_r}\sum_{k=1}^{\infty}\sum_{j\in I_r} a_{\sigma^j(n),k}x_k \longrightarrow \sum_{k=1}^{\infty} u_k x_k.$$

Hence $Ax\in V_\sigma(\theta)$ with (σ,θ)-limit $\sum_{k=1}^{\infty} u_k x_k$.

Necessity. Let A be (σ,θ)-coercive matrix. This implies that A is (σ,θ)-conservative, then we have condition (i) and (ii) from Theorem 3.1. Now we have to show that (iii) holds.

Suppose that for some n, we have $\limsup_r \sum_{k=1}^{\infty}\left|\sum_{j\in I_r}[a_{\sigma^j(n),k}-u_k]\right|\Big/ h_r = N>0$.Since $\|A\|$ is finite, therefore N is also finite. We observe that since $\sum_{k=1}^{\infty}|u_k|<+\infty$ and A is (σ,θ)-coercive, the matrix $B=(b_{nk})$, where $b_{nk}=a_{nk}-u_k$, is also (σ,θ)-coercive matrix. By an argument similar to that of Theorem 2.1 in [6], one can find $x\in\ell_\infty$ for which $Bx\notin V_\sigma(\theta)$. This contradiction implies the necessity of (iii).

Now, we use Lemma 4.1 to show that this convergence is uniform in n. Let $t_{rk}(n)=\sum_{j\in I_r}[a_{\sigma^j(n),k}-u_k]\Big/ h_r$and let $T(n)$ be the matrix $(t_{rk}(n))$. It is easy to see that $\|H(n)\|\le 2\|A\|$ for every n; and from condition (ii) $\lim_r t_{rk}(n)=0 \quad for\ each \quad k, \quad uniformly\ in \quad n$.For any $x\in\ell_\infty$ $\lim_r \sum_{j\in I_r} t_{rk}(n)x_k = (\sigma,\theta)\text{-}\lim Ax-\sum_{k=1}^{\infty}u_k x_k$and the limit exists uniformly in n, since $Ax\in V_\sigma(\theta)$. Moreover, this limit is zero since $\left|\sum_{k=1}^{\infty} t_{rk}(n)x_k\right|\le$

$\|x\| \sum_{k=1}^{\infty} \left| \sum_{j \in I_r} [a_{\sigma^j(n),k} - u_k] \right| \Big/ h_r$.Hence $\lim_r \sum_{k=1}^{\infty} \left| t_{rk}(n) \right| = 0$ *uniformly in* n; i.e. the condition (iii) holds.

This completes the proof of the theorem.

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