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# Some separation axioms in $L$-topological spaces

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**Abstract**

*In this paper, under the idea of L-$T_0$ or sub-$T_0$, we propose a set of new separation axioms in L-topological spaces, namely sub-separation axioms. And some of their properties are studied. In addition, the relation between the sub-separation axioms defined in the paper and other separation axioms is discussed. The results show that the sub-separation axioms in this paper are weaker than other separation axioms that had appeared in literature.*

**Keywords :** *L-topology; sub-separation axioms; sub-$T_1$; sub-$T_2$; sub-$T_{2\frac{1}{2}}$; sub-$T_3$; sub-$T_4$.*

## 1. Introduction and preliminaries

Since Chang [1] introduced fuzzy theory into topology, Wong, Lowen, Hutton etc., discussed respectively various aspects of fuzzy topology (Wong [17], Lowen [12], Hutton [6]).

Separation is an essential part of fuzzy topology, on which a lot of work have been done [2–19]. In 1983, Liu [9] introduced the sub-$T_0$ axiom, for underlying lattice $L$ being a completely distributive DeMorgan algebra, in terms of closed sets and proved that the fuzzy real line and the fuzzy unit interval satisfy this axiom. Wuyts and Lowen [18] and Rodabaugh [13] independently gave a more general $L$-$T_0$ axioms, the latter for $L$ being a complete lattice, using only open sets and equivalent to the sub-$T_0$ when $L$ is a completely distributive DeMorgan algebra. The aim of this paper is to study some separation axioms on the basis of the thought of the sub-$T_0$ and the layer of $L$-topology.

Now we recall some the concepts required in the sequel.

Throughout this paper, $(L, \bigvee, \bigwedge, ')$ is a completely distributive DeMorgan algebra, i.e., a complete and completely distributive lattice with an order-reversing involution $(\ )'$, and with the smallest element $\bot$ and the largest element $\top$ ($\bot \neq \top$). Obviously, for every nonempty set $X$, $L^X$, the family of all $L$-sets, i.e., all mappings from $X$ to $L$, is also a complete and completely distributive lattice under the pointwise order. we denote the smallest element and the largest element of $L^X$ by $\bot_X$ and $\top_X$, respectively. For any $A \in L^X$, the set $\{x : A(x) \neq \bot\}$ is called the support of $A$ and denoted by $suppA$, i.e., $suppA = \{x : A(x) \neq \bot\}$

An $L$-topological space, briefly $L$-ts, is a pair $(L^X, \delta)$, where $\delta$, called an $L$-topology on $L^X$, a subfamily of $L^X$ closed under the operation of finite intersections and arbitrary unions, and $\delta' = \{A' : A \in \delta\}$; the member of $\delta$ (resp. $\delta'$) is called open (resp., closed) $L$-sets, and for each $B \in L^X$, the $L$-set $B^\circ = \bigvee\{U \in \delta : U \leq B\}$ (resp. $B^- = \bigwedge\{C \in \delta' : B \leq C\}$) is called the interior (resp., closure) of $B$. An element $\lambda \in L$ is called a molecule if $\lambda \neq \bot$ and $\lambda \leq a \vee b$ implies $\lambda \leq a$ or $\lambda \leq b$. The set of all molecules of $L$ (resp., $L^X$) will be denoted by $M(L)$ (resp., $M(L^X)$); obviously, $M(L^X) = \{x_\lambda : x \in X, \lambda \in M(L)\}$. For any $x_\lambda \in M(L^X)$, a closed $L$-set $P \in \delta'$ is called a closed remote neighborhood of $x_\lambda$ if $x_\lambda \not\leq P$. The set of all closed remote neighborhood of $x_\lambda$ is denoted by $\eta^-(x_\lambda)$. For any $A \in L^X$, a closed $L$-set $P \in \delta'$ is called a closed remote neighborhood of $A$ if for any $x \in suppA$ such that $A(x) \not\leq P(x)$. The set of all closed remote neighborhood of $A$ is denoted by $\eta^-(A)$. For any $A \in L^X$, $A$ is a called pseudo-crisp closed set if $\exists\ a \in L - \{\perp\}$ such that $A(x) > \perp$ if and only if $\forall x \in X, A(x) \geq a$.

Let $f : X \longrightarrow Y$ be an ordinary mapping. Based on $f : X \longrightarrow Y$ define an mapping $f^{\rightarrow} : L^X \longrightarrow L^Y$ which is called a function of Zadeh's type and its right adjoint mapping $f^{\leftarrow} : L^Y \longrightarrow L^X$ by

$$\forall A \in L^X,\ \forall y \in Y,\quad f^{\rightarrow}(A)(y) = \bigvee\{A(x) : x \in X, f(x) = y\},\ \text{and}$$

$$\forall B \in L^Y,\ \forall x \in X,\quad f^{\leftarrow}(B)(x) = B(f(x)),\ \text{respectively.}$$

For other undefined notions and symbols in this paper, please refer to Wang [16].

**Definition 1.1** (Liu [9]). An $L$-ts $(L^X, \delta)$ is called a sub-$T_0$ space if for any $x, y \in X$ with $x \neq y$, there exists $\lambda \in M(L)$, either there is $P \in \eta^-(x_\lambda)$ such that $y_\lambda \leq P$ or there is $Q \in \eta^-(y_\lambda)$ such that $x_\lambda \leq Q$.

**Definition 1.2** (Chen and Meng [2]). An $L$-ts $(L^X, \delta)$ is called a $T_{2\frac{1}{2}}$ or $L$-Urysohn space if for any $x_\lambda, y_\mu \in M(L^X)$ with $x \neq y$, there exist $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\mu)$ such that $P^\circ \vee Q^\circ = \top_X$.

**Definition 1.3** (Wang [16]). Let $(L^X, \delta)$ be an $L$-ts. Then,

(1) $(L^X, \delta)$ is said to be $T_1$ if for any $x_\lambda, y_\mu \in M(L^X)$ with $x_\lambda \not\leq y_\mu$, there exists $P \in \eta^-(x_\lambda)$ such that $y_\mu \leq P$.

(2) $(L^X, \delta)$ is said to be $T_2$ (or Hausdorff) if for any $x_\lambda, y_\mu \in M(L^X)$ with $x_\lambda \not\leq y_\mu$, there exist $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\mu)$ such that $P \vee Q = \top_X$.

(3) $(L^X, \delta)$ is said to be regular if for each $x_\lambda \in M(L^X)$ and each nonempty pseudo-crisp closed set $A$ with $x \notin suppA$, there exist $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(A)$ such that $P \vee Q = \top_X$. $(L^X, \delta)$ is said to be $T_3$ if it is regular and $T_1$.

(4) $(L^X, \delta)$ is said to be normal if for each pair of nonempty pseudo-crisp closed set $A$ and $B$ with $suppA \cap suppB = \emptyset$, there exist $P \in \eta^-(A)$ and $Q \in \eta^-(B)$ such that $P \vee Q = \top_X$. $(L^X, \delta)$ is said to be $T_4$ if it is normal and $T_1$.

**Theorem 1.4** (Wang [16], You [19]). *$T_i (i = 1, 2, 2\frac{1}{2}, 3, 4)$ is $L$-good extension in Lowen's sense.*

**Definition 1.5** (Gu and Zhao [4]). An $L$-ts $(L^X, \delta)$ is said to be layer $T_0$ if for any $\alpha \in M(L), (X, (\tau_\alpha(\delta'))')$ is $T_0$, where $\tau_\alpha(\delta) = \{\tau_\alpha(A) : A \in$

$\delta'\}, \tau_\alpha(A) = \{x \in X : A(x) \geq \alpha\}$. In the same way, layer $T_i (i = 1, 2, 3, 4)$ and layer regular (completely regular, normal) are defined.

**Definition 1.6** (Kubiak [7]). An $L$-ts $(L^X, \delta)$ is said to be

(i) Kubiak-$T_1$ (or $L$-$T_1$) if for all $x, y \in X$ with $x \neq y$, there exist $U, V \in \delta$ such that $U(x) \nleq U(y)$ and $V(y) \nleq V(x)$.

(ii) Kubiak-$T_2$ if for all $x, y \in X$ with $x \neq y$, there exist $U, V \in \delta$ such that $U(x) \nleq U(y)$ , $V(y) \nleq V(x)$ and $U \leq V'$.

**Lemma 1.7** (Liu and Luo [10]). Let $(L^X, \delta)$ be an $L$-ts, where $\delta$ is generated by a classical topology, then for any $A \in L^X$ such that $A^\circ = \bigvee\{\alpha\chi_{[\tau_\alpha(A)]^\circ} : \alpha \in M(L)\}$.

**Definition 1.8** (Shi [14]). An $L$-ts $(L^X, \delta)$ is called $L$-$T_2$ if for all $x, y \in X$ with $x \neq y$, there exist $P \in \delta'$ and $Q \in \delta$ such that $Q \leq P$ and $Q(x) \nleq P(y)$.

## 2. Definitions and characterizations

In this section, we introduce the concept of sub-$T_1$, sub-$T_2$, sub-$T_{2\frac{1}{2}}$, sub-$T_3$ and sub-$T_4$ separation axioms in $L$- topological spaces and establish the characteristic theorems of these sub-separation axioms. First, some definitions are given as follows:

**Definition 2.1.** Suppose that $(L^X, \delta)$ is an $L$-ts. Then,

(1) $(L^X, \delta)$ is said to be sub-$T_1$ if for any $x, y \in X$ with $x \neq y$, there exists $\lambda \in M(L)$, both there is $P \in \eta^-(x_\lambda)$ such that $y_\lambda \leq P$ and there is $Q \in \eta^-(y_\lambda)$ such that $x_\lambda \leq Q$ .

(2) $(L^X, \delta)$ is said to be sub-$T_2$ if for any $x, y \in X$ with $x \neq y$, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$.

(3) $(L^X, \delta)$ is said to be sub-$T_{2\frac{1}{2}}$ if for any $x, y \in X$ with $x \neq y$, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P^\circ \vee Q^\circ = \top_X$.

(4) $(L^X, \delta)$ is said to be sub-regular if for each $x \in X$ and each nonempty pseudo-crisp closed set $A$ with $x \notin suppA$, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(\lambda A)$ such that $P \vee Q = \top_X$. $(L^X, \delta)$ is said to be sub-$T_3$ if it is sub-regular and sub-$T_1$.

(5) $(L^X, \delta)$ is said to be sub-normal if for each pair of nonempty pseudo-

crisp closed set $A$ and $B$ with $suppA \cap suppB = \emptyset$, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(\lambda A)$ and $Q \in \eta^-(\lambda B)$ such that $P \vee Q = \top_X$, where $\lambda A(x) = \lambda \wedge A(x)$ for any $x \in X$. $(L^X, \delta)$ is said to be sub-$T_4$ if it is sub-normal and sub-$T_1$.

By Definition 2.1, we have:

**Corollary 2.2**. The following implications hold: $T_i \Longrightarrow$ sub-$T_i$, where $i$= 1, 2, $2\frac{1}{2}$,3,4.

Now we introduce the convergence of molecular nets. Let $(L^X, \delta)$ be an $L$-ts, $S = \{S(n) : n \in D\}$ a molecular net and $e \in M(L^X)$ , $e$ is said to be a limit point of $S$, (or $S$ converges to $e$ ; in symbols, $S \to e$), if for $\forall P \in \eta^-(e)$, $S(n) \not\leq P$ is eventually true, that is there exists $m \in D$ such that $S(n) \not\leq P$ for all $n \in D$ with $n \geq m$. The following results show that the convergence of molecular nets is unique under a certain condition for the sub-$T_2$ space.

**Theorem 2.3**. Let $(L^X, \delta)$ be a sub-$T_2$ space, then for each molecular net $S$ such that $|K_S| \leq 1$, where $K_S = \{x \in X : \lim S(x) = \top\}$.

**Proof**. Let $(L^X, \delta)$ be a sub-$T_2$ space and $S = \{S(n) : n \in D\}$ be a molecular net. Assume that $|K_S| \geq 2$, for any $x, y \in K_S$ with $x \neq y$, since $(L^X, \delta)$ is sub-$T_2$, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. Then we have $S \to x_\lambda$ and $S \to y_\lambda$ from $x_\lambda \leq \lim S$ and $y_\lambda \leq \lim S$ by Theorem 2.3.4 (Wang [16]). So there exists an $n_1 \in D$ such that $S(n) \not\leq P$ for all $n \in D$ with $n \geq n_1$ and there exists an $n_2 \in D$ such that $S(n) \not\leq Q$ for all $n \in D$ with $n \geq n_2$. Taking $n_3 \in D$ such that $n_3 \geq n_1$ and $n_3 \geq n_2$, hence we have $S(n) \not\leq P \vee Q$ when $n \geq n_3$.This implies that we must have $P \vee Q \neq \top_X$. This is a contradiction. □

If $\top$ is a molecule, the inverse of Theorem 2.3 is also true.

**Theorem 2.4**. $(L^X, \delta)$ is an $L$-ts, if for each molecular net $S$ with $|K_S| \leq 1$, where $K_S = \{x \in X : \lim S(x) = \top\}$, then $(L^X, \delta)$ is a sub-$T_2$ space.

**Proof**. Suppose that $(L^X, \delta)$ is not a sub-$T_2$ space, then there exist $x, y \in X$ satisfying $x \neq y$, $\forall \lambda \in M(L), \forall P \in \eta^-(x_\lambda)$ and $\forall Q \in \eta^-(y_\lambda)$ such that $P \vee Q \neq \top_X$. Let $D(\lambda) = \eta^-(x_\lambda) \times \eta^-(y_\lambda)$ and $D(\lambda)$ be a directed set by product order. For each $m=(P,Q)\in D(\lambda)$, we can take a molecule $S^\lambda(m)$ such that $S^\lambda(m)\not\leq P\vee Q$. Let $S^\lambda=\{S^\lambda(m): m\in D(\lambda)\}$, hence it is easy to prove $S^\lambda\to x_\lambda$ and $S^\lambda\to y_\lambda$ . Therefore, $\lim S^\lambda\geq x_\lambda\vee y_\lambda$. Since $\top$ is a molecule, the standard minimal set $\beta^*(\top)$ is a directed set(Wang [16]). We denote $\beta^*(\top)$ by $E$, i.e., $E=\beta^*(\top)$. Noticing that $\{x_\lambda\}_{\lambda\in E}$, $\{y_\lambda\}_{\lambda\in E}$ are molecular nets and $\{x_\lambda\}_{\lambda\in E}\to x_\top$, $\{y_\lambda\}_{\lambda\in E}\to y_\top$, we can make a molecular net $\bar{S}: E\times\prod\limits_{\lambda\in E}D(\lambda)\longrightarrow M(L^X)$ such that

$$\bar{S}(\lambda,f)=S^\lambda(f(\lambda))\ ,\ \ \forall(\lambda,f)\in E\times\prod_{\lambda\in E}D(\lambda).$$

Then, $\bar{S}\to x_\top$, $\bar{S}\to y_\top$. In fact, for every $P\in\eta^-(x_\top)$, since $\{x_\lambda\}_{\lambda\in E}\to x_\top$, there exists a $\lambda_0\in E$ such that $x_\lambda\not\leq P$ for all $\lambda\in E$ with $\lambda\geq\lambda_0$. It follows from $S^\lambda\to x_\lambda$ for $\lambda\in E$ that there exists $m_\lambda\in D(\lambda)$ s.t. $S^\lambda(m)\not\leq P$ for all $m\in D(\lambda)$ with $m\geq m_\lambda$. We define $f_0\in\prod\limits_{\lambda\in E}D(\lambda)$ as follows:

$$f_0(\lambda)=\begin{cases} m_\lambda, & \lambda\leq\lambda_0.\\ \text{any fixed element in } D(\lambda), & \lambda\not\leq\lambda_0.\end{cases}$$

Then we can prove for every pair $(\lambda,f)\in E\times\prod\limits_{\lambda\in E}D(\lambda)$ with $(\lambda,f)\geq(\lambda_0,f_0)$ such that $\bar{S}(\lambda,f)\not\leq P$, i.e., $\bar{S}$ is not in any closed remote neighborhood $P$ of $x_\top$ eventually. So we have $\bar{S}\to x_\top$ . Similarly, we can prove $\bar{S}\to y_\top$. Therefore, $|K_{\bar{S}}|\geq 2$. This contradicts to $|K_{\bar{S}}|\leq 1$. Thus, we conclude that $(L^X,\delta)$ is a sub-$T_2$ space. □

With Theorem 2.3 and Theorem 2.4, we have:

**Corollary 2.5**. Let $\top$ be a molecule, then $(L^X,\delta)$ is a sub-$T_2$ space iff for each molecular net $S$ such that $|K_S|\leq 1$, where $K_S=\{x\in X:\lim S(x)=\top\}$.

For the sub-$T_2$ space, we have the following theorem:

**Theorem 2.6**. Let $(L^X,\delta)$ be a sub-$T_2$ space and $\top$ be a molecule, then super F-compactness, N-compactness, strongly F-compactness and F-compactness are equivalent.

**Proof**. The proof is similar to that of Wang's Theorem 6.4.29 in [16].

**Theorem 2.7**. Suppose that $(L^X, \delta)$ is a weakly induced $L$-ts. If it is a sub-$T_2$ space, then $(X, [\delta])$ is a $T_2$ space.

**Proof**. Let $(L^X, \delta)$ be a sub-$T_2$ space and $x, y \in X$ with $x \neq y$. Then there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. We put

$$U = \{t \in X : P^{'}(t) \not\leq \lambda^{'}\} = \{t \in X : P(t) \not\geq \lambda\}, \text{ and}$$

$$V = \{t \in X : Q^{'}(t) \not\leq \lambda^{'}\} = \{t \in X : Q(t) \not\geq \lambda\}.$$

Then it is easy to know that $\chi_U, \chi_V \in \delta$ i.e., $U, V \in [\delta]$. Obviously $x \in U, y \in V$. Thus it remains only to show that $U \bigcap V = \emptyset$. In fact, if there were a $z \in U \bigcap V$, then we have $\lambda \not\leq P(z)$ and $\lambda \not\leq Q(z)$. Hence, $\lambda \not\leq (P \vee Q)(z)$, which contradicts to $P \vee Q = \top_X$. Therefore, $(X, [\delta])$ is a $T_2$ space.

## 3. Properties

In this section,we will investigate some nice properties of sub-separation axioms. At first, we show that sub-separation axioms are good extensions in the sense of Lowen.

**Theorem 3.1**. Let $(X, \mathcal{T})$ be a crisp topological space. Then $(L^X, \omega_L(\mathcal{T}))$ is a sub-$T_i$ space iff $(X, \mathcal{T})$ is a $T_i$ space, where $i$= 1, 2, $2\frac{1}{2}$, 3, 4.

**Proof**. Sufficiency. Let $(X, \mathcal{T})$ be a $T_i$ space ($i$=1, 2, $2\frac{1}{2}$, 3, 4). Then $(L^X, \omega_L(\mathcal{T}))$ is a $T_i$ space by Theorem 1.4. Thus from Corollary 2.2, we know that $(L^X, \omega_L(\mathcal{T}))$ is a sub-$T_i$ space.

Necessity. Case $i$=1: Let $(L^X, \omega_L(\mathcal{T}))$ be a sub-$T_1$ space. For any $x \in X$ and taking $y \in X$ with $x \neq y$, since $(L^X, \omega_L(\mathcal{T}))$ is a sub-$T_1$ space, there exists $\lambda \in M(L)$, both there is $P \in \eta^-(y_\lambda)$ such that $x_\lambda \leq P$ and there is $Q \in \eta^-(x_\lambda)$ such that $y_\lambda \leq Q$. We put

$$U = \{z \in X : P^{'}(z) \not\leq \lambda^{'}\}.$$

It is clear that $U \in \mathcal{T}, x \notin U$ and $y \in U$. Hence $y \notin \{x\}^-$, where $\{x\}^-$ is the closure of $\{x\}$. Therefore, $(X, \mathcal{T})$ is a $T_1$ space.

Case $i=2$: For any two distinct points $x, y \in X$ with $x \neq y$, since $(L^X, \omega_L(\mathcal{T}))$ is a sub-$T_2$ space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. We put

$$U = \{z \in X : P'(z) \not\leq \lambda'\}, \quad V = \{z \in X : Q'(z) \not\leq \lambda'\}.$$

Noticing that $P', Q' \in \omega_L(\mathcal{T})$ and $x_\lambda \not\leq P$ , $y_\lambda \not\leq Q$, hence $U, V \in \mathcal{T}$ and $x \in U$, $y \in V$. Thus it remains only to show that $U \cap V = \emptyset$. In fact, if there were a $z \in U \cap V \neq \emptyset$, then we have $\lambda \not\leq P(z)$ and $\lambda \not\leq Q(z)$. Hence $\lambda \not\leq (P \vee Q)(z) = \top$, which contradicts to $\lambda \leq \top$. Therefore, $(X, \mathcal{T})$ is a $T_2$ space.

Case $i=2\frac{1}{2}$: For any $x, y \in X$ with $x \neq y$, since $(L^X, \omega_L(\mathcal{T}))$ is a sub-$T_2\frac{1}{2}$ space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P^\circ \vee Q^\circ = \top_X$. Clearly, $P \vee Q = \top_X$. From $\lambda \not\leq P(x)$ and $\lambda \not\leq Q(y)$, we know that there exist $\lambda_P \in \beta^*(\lambda)$ and $\lambda_Q \in \beta^*(\lambda)$ such that $\lambda_P \not\leq P(x)$ and $\lambda_Q \not\leq Q(y)$. Since $\lambda$ is a molecule, the standard minimal set $\beta^*(\lambda)$ is a directed set (Wang [16]). Taking $\gamma \in \beta^*(\lambda)$ such that $\gamma \geq \lambda_P \vee \lambda_Q$. We put

$$E = \tau_\gamma(P) = \{z : P(z) \geq \gamma\}, \ F = \tau_\gamma(Q) = \{z : Q(z) \geq \gamma\}.$$

It is clear that $E, F \in \mathcal{T}'$, $x \notin E$, $y \notin F$ and $E \cup F = X$. In order to prove that $(X, \mathcal{T})$ is a $T_2\frac{1}{2}$ space, we need only to verify $E^\circ \cup F^\circ = X$ by the definition of $T_2\frac{1}{2}$. For this purpose, we firstly prove that $\tau_\lambda(P^\circ) \subseteq [\tau_\gamma(P)]^\circ$ and $\tau_\lambda(Q^\circ) \subseteq [\tau_\gamma(Q)]^\circ$. In fact, taking $z \in \tau_\lambda(P^\circ)$, from Lemma 1.7, we have

$$\bigvee_{\alpha \in M(L)} \alpha \chi_{[\tau_\alpha(P)]^\circ}(z) = P^\circ(z) \geq \lambda \ .$$

Therefore, there exists $\alpha \in M(L)$ such that $z \in [\tau_\alpha(P)]^\circ$ and $\alpha \geq \gamma$. i.e., $z \in [\tau_\alpha(P)]^\circ \subseteq [\tau_\gamma(P)]^\circ$. Hence, we obtain that $\tau_\lambda(P^\circ) \subseteq [\tau_\gamma(P)]^\circ$ from the arbitrariness of $z$. Similarly, we can get $\tau_\lambda(Q^\circ) \subseteq [\tau_\gamma(Q)]^\circ$, as desired. Naturally, we have

$$E^\circ \cup F^\circ = [\tau_\gamma(P)]^\circ \cup \ [\tau_\gamma(Q)]^\circ \supseteq \tau_\lambda(P^\circ) \cup \tau_\lambda(Q^\circ) = \tau_\lambda(P^\circ \vee Q^\circ) = X,$$

i.e., $(X, \mathcal{T})$ is a $T_{2\frac{1}{2}}$ space.

Case $i=3$: Since sub-$T_1$ separation axiom is an $L$-good extension, we prove this theorem only for the sub-regular case.

For any $x \in X$, suppose that $E \in \mathcal{T}'$ with $x \notin E$. Clearly, $\chi_E$ is a nonempty pseudo-crisp closed set in $(L^X, \omega_L(\mathcal{T}))$ and $x \notin supp(\chi_E)$. Since

$(L^X, \omega_L(\mathcal{T})$ is a sub-regular space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(\lambda\chi_E)$ such that $P \vee Q = \top_X$.

Let

$$U = \{z : P^{'}(z) \not\leq \lambda^{'}\},\ V = \{z : Q^{'}(z) \not\leq \lambda^{'}\}.$$

It is easy to verify that $x \in U$, $E \subseteq V$ and $U \cap V = \emptyset$ $(U, V \in \mathcal{T})$. Therefore, $(X, \mathcal{T})$ is a regular space.

Case $i$= 4: the proof is similar to that of the case $i$= 3. □

Now we consider the heredity of the sub-separation. The following results show that sub-$T_i$ ($i$= 1, 2, $2\frac{1}{2}$) separation axioms are hereditary. Firstly, the concept of the extension is introduced. Let $Y \subseteq X, A \in L^Y$. $A^* \in L^X$ is defined as follows: $\forall x \in X$,

$$A^*(x) = \begin{cases} A(x), & x \in Y. \\ 0, & x \notin Y. \end{cases}$$

Then $A^*$ is called the extension of $A$.

**Theorem 3.2**. Let $(L^X, \delta)$ be $L$-ts and $Y$ be a nonempty crisp subset of X. If $(L^X, \delta)$ is a sub-$T_i$ space, then the subspace $(L^Y, \delta|Y)$ is also a sub-$T_i$ space, where $\delta|Y = \{G|Y : G \in \delta\}$, $i$= 1, 2, $2\frac{1}{2}$.

**Proof.** We only prove the case $i$=2 and $i$=$2\frac{1}{2}$.

Case $i$= 2: Let $x, y \in Y$ with $x \neq y$. Since $(L^X, \delta)$ is a sub-$T_2$ space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda{}^*)$ and $Q \in \eta^-(y_\lambda{}^*)$ such that $P \vee Q = \top_X$, where $x^*_\lambda$, $y^*_\lambda$ are the extensions of $x_\lambda$, $y_\lambda$, respectively. Notice that $P \in \eta^-(x_\lambda{}^*)$ implies that $P|Y \in \eta^-(x_\lambda)$. Similarly, $Q|Y \in \eta^-(y_\lambda)$. Therefore, there exists $\lambda \in M(L)$ and there are $P|Y \in \eta^-(x_\lambda)$ and $Q|Y \in \eta^-(y_\lambda)$ such that $(P|Y) \vee (Q|Y) = \top_Y$, i.e., $(L^Y, \delta|Y)$ is also a sub-$T_2$ space.

Case $i$= $2\frac{1}{2}$: Suppose that $x, y \in Y$ with $x \neq y$. Since $(L^X, \delta)$ is a sub-$T_{2\frac{1}{2}}$ space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x^*_\lambda)$ and $Q \in \eta^-(y^*_\lambda)$ such that $P^\circ \vee Q^\circ = \top_X$, where $x^*_\lambda$, $y^*_\lambda$ is the extensions of $x_\lambda$, $y_\lambda$, respectively. Noticing that $P \in \eta^-(x^*_\lambda)$ implies that $P|Y \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y^*_\lambda)$ implies that $Q|Y \in \eta^-(y_\lambda)$, then we have that there exists $\lambda \in M(L)$ and there are $P|Y \in \eta^-(x_\lambda)$ and $Q|Y \in \eta^-(y_\lambda)$. Hence we need only to show that $(P|Y)^\circ \vee (Q|Y)^\circ = \top_Y$. In fact, from $(P|Y)^\circ \geq (P^\circ|Y)$, we get that

$$(P|Y)^\circ \vee (Q|Y)^\circ \geq (P^\circ|Y) \vee (Q^\circ|Y) = \top_Y.$$

Thus $(L^Y, \delta|Y)$ is also a sub-$T_2\frac{1}{2}$ space. □

**Theorem 3.3**. Let $(L^X, \delta)$ be $L$-ts , $Y$ be a nonempty crisp subset of X and $\chi_Y \in \delta'$. If $(L^X, \delta)$ is a sub-$T_i$ space, then the subspace $(L^Y, \delta|Y)$ is also a sub-$T_i$ space, where $i$= 3, 4.

**Proof.** We only prove this theorem only for the case $i$= 3. Since sub-$T_1$ separation axiom is hereditary, we prove the theorem only for the sub-regular case.

Let $B$ be a nonempty pseudo-crisp closed set in $(L^Y, \delta|Y)$ and $y \in Y$ with $y \notin suppB$. Since $B \in (\delta|Y)' = \delta'|Y$, there exists $A \in \delta'$ such that $B = A|Y$. And we have $B = B^*|Y$, where $B^*$ is the extension of B. It is easy to prove that $B^* = A \wedge \chi_Y$ and $B^*$ is a nonempty pseudo-crisp closed set. By the sub-regularity of $(L^X, \delta)$ and $y \notin suppB^*$, there exists $\lambda \in M(L)$ , and there are $P \in \eta^-(y^*_\lambda)$ and $Q \in \eta^-(\lambda B^*)$ such that $P \vee Q = \top_X$, where $y^*_\lambda$ is the extension of $y_\lambda$. Then we know that $P|Y \in \eta^-(y_\lambda)$ and $Q|Y \in \eta^-(\lambda B)$ such that $(P|Y) \vee (Q|Y) = \top_Y$. This shows $(L^Y, \delta|Y)$ is also a sub-regular space. Therefore, the proof of the theorem is complete.□

In the end of this section, we show that sub-$T_i$ ($i$= 1, 2, $2\frac{1}{2}$) separation axioms are productive. First, a lemma is needed.

**Lemma 3.4**. Let $(L^X, \delta)$, $(L^Y, \mu)$ be two $L$-ts, $f^\rightarrow : (L^X, \delta) \rightarrow (L^Y, \mu)$ be a closed bijection and $f^\rightarrow, f^\leftarrow$ be continuous. If $(L^X, \delta)$ is a sub-$T_i$ space, then so is $(L^Y, \mu)$, where $i$= 1, 2, $2\frac{1}{2}$, 3, 4.

**Proof.** We prove the theorem only for the case $i$=1 and $i$=3.

Case $i$= 1: For any $y, z \in Y$ with $y \neq z$, since $f$ is a closed bijection, there are $u, v \in X$ with $u \neq v$ such that $f(u) = y, f(v) = z$. Since $(L^X, \delta)$ is a sub-$T_1$ space, there exists $\lambda \in M(L)$, both there is $P \in \eta^-(u_\lambda)$ such that $v_\lambda \leq P$ and there is $Q \in \eta^-(v_\lambda)$ such that $u_\lambda \leq Q$ . Therefore,

$$y_\lambda = f^\rightarrow(u_\lambda) \not\leq f^\rightarrow(P),\ z_\lambda = f^\rightarrow(v_\lambda) \leq f^\rightarrow(P),\ \text{and}$$

$$z_\lambda = f^\rightarrow(v_\lambda) \not\leq f^\rightarrow(Q),\ y_\lambda = f^\rightarrow(u_\lambda) \leq f^\rightarrow(Q).$$

From $P, Q \in \delta'$ and $f^\leftarrow$ is continuous, we have $f^\rightarrow(P) \in \eta^-(y_\lambda)$ and $f^\rightarrow(Q) \in \eta^-(z_\lambda)$. Hence, $(L^Y, \mu)$ is a sub-$T_1$ space.

Case $i$= 3: For any $y \in Y$ and nonempty pseudo-crisp closed set $A \in \mu^{'}$ with $y \notin suppA$. Since $f$ is closed bijection, there exist $x \in X, B \in \delta^{'}$ such that $f(x) = y, f(B) = A$ i.e., $x = f^{-1}(y), B = f^{\leftarrow}(A)$. We have $B \in \delta^{'}$ from $f^{\rightarrow}$ is continuous. It is easy to prove that $B$ is a nonzero pseudo-crisp closed set. From $y \notin suppA$, we have:

$$\begin{aligned} y \notin suppA \Rightarrow A(y) = \bot \ &\Rightarrow A(f(x)) = \bot \ \ (f(x) = y) \\ &\Rightarrow f^{\leftarrow}(A)(x) = \bot \text{ (by the definition of } f^{\leftarrow}) \\ &\Rightarrow x \notin suppf^{\leftarrow}(A) = suppB. \end{aligned}$$

For $x \in X$ and $B \in \delta^{'}$ with $x \notin suppB$, since $(L^X, \delta)$ is a sub-$T_3$ space, there exists $\lambda \in M(L)$, and there are $P \in \eta^{-}(x_\lambda)$ and $Q \in \eta^{-}(\lambda B)$ such that $P \vee Q = \top_X$.

Therefore, $y_\lambda = f^{\rightarrow}(x_\lambda) \not\leq f^{\rightarrow}(P)$, $\lambda A = \lambda f^{\rightarrow}(B) \not\leq f^{\rightarrow}(Q)$, $f^{\rightarrow}(P) \vee f^{\rightarrow}(Q) = \top_Y$. From $P \in \delta^{'}$ and $f^{\leftarrow}$ is continuous, we have $f^{\rightarrow}(P) \in \eta^{-}(y_\lambda)$ and $f^{\rightarrow}(Q) \in \eta^{-}(\lambda A)$. Thus, $(L^Y, \mu)$ is a sub-$T_3$ space. □

**Theorem 3.5**. Let $\{(L^{X_t}, \delta_t)\}_{t \in T}$ be a family of $L$-ts and $(L^X, \delta)$ be a product topological space. If for any $t \in T$, $(L^{X_t}, \delta_t)$ is a sub-$T_i$ space , then so is $(L^X, \delta)$. If $(L^X, \delta)$ is a sub-$T_i$ space and $(L^{X_t}, \delta_t)$ is a fully stratified space, then so is $(L^{X_t}, \delta_t)$ , where $i$= 1, 2, $2\frac{1}{2}$.

**Proof.** We only prove the case $i$=2, other cases are obtained in the similar way.

Necessity. Suppose that $\{(L^{X_t}, \delta_t)\}_{t \in T}$ is a family of sub-$T_2$ space. Let $\forall x = \{x^t\}_{t \in T}$, $y = \{y^t\}_{t \in T} \in X$ with $x \neq y$, then there exists a $r \in T$ such that $x^r \neq y^r$. Since $(L^{X_r}, \delta_r)$ is a sub-$T_2$ space, there exists $\lambda \in M(L)$, and there are $B_r \in \eta^{-}(x^r_\lambda)$, $C_r \in \eta^{-}(y^r_\lambda)$ such that $B_r \vee C_r = \top_{X_r}$. Clearly, ${P_r}^{\leftarrow}(B_r), {P_r}^{\leftarrow}(C_r) \in \delta^{'}$, ${P_r}^{\leftarrow}(B_r)(x) = B_r(x^r) \not\geq \lambda$ and ${P_r}^{\leftarrow}(C_r)(y) = C_r(y^r) \not\geq \lambda$. Furthermore, $x_\lambda \not\leq {P_r}^{\leftarrow}(B_r)$, $y_\lambda \not\leq P_r^{\leftarrow}(C_r)$ and ${P_r}^{\leftarrow}(B_r) \vee {P_r}^{\leftarrow}(C_r) = \top_X$. Hence, we prove that $(L^X, \delta)$ is a sub-$T_2$ space.

Sufficiency. Let $(L^X, \delta)$ be a sub-$T_2$ space and $(L^{X_r}, \delta_r)$ be a fully stratified space, where $r \in T$. For any $x = \{x^t\}_{t \in T} \in X$, from Theorem 2.8.9 (Wang [16]), $(L^{\tilde{X}_r}, \delta|\tilde{X}_r)$ which is parallel to $(L^{X_r}, \delta_r)$ through $x$ is homeomorphic to $(L^{X_r}, \delta_r)$. Since $(L^{\tilde{X}_r}, \delta|\tilde{X}_r)$ is a sub-$T_2$ space as a subspace of $(L^X, \delta)$, $(L^{X_r}, \delta_r)$ is a sub-$T_2$ space from Lemma 3.4. □

The next result follows from the above Theorem.

**Corollary 3.6**. Let $\{(L^{X_t}, \omega_L(\mathcal{T}_t))\}_{t\in T}$ be a family of $L$-ts topologically generated by a family of topological spaces $\{(X_t, \mathcal{T}_t)\}_{t\in T}$ and $(L^X, \omega_L(\mathcal{T}))$ be a product $L$-ts of $\{(L^{X_t}, \omega_L(\mathcal{T}_t))\}_{t\in T}$. Then $(L^X, \omega_L(\mathcal{T}))$ is a sub-$T_i$ space iff for $\forall t \in T, (L^{X_t}, \omega_L(\mathcal{T}_t))$ is a sub-$T_i$ space, where $\mathcal{T} = \prod\limits_{t\in T} \mathcal{T}_t$, $i=$ 1, 2, $2\frac{1}{2}$.

## 4. The relations with respect to other separation axioms.

In this section, we make a comparison between separation axioms defined in this paper and those presented by Chen and Meng [2], Fang and Ren [3], Gu and Zhao [4], Ganguly and Saha [5], Kubiak [7], Kandil and El-Shafee [8], Shi [14],Shi and Chen [15] and Wang [16], and offer a lot of examples to show the relations between them. At first, we show that the sub-separation axioms defined in this paper are harmonious.

From Definition 2.1, the following theorem is obvious .

**Theorem 4.1**. Let $(L^X, \delta)$ be an $L$-ts. Then the following implications hold:

(1) sub-$T_1 \Rightarrow$ sub-$T_0$

(2) sub-$T_4 \Rightarrow$ sub-$T_3 \Rightarrow$ sub-$T_2$

**Theorem 4.2**. Let $(L^X, \delta)$ be an $L$-ts. Then sub-$T_2 \Rightarrow$ sub-$T_1$.

**Proof.** Suppose that $(L^X, \delta)$ is a sub-$T_2$ space. For any $x, y \in X$ with $x \neq y$, since $(L^X, \delta)$ is a sub-$T_2$ space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. From $y_\lambda \leq P \vee Q$ and $y_\lambda \not\leq Q$, we have $y_\lambda \leq P$. Similarly, on account of $x_\lambda \leq P \vee Q$ and $x_\lambda \not\leq P$, then $x_\lambda \leq Q$. Hence, for any $x \in X$ with $x \neq y$, there exists $\lambda \in M(L)$, both there is $P \in \eta^-(x_\lambda)$ such that $y_\lambda \leq P$ and there is $Q \in \eta^-(y_\lambda)$ such that $x_\lambda \leq Q$, i.e., $(L^X, \delta)$ is a sub-$T_1$ space. □

From Theorem 4.1 and 4.2, we obtain the following result that shows the sub-separation axioms are harmonious.

**Corollary 4.3**. sub-$T_4 \Rightarrow$ sub-$T_3 \Rightarrow$ sub-$T_2 \Rightarrow$ sub-$T_1 \Rightarrow$ sub-$T_0$.

**Theorem 4.4**. Let $(L^X, \delta)$ be an $L$-ts. Then sub-$T_2\frac{1}{2} \Rightarrow$ sub-$T_2$.

**Proof.** Let $(L^X, \delta)$ be a sub-$T_2\frac{1}{2}$ space. For any $x, y \in X$ with $x \neq y$, since $(L^X, \delta)$ is a sub-$T_2\frac{1}{2}$ space, there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $P^\circ \vee Q^\circ = \top_X$. Noticing that $P^\circ \vee Q^\circ \leq P \vee Q$, we get that $P \vee Q = \top_X$. Therefore, $(L^X, \delta)$ is a sub-$T_2$ space. □

The following example shows that the $L$-unit interval $I(L)$ need not satisfy the sub-$T_1$ axiom.

**Example 4.5**. The [0,1]-real line $[0,1](I)$ does not satisfy sub-$T_1$ axiom. In fact, take $x, y \in [0,1](I)$ satisfying $\forall t \in R$

$$x(t) = \begin{cases} 1, & t \in (-\infty, 0), \\ 0.5, & t \in [0,1], \\ 0, & t \in (1, +\infty), \end{cases} \qquad y(t) = \begin{cases} 1, & t \in (-\infty, 0), \\ 0.6, & t \in [0,1], \\ 0, & t \in (1, +\infty). \end{cases}$$

For convenience, we only consider $P$ or $Q$ which has the forms of $R_s^{'} \vee L_t^{'}$.
Case I: When $0 < \lambda \leq 0.4$, we have that

$$\varepsilon(x_\lambda) = \vee\{t \mid x_\lambda \leq L_t^{'}\} = \vee\{t \mid \lambda \leq x(t-)\} = 1;$$

$$\sigma(x_\lambda) = \wedge\{s \mid x_\lambda \leq R_s^{'}\} = \wedge\{s \mid \lambda \leq x(s+)^{'}\} = 0.$$

Hence,

$$x_\lambda \not\leq L_t^{'} \vee R_s^{'} \Rightarrow t > 1, s < 0.$$

Then we get that

$$(L_t^{'} \vee R_s^{'})(y) = 0 \ \text{ from } \ t > 1, s < 0.$$

Naturally, we obtain that $y_\lambda \not\leq L_t^{'} \vee R_s^{'}$.
Case II: When $0.4 < \lambda \leq 0.5$, we have that

$$\varepsilon(x_\lambda) = \vee\{t \mid x_\lambda \leq L_t^{'}\} = 1; \quad \sigma(x_\lambda) = \wedge\{s \mid x_\lambda \leq R_s^{'}\} = 0.$$

Hence,

$$x_\lambda \not\leq L_t^{'} \vee R_s^{'} \Rightarrow t > 1, s < 0.$$

Then we get that

$$(L_t^{'} \vee R_s^{'})(y) = 0 \ \text{ from } \ t > 1, s < 0 \Rightarrow y_\lambda \not\leq L_t^{'} \vee R_s^{'}.$$

Case III: When $0.5 < \lambda \leq 0.6$, we have that

$$\varepsilon(y_\lambda) = \vee\{t \mid x_\lambda \leq L_t^{'}\} = 1; \ \ \sigma(y_\lambda) = \wedge\{s \mid y_\lambda \leq R_s^{'}\} = 1.$$

Hence,

$$y_\lambda \not\leq L_t^{'} \vee R_s^{'} \Rightarrow t > 1, s < 1.$$

Then we get that

$$(L_t^{'} \vee R_s^{'})(x) \leq 0 \vee 0.5 = 0.5 \ \text{ from } \ t > 1, s < 1 \Rightarrow x_\lambda \not\leq L_t^{'} \vee R_s^{'}.$$

Case IV: When $0.6 < \lambda \leq 1$, we have that

$$\varepsilon(x_\lambda) = \vee\{t \mid x_\lambda \leq L_t^{'}\} = 0; \ \ \sigma(x_\lambda) = \wedge\{s \mid x_\lambda \leq R_s^{'}\} = 1.$$

Hence,

$$x_\lambda \not\leq L_t^{'} \vee R_s^{'} \Rightarrow t > 0, s < 1.$$

Then we get that

$$(L_t^{'} \vee R_s^{'})(y) \leq 0.5 \vee 0.4 = 0.5 \ \text{ from } \ t > 0, s < 1 \Rightarrow y_\lambda \not\leq L_t^{'} \vee R_s^{'}.$$

From case I, II, III and IV, we have that $[0,1](I)$ does not satisfy the sub-$T_1$ axiom. □

**Remark 4.6**. From the above example , we know that the $L$-unit interval need not satisfy the sub-$T_1$ axiom. So the $L$-unit interval is not compatible with the sub-separation axioms proposed in this paper.

Next we make a comparison between the sub-separation axioms and those presented by Kubiak [7].

**Theorem 4.7**. Let $(L^X, \delta)$ be an $L$-ts. Then sub-$T_1 \Rightarrow$ Kubiak-$T_1$.

**Proof**. Let $(L^X, \delta)$ be sub-$T_1$. In order to prove that $(L^X, \delta)$ is Kubiak-$T_1$, take $x, y \in X$ with $x \neq y$. Then there exists $\lambda \in M(L)$, and there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that $y_\lambda \leq P$, $x_\lambda \leq Q$ , respectively. Taking $U = P^{'}$, $V = Q^{'}$, we have that $U, V \in \delta$ , $U(x) \not\leq U(y)$ and

$V(y) \not\leq V(x)$. Therefore, $(L^X, \delta)$ is Kubiak-$T_1$. □

In general, Kubiak-$T_1$ need not imply our sub-$T_1$. This can be seen from the following example.

**Example 4.8**. Let $L$=$\{\perp, a, b, \top\}$ satisfy $a \vee b = \top$, $a \wedge b = \perp$, $a^{'} = b$ and $X = \{x, y\}$ with $x \neq y$. Take $\delta = \{\perp_X, \top_X, x_a, y_b, x_a \vee y_b\}$, then $\delta^{'} = \{\perp_X, \top_X, M, N, R\}$, where $M, N$ and $R$ are defined as follows:

$$M(x) = b,\ M(y) = \top;\quad N(x) = \top,\ N(y) = a;\quad R(x) = b,\ R(y) = a.$$

We can prove that $(L^X, \delta)$ is not sub-$T_1$ , but it is Kubiak-$T_1$. Now we show that $(L^X, \delta)$ is not sub-$T_1$. We need to show that $\forall \lambda \in M(L)$, $\forall P \in \eta^{-}(x_\lambda)$ such that $y_\lambda \not\leq P$, or $\forall Q \in \eta^{-}(y_\lambda)$ such that $x_\lambda \not\leq Q$. In fact, we have that $P \in \eta^{-}(x_\lambda) = \{\perp_X, M, R\}$, $Q \in \eta^{-}(y_\lambda) = \{\perp_X\}$ when $\lambda = a$. Hence we get that $x_\lambda \not\leq Q$. Similarly, we have that $P \in \eta^{-}(x_\lambda) = \{\perp_X\}$, $Q \in \eta^{-}(y_\lambda) = \{\perp_X, N, R\}$ when $\lambda = b$. Then we obtain that $y_\lambda \not\leq P$. Therefore $(L^X, \delta)$ is not sub-$T_1$. Next we show that $(L^X, \delta)$ is Kubiak-$T_1$. Taking $U = x_a, V = y_b$, we get that $U, V \in \delta$, $U(x) \not\leq U(y)$ and $V(y) \not\leq V(x)$. □

**Theorem 4.9**. Let $(L^X, \delta)$ be an $L$-ts and $\top$ be a molecule. Then sub-$T_2$ $\Rightarrow$ Kubiak-$T_2$.

**Proof**. For any $x, y \in X$ with $x \neq y$, since $(L^X, \delta)$ is a sub-$T_2$ space, there exists $\lambda \in M(L)$ , there are $P \in \eta^{-}(x_\lambda)$ and $Q \in \eta^{-}(y_\lambda)$ such that $P \vee Q = \top_X$. Taking $U = P^{'}, V = Q^{'}$, we have that $U, V \in \delta, U(x) \not\leq U(y)$ and $V(y) \not\leq V(x)$. Now we only need to prove that $U \leq V^{'}$ i.e. $P^{'} \leq Q$. In fact, since $P \vee Q = \top_X$ and $\top$ is a molecule, we have that $P(x) = \top$ or $Q(x) = \top$ for $\forall x \in X$. Easily we get $P^{'} \leq Q$ i.e. $U \leq V^{'}$. Therefore $(L^X, \delta)$ is a Kubiak-$T_2$ space.

**Remark 4.10**. If $\top$ is not a molecule, is Theorem 4.9 true? We leave it as an open problem. Generally, Kubiak-$T_2$ need not imply sub-$T_2$ (seeing Example 4.11 below). □

**Example 4.11**. Let $L$=[0,1] and $X = \{x, y\}$. Take $\delta$=$\{\perp_X, \top_X, A, B, 0.3^*, 0.7^*\}$, where $A, B$ are defined as follows:

$$A(x) = 0.7,\ A(y) = 0.3;\quad B(x) = 0.3,\ B(y) = 0.7$$

Then we get that $(L^X, \delta)$ is a Kubiak-$T_2$ space. In fact, taking $U = A, V = B$, we get that $U(x) \nleq U(y)$ $V(y) \nleq V(x)$ and $U \leq V'$. It is easy to check that $(L^X, \delta)$ is not a sub-$T_2$ space. For $\forall \lambda \in M(L)$, we get that $\eta^-(x_\lambda)$, $\eta^-(y_\lambda) \subseteq \{\perp_X, A, B, 0.3^*, 0.7^*\}$. Therefore for $\forall P \in \eta^-(x_\lambda)$, $\forall Q \in \eta^-(y_\lambda)$, we have $P \vee Q \leq 0.7^*$. Hence $(L^X, \delta)$ is not a sub-$T_2$ space.□

Now, we discuss the relation between the sub-separation axioms and other separation axioms presented by Shi[14], Wang[16], Gu and Zhao[4]. The following two examples show that sub-$T_2$ need not imply $L$-$T_2$ and $L$-$T_2$ also need not imply sub-$T_2$.

**Example 4.12**. Let $L$=[0,1] and $X = \{x, y\}$. Take $\delta = \{\perp_X, \top_X, C_1, C_2, C_1 \vee C_2\}$, where $C_i$ is defined as follows:

$$C_1(x) = 0.5,\ C_1(y) = 0;\quad C_2(x) = 0,\ C_2(y) = 0.5.$$

Easily we get that $(L^X, \delta)$ is a sub-$T_2$ space. In fact, taking $\lambda = \frac{2}{3}$,$P = C_1'$ and $Q = C_2'$, we get that $P \in \eta^-(x_\lambda)$, $Q \in \eta^-(y_\lambda)$ and $P \vee Q = \top_X$. It is easy to check that $(L^X, \delta)$ is not $L$-$T_2$. □

**Example 4.13**. Let $L$, $X$ and $\delta$ be defined as in Example 4.11. From Example 4.11, we know that $(L^X, \delta)$ is not a sub-$T_2$ space. Next we prove that $(L^X, \delta)$ is $L$-$T_2$. Take $Q = A, P = B'$, then $Q \in \delta$, $P \in \delta'$,$Q \leq P$ and $Q(x) \nleq P(y)$. □

**Lemma 4.14** (Wang [16]). If $(L^X, \delta)$ is N-compact and $T_2$, then it is $T_4$. □

Obviously, we have the following result.

**Theorem 4.15**. If $(L^X, \delta)$ is N-compact and $T_2$, then it is sub-$T_i$ , where $i$= 1, 2, $2\frac{1}{2}$, 3, 4. □

**Lemma 4.16** (Gu and Zhao [4]). Let $(L^X, \delta)$ be an $L$-ts. Then,

(1) $(L^X, \delta)$ is Layer $T_0$ iff for any $x_\lambda, y_\lambda \in M(L^X)$ with $x \neq y$, there exists $P \in \delta'$ such that $x_\lambda \nleq P$ and $y_\lambda \leq P$ or $x_\lambda \leq P$ and $y_\lambda \nleq P$ .

(2) $(L^X, \delta)$ is Layer $T_1$ iff for any $x_\lambda, y_\lambda \in M(L^X)$ with $x \neq y$, there exists $P \in \delta'$ such that $x_\lambda \nleq P$ and $y_\lambda \leq P$.

(3) $(L^X, \delta)$ is Layer $T_2$ iff for any $x_\lambda, y_\lambda \in M(L^X)$ with $x \neq y$, there exist $P, Q \in \delta^{'}$ such that $P \in \eta^{-}(x_\lambda)$ ,$Q \in \eta^{-}(y_\lambda)$ and $P \vee Q \geq [\lambda]$. □

By Lemma 4.16, we have the following conclusion.

**Theorem 4.17**. Let $(L^X, \delta)$ be an $L$-ts. If $(L^X, \delta)$ is a layer $T_i$ space, then it is a sub-$T_i$ space, where $i$= 0, 1.□

**Lemma 4.18** (Gu and Zhao [4]). Let $(L^X, \delta)$ be an $L$-ts. If $(L^X, \delta)$ is a $T_i$ space, then it is a layer $T_i$ space, where $i$= 0, 1, 2.□

Therefore, by Theorem 4.17, Lemmas 4.18, 4.16 (3), the following results hold.

**Corollary 4.19**. (1) $T_i \Rightarrow$ layer $T_i \Rightarrow$ sub-$T_i$, where $i$= 0, 1.

(2) layer $T_2 \Rightarrow$ sub-$T_2$ whenever the largest element $\top$ is a molecule. □

In the following, we give an example showing that a sub-$T_2$ space need not be a layer $T_2$ space, to say nothing of being $T_2$.

**Example 4.20**. Let $L$,$X$ and $\delta$ be defined as in Example 4.12. From Example 4.12, we know that $(L^X, \delta)$ is a sub-$T_2$ space. But $(L^X, \delta)$ is not a layer $T_2$ space. In fact, taking $\alpha = \frac{1}{3}$, then we get that $(X, (\tau_\alpha(\delta^{'}))^{'}) = \{\emptyset, X\}$. Clearly, $(X, (\tau_\alpha(\delta^{'}))^{'})$ is not a $T_2$ space. From Definition 1.5, we know that $(L^X, \delta)$ is not a layer $T_2$ space. It is easy to prove that $(L^X, \delta)$ is not a $T_2$ space. □

From all of examples above, we find that, in general, all sub-separation axioms in this paper are weaker than other separation axioms that had appeared in literature. Indeed, there are many $L$-topological spaces which satisfy sub-separation axioms, but doesn't fulfill other separation axioms. This is one of differences between sub-separation axioms and other separation axioms. For examples, there are good work on separation axioms of $L$-topological spaces in [2], [3], [5] and [8]. In the following, we will offer more examples to show that our sub-separation axioms is very different from separation axioms established in these papers. For simplicity, we only consider $T_{2\frac{1}{2}}$ and $T_2$ separation axiom therein. Recall the definition of $WT_2$ in [3] as follows.

**Definition 4.21.** An $L$-ts $(L^X, \delta)$ is called a $WT_2$-space if for any $x_\lambda$, $y_\mu \in M(L^X)$ with $x \neq y$, there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\mu)$ such that $P \vee Q \geq (\lambda \vee \mu)^*$. □

**Remarks 4.22.** (1) For an $L$-ts $(L^X, \delta)$, it is easy to check its $T_{2\frac{1}{2}}$ [2] (seeing Definition 1.2 introduced in the paper) means sub-$T_{2\frac{1}{2}}$, but Example 4.23 below shows that the converse needn't be true.

(2) For an $L$-ts $(L^X, \delta)$, it is easy to check its $WT_2$ implies sub-$T_2$ whenever the largest element $\top$ is a molecule, and Example 4.23 below proves that the converse needn't be true. □

**Example 4.23.** Let $L$ be the completely distributive De Morgan algebra having four elements: $\bot$, $a$, $b$, $\top$ satisfying $a \vee b = \top$, $a \wedge b = \bot$, $a' = b$, and $X = \{x, y\}\,(x \neq y)$. Take $\delta' = \{\bot_X, \top_X, x_a \vee y_b, x_b \vee y_a\}$. We will show the following conclusions.

($i$) The $L$-ts $(L^X, \delta)$ is sub-$T_2$. In fact, now $M(L) = \{a, b\}$. For $x, y \in X$ with $x \neq y$, there is $\lambda = a$ , and also $P = x_b \vee y_a \in \eta^-(x_\lambda)$, $Q = x_a \vee y_b \in \eta^-(y_\lambda)$ such that $P \vee Q = \top_X$. Thus $(L^X, \delta)$ is sub-$T_2$, as desired.

($ii$) The $L$-ts $(L^X, \delta)$ is sub-$T_0$ and also sub-$T_1$. These can be obtained by ($i$) and Theorems 4.1(1), 4.2.

($iii$) The $L$-ts $(L^X, \delta)$ isn't $WT_2$ in the sense of [3]. Letting $x_a$, $y_b \in M(L^X)$, it is easy to check $\eta^-(x_a)=\{\bot_X, x_b \vee y_a\}$ and $\eta^-(y_b)=\{\bot_X, x_b \vee y_a\}$. Hence for any $P \in \eta^-(x_a)$ and $Q \in \eta^-(y_b)$, it cannot be true that

$$P \vee Q \geq (a \vee b)^* = \top_X,$$

where $(a \vee b)^*$ is the constant $L$-set with its value $(a \vee b)$ in the sense of [3]. In fact, for any $P \in \eta^-(x_a)$ and $Q \in \eta^-(y_b)$, $P \vee Q \leq x_b \vee y_a$, but $x_b \vee y_a \leq \top_X$ and $x_b \vee y_a \neq \top_X$. By Definition 4.21 of $WT_2$, the $L$-ts $(L^X, \delta)$ isn't $WT_2$.

($iv$) The $L$-ts $(L^X, \delta)$ is sub-$T_{2\frac{1}{2}}$. Let $P = x_b \vee y_a$ and $Q = x_a \vee y_b$. It is observed that both $P$ and $Q$ are open since $P' = Q$ and $Q' = P$. For the unique pair of $x$ and $y$ with $x \neq y$, putting $\lambda = a \in M(L)$, there are $P \in \eta^-(x_\lambda)$ and $Q \in \eta^-(y_\lambda)$ such that

$$P^\circ \vee Q^\circ = P \vee Q = \top_X.$$

Thus , $(L^X, \delta)$ is sub-$T_{2\frac{1}{2}}$ by Definition 2.1 (3).

($v$) The $(L^X, \delta)$ isn't $T_{2\frac{1}{2}}$ in the sense of [2] (see Definition 1.2 also).

Indeed, $\eta^-(x_a) = \{x_b \vee y_a\}$ and $\eta^-(y_b) = \{x_b \vee y_a\}$. We find that there are two points $x_a, y_b \in M(L^X)$ with $x \neq y$ such that for any $P \in \eta^-(x_a) = \{x_b \vee y_a\}$ and $Q \in \eta^-(y_b) = \{x_b \vee y_a\}$ (it must be $P = x_b \vee y_a$ and $Q = x_b \vee y_a$)

$$P^\circ \vee Q^\circ = P \vee Q = x_b \vee y_a \neq \top_X.$$

It implies that $(L^X, \delta)$ isn't $T_{2\frac{1}{2}}$ by Definition 1.2, as desired. □

In [15],Shi and Chen redefined Urysohn in L-topology, where it was called Shi-Urysohn. Now we consider the relation between the Shi-Urysohn axiom and our sub-$T_{2\frac{1}{2}}$ axiom. Recall the definition of Shi-Urysohn axiom in [15] as follows.

**Definition 4.24.** (Shi and Chen [15]).

An $L$-ts $(L^X, \delta)$ is said to be Urysohn if any $x_\lambda, y_\mu \in M(L^X)$ with $x_\lambda \not\leq y_\mu$, there exist $P \in \eta^-(x_\lambda)$ and $Q \in \aleph^\circ(y_\mu)$ such that$P^\circ \geq Q^-$, where $\aleph^\circ(y_\mu) = \{V : y_\mu \leq V,\ V \in \delta\}$. □

The following example shows that our sub-$T_{2\frac{1}{2}}$ need not imply Shi-Urysohn.

**Example 4.25.** Let $X = L = [0, 1]$, and $\delta = \{\chi_E : E \subset X\}$, where $\chi_E$ is the characteristic function of E. Then $\delta$ is a [0,1]-topology on $X$. It is easy to check that $(L^X, \delta)$ is $T_{2\frac{1}{2}}$(Urysohn), then it is sub-$T_{2\frac{1}{2}}$ . But it is not Shi-Urysohn . In fact, for any $x \in X$ and any $P \in \eta^-(x_\top)$, it follows that $P^\circ(x) = \bot$. But there is no $Q \in \aleph^\circ(x_{0.5})$ such that $P^\circ \geq Q^-$. □

**Remark 4.26.** Does the Shi-Urysohn axiom imply our sub-$T_{2\frac{1}{2}}$? We can't solve it. So we leave it as an open problem. □

To discuss the relation between sub-$T_2$ separation axiom and other $T_2$ separation axiom introduced in [5] and [8]. Note that we consider the case of $L = I = [0, 1]$, the unit interval, so that the conclusions is available for the membership valued lattice using in the published papers [6] and [9]. We introduce some definitions for the convenience of readers.

**Definition 4.27.** (Liu and Luo[11]). Let $x_\lambda \in M(I^X)$ and $A, B \in I^X$. We say $x_\lambda$ quasi-coincides with $A$, or say $x_\lambda$ is quasi-coincident with $A$, denoted by $x_\lambda q A$, if $A(x) + \lambda > \top$; say $A$ quasi-coincides with $B$ at $x \in X$, or say $A$ is quasi-coincident with $B$ at $x$, $AqB$ at $x$ for short, if $A(x) + B(x) > \top$; say $A$ quasi-coincides with $B$, or say $A$ is quasi-coincident with $B$, denoted by $AqB$, if $A$ quasi-coincides with $B$ at some point $x \in X$. Relation "does not quasi-coincides with" or "is not quasi-coincident with" is denoted by $\overline{q}$. □

**Definitio 4.28.** (Liu [9]). Let $(I^X, \delta)$ be $I$-ts and $x_\lambda \in M(I^X)$. A fuzzy set $U$ is called a quasi-coincident neighborhood (q-nbd, for short) of $x_\lambda$ if there exists $V \in \delta$ such that $x_\lambda q V$ and $V \leq U$. □

**Definition 4.29.** (C.K. Wong [16]). Let $(I^X, \delta)$ be an $I$-ts, $A \in I^X$ and $x_\lambda \in M(I^X)$.

A is said to be a neighborhood (nbd, in short) in $(X, \delta)$ iff there is a $B \in \delta$ such that $x_\lambda \leq B \leq A$. Therefore, an open set $U \in \delta$ is the nbd of each of its points. □

**Definition 4.30.** (S. Ganguly and S. Saha [5]). An $I$-ts $(I^X, \delta)$ is $GS$-$T_2$ (Originally, $T_2$) iff for any two distinct points $x_\lambda$ and $y_\mu$:

Case $I$. When $x \neq y$, $x_\lambda$ and $y_\mu$ have open nbds which are not quasi-coincident.

Case $II$. When $x = y$ and $\lambda < \mu$ , then $y_\mu$ has an open $q$-nbd $V$ and $x_\lambda$ has an open nbd $U$ such that $V\overline{q}U$. □

**Definition 4.31.** (A. Kandil and M.E. El-Shafee [8]). An $I$-ts $(I^X, \delta)$ is $FT_2$ if $\forall x_\lambda, y_\mu \in M(I^X)$ with $x_\lambda \overline{q} y_\mu$, there exist $Q_{x_\lambda} \in \delta$ and $Q_{y_\mu} \in \delta$ such that

$$\lambda \leq Q_{x_\lambda}(x),\ \mu \leq Q_{y_\mu}(y) \text{ and } Q_{x_\lambda} \overline{q} Q_{y_\mu}. \square$$

**Remark 4.32.** The following Example 4.33 shows that our sub-$T_2$ needn't be $GS$-$T_2$ in the sense of Definition 4.30. □

**Example 4.33.** Let $(I^X, \delta)$ be the $I$-ts defined in Example 4.12. We have showed that $(I^X, \delta)$ is a sub-$T_2$ space in Example 4.12. Now we assert that $(I^X, \delta)$ isn't $GS$-$T_2$ in the sense of Definition 4.30. Taking $\lambda = \frac{1}{8}$ and $\mu = \frac{1}{4}$, then $x_\lambda$ and $x_\mu$ are different points with $\lambda < \mu$. Moreover, $\top_X$ is the unique open $Q$-neighborhood of $x_\mu$ and the set of open neighborhood of $x_\lambda$ is $A = \{C_1, C_1 \vee C_2, \top_X\}$. Obviously, for each $V \in A$, we cannot have $V \overline{q} \top_X$. Thus, $(I^X, \delta)$ isn't $GS$-$T_2$ in the sense of Definition 4.30. □

**Remark 4.34.** The following Example 4.35 shows that our sub-$T_2$ needn't be $FT_2$ in the sense of Definition 4.31. □

**Example 4.35.** Let $(I^X, \delta)$ be the $I$-ts defined in Example 4.12. We have showed that $(I^X, \delta)$ is a sub-$T_2$ space in Example 4.12. Now we assert that $(I^X, \delta)$ isn't $FT_2$. For the $x, y \in X$ with $x \neq y$, taking $\lambda = \frac{1}{3}$, then $x_\lambda \overline{q} y_\lambda$. The unique neighborhood of $x_\lambda$ and $y_\lambda$ is $\top_X$, moreover $\top_X \overline{q} \top_X$ never is true. Hence $(I^X, \delta)$ isn't $FT_2$. □

## References

[1] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl., 24, pp. 182–190, (1968).

[2] S. L. Chen, G. W. Meng, U-separation axioms and characterizations in $L$-fuzzy topological spaces, J. Liaochen. Sci. Technol. Univ., 11(1), pp. 1–6, (1998).

[3] J.X. Fang, B. Ren, A set of new separation axioms in $L$-fuzzy topological spaces, Fuzzy sets and systems, 96, pp. 359–366, (1998).

[4] M. Gu, B. Zhao, Layer separation axioms in $L$-fuzzy topological spaces, Fuzzy Systems and Mathematics, 17, pp. 12–18 (in Chinese), (2003).

[5] S. Ganguly and S. Saha, On separation axioms and $T_i$-fuzzy continuity, Fuzzy Sets and Systems, 16, pp. 265–275, (1985).

[6] B. Hutton, Normality in fuzzy topoligical spaces, J. Math. Anal. Appl., 50, pp. 74–79, (1975).

[7] T. Kubiak, On $L$-Tychonoff spaces, Fuzzy Sets and Systems, 73, pp. 25–53, (1995).

[8] A. Kandil, M.E. El-Shafee, Regularity axioms in fuzzy topological spaces and $FR_i$-proximities, Fuzzy Sets and Systems, 27, pp. 217–231, (1988).

[9] Y. Liu, Pointwise characterizations of complete regularity and embeding thorem in fuzzy topological space, Sci. Sinica. Ser. A 26, pp. 138–147, (1983).

[10] Y. Liu, M. Luo, Separation in latticed induced spaces, Fuzzy Sets and Systems, 36, pp. 55–66, (1990).

[11] Y. Liu, M. Luo, Fuzzy topology, World Scienctific Publishing, Singapore, (1997).

[12] R. Lowen, Fuzzy topological spaces and compactness, J. Math. Anal. Appl., 56, pp. 621–633, (1976).

[13] S. E. Rodabaugh, Categorical frameworks for stone representation theorems, in: S. E. Rodabaugh, et al., (Eds.), Applications of category theory to Fuzzy Subsets, Kluwer Academic Publishers, Netherlands, pp. 177–231, (1992).

[14] F. G. Shi, A new approach to $L$-$T_2$, $L$-Urysohn, and $L$-completely Hausdorff axioms, Fuzzy Sets and Systems, 157, pp. 794–803, (2006).

[15] F. G. Shi and P. Chen, The Urysohn axiom and the completely Hausdorff axiom in L-topological spaces, Iranian Journal of Fuzzy Systems, Vol. 7, No. 1, pp. 33-45, (2010).

[16] G. Wang, Theory of $L$-fuzzy topolgical spaces, Sha'anxi Normal University Xi'an, (1988) (in Chinese).

[17] C. K. Wong, Fuzzy point and local properties of fuzzy topology, J. Math. Anal. Appl., 46, pp. 316–328, (1974).

[18] P. Wuyts, R. Lowen, On local and global measures of separation in fuzzy topological spaces, Fuzzy Sets and Systems, 19, pp. 51–80, (1986).

[19] F. You, The separation axioms of $T_2\frac{1}{2}$ $L$-fts and $ST_2\frac{1}{2}$ $L$-fts, Fuzzy Systems and Mathematics, 15, pp. 73–76 (in Chinese), (2001).

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