Proyecciones Journal of Mathematics
Vol. 31, N^o 2, pp. 103-123, June 2012.
Universidad Católica del Norte
Antofagasta - Chile

Polar topologies on sequence spaces in non-archimedean analysis

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Received : May 2011. Accepted : January 2012

Abstract

The purpose of the present paper is to develop a theory of a duality in sequence spaces over a non-archimedean vector space. We introduce polar topologies in such spaces, and we give basic results characterizing compact, C-compact, complete and $AK-$complete subsets related to these topologies.

Key words : *Locally K-convex topologies, non archimedean sequence spaces, Schauder basis, separated duality.*

*MSC*2010 *: 11F85 - 46A03 - 46A20 - 46A22 - 46A35 - 46A45 - 464A50.*

1. Introduction

The duality $\langle \lambda, \lambda^{\alpha} \rangle$, where λ is a scalar sequence space, was studied by Köthe and Toeplitz [7] and it has been reformulated by Köthe [6] using the theory of locally convex spaces. After, the duality $\left\langle \lambda, \lambda^{\beta} \right\rangle$ has been studied by Chillingworth [2], Matthews [8], T. Komura and Y. Komura [4]. In this work, we are interested to a duality in non-archimedean sequence spaces. We consider a separated duality $\langle X, Y \rangle$ of vector spaces over a non-archimedean valued field K $(n.a)$; in [1] Ameziane and Babahmed gave a fundamental properties of this duality. Afterwards we take $E(X)$ and $E(Y)$ two vector-valued sequence spaces over X and Y respectively such that $E(Y) \subset E(X)^{\beta}$ that are endwed with the separated duality $\langle E(X), E(Y) \rangle$ by the canonic bilinear form (p.108). We introduce the notion of polar topoogies over $E(X)$; and by the linear maps π_j^X and δ_j^X which we define in this paper; we study the polar topologies compatible with the duality $\langle E(X), E(Y) \rangle$ using the basic duality $\langle X, Y \rangle$. Finally we characterize $C-$ compact, $AK-$complete and complete subsets of $E(X)$ relatively at these topologies. This study was useful in the study that we made in [3].

Throughout this paper, K is a non-archimedean $(n.a)$ non trivially valued complete field with valuation $|.|$, X and Y are two $n.a$ topological vector spaces over K (or K vector spaces) that are in separated duality $\langle X, Y \rangle$. The duality theory for locally $K-$convex spaces can be found more extensively in [1], [9] , [11] and [12].

2. Preliminary

A nonempty subset A of a $K-$vector space X is called $K-$convex if $\lambda x + \mu y + \gamma z \in A$ whenever $x, y, z \in A$, $\lambda, \mu, \gamma \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1, |\gamma| \leq 1$ and $\lambda + \mu + \gamma = 1$. A is said to be absolutely $K-$convex if $\lambda x + \mu y \in A$ whenever $x, y \in A$, $\lambda, \mu \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$. For a nonempty set $A \subset X$ its $K-$convex hull $c(A)$ and absolutely $K-$convex hull $c_0(A)$ are respectively the smallest $K-$convex and absolutely $K-$convex set that contains A. If A is a finite set $\{x_1, ..., x_n\}$ we sometimes write $c_0(x_1, ..., x_n)$ instead of $c_0(A)$.

An absolutely $K-$convex subset of a locally $K-$convex space X is called $K-$ closed if for every $x \in X$ the set $\{|\lambda| : \lambda \in K, \lambda x \in A\}$ is closed in $|K|$. If the valuation on K is discrete every absolutely $K-$convex set A is $K-$closed. If K has a dense valuation an absolutely K convex set A is

$K-$closed if and only if from $x \in E$, $\lambda x \in A$ for all $\lambda \in K, |\lambda| \prec 1$ it follows that $x \in A$. Intersections of $K-$closed sets are $K-$closed. For an absolutely $K-$convex set A the $K-$closed hull of A is the smallest subset of X that is $K-$closed and contains A, it is denoted by $K_c(A)$. If K is discrete we have $K_c(A) = A$ and if K is dense, $K_c(A) = \cap \{\lambda A : \lambda \in K \text{ } and \, |\lambda| \succ 1\}$ ([1] p. 220).

A topological vector space X over K is called locally $K-$convex space if X has a base of zero consisting of locally $K-$convex sets.

Let (X, τ) a locally $K-$convex space, τ is define by a family of n.a. semi-norms $\tau-$ continuous over X, and if K is discrete, we can suppose that $N_p = \{p(x)/x \in X\} \subset |K|$ for every $p \in \mathcal{P}$ ([9]); where $(\mathcal{P})$ is a family of $n.a$ semi-norms which define the topology τ.

If p is a $(n.a)$ semi-norm over X, $B_p(0,1)$ is the set $\{x \in X : p(x) \leq 1\}$.

A sequence $(e_i)_i$ is a Schauder basis for X if every $x \in X$ can be written uniquely as $x = \sum_{i=1}^{\infty} \lambda_i x_i$ where the coefficient functionals $f_j : x \longmapsto \lambda_j$ are continuous.

Let X a $K-$vector space and M a subset of X, a $K-$convex filter over M, is a filter $\mathcal{F}$ over M having a basis $\mathcal{B}$ consisting of $K-$convex subsets of M; this basis is called $K-$convex basis of $K-$convex filter $\mathcal{F}$.

The order of all filters on M induces an order on all $K-$convex filters on M. A maximal element of the ordered set of $K-$convex filter on M is called maximal $K-$convex filter of M.

Let $(x_i)_{i\in I}$ a net on M; for all $i \in I$, put $F_i = \{x_j / j \geq i\}$. $(F_i)_{i\in I}$ is a filter over M called filter associated to a net $(x_i)_{i\in I}$. Conversely, if $\mathcal{F}=(F_i)_{i\in I}$ is a filter over M, for all $i \in I$ let $x_i \in F_i$; over I we define the following order: $i \leq j \Leftrightarrow F_j \subset F_i$. $(x_i)_{i\in I}$ is a net in M called a net associated to a filter $\mathcal{F}$.

Proposition 1. *Let X a locally $K-$convex space, M a subset of X and $\mathcal{F}=(F_i)_{i\in I}$ a maximal $K-$convex filter over M.*

1. $\mathcal{F}$ converges or not having any clusterpoint .

2. Let $(x_i)_{i\in I}$ a net associated to a $\mathcal{F}$; if $(x_i)_{i\in I}$ converges to $x_0, \mathcal{F}$ converges to x_0.

Proof. 1. Let x_0 a cluster point of $\mathcal{F}$ and $(U_j)_{j\in J}$ a $K-$convex neighbourhood base of x_0, $\mathcal{F}' = \{F_i \cap U_j / i \in I \text{ } and \text{ } j \in J\}$ is a $K-$convex filter which converges to x_0 and it is coarsest than $\mathcal{F}$, then $\mathcal{F} = \mathcal{F}'$.

2. x_0 is a clusterpoint of $(x_i)_{i\in I}$, then it is a clusterpoint of $\mathcal{F}$, and so $\mathcal{F}$ converges to x_0. ∎

Proposition 2. *Let X, Y two $K-$vector spaces, $f : X \longrightarrow Y$ a linear map and $\mathcal{F} = (F_i)_{i\in I}$ a maximal $K-$convex filter over X that having $\mathcal{B}$ us a $K-$convex basis; $f(\mathcal{B})$ is a $K-$convex basis of a maximal $K-$convex filter over Y.*

A subset A of a locally $K-$convex space X is compactoid if for each neighbourhood U of zero there exist $x_1, ..., x_n \in X$ such that $A \subset U + c_0 (x_1, ..., x_n)$. An absolutely $K-$convex subset A of X is said to be $C-$compact if every convex filter on A has a clusterpoint on A.
K is $C-$compact if and only if K is spherically complete.

Proposition 3. *Let M be a subset of X. The following are equivalent:*

(i). *M is $C-$compact;*

(ii). *Every maximal $K-$convex filter over M converges;*

(iii). *Any family of closed and $K-$convex subsets of M whose intersection is empty contains a finite subfamily whose intersection is empty.*

Let $\mathcal{B}$ a basis of a filter $\mathcal{F}$ on a subset M of X; the smallest $K-$convex filter containing $\mathcal{B}$, is called $K-$convex filter generated by $\mathcal{B}$ and is denoted by $\mathcal{F}_c(\mathcal{B})$. We show that $\mathcal{F}_c(\mathcal{B}) = \{F \subset M/there\ exists\ B \in \mathcal{B} : c(B) \subset F\}$, and $c(\mathcal{B})$ is $K-$convex basis of $\mathcal{F}_c(\mathcal{B})$, that is to say $\mathcal{F}_c(\mathcal{B}) = \mathcal{F}(c(\mathcal{B}))$.

If $(x_i)_{i\in I}$ is a net in X; $(x_i)_{i\in I}$ converges to x_0 if and only if the filter $K-$convex associated with $(x_i)_{i\in I}$ converges to x_0.

Proposition 4. *Let X, Y two $K-$vector spaces, $f : X \longrightarrow Y$ a linear map, M a subset of X and $\mathcal{B}$ a base of filter on M. Then $f(\mathcal{B})$ is a base of filter on $f(M)$, and we have $\mathcal{F}_c(f(\mathcal{B})) = f(\mathcal{F}_c(\mathcal{B}))$.*

$(\omega (X), \tau_\omega (X))$ = the linear space of all sequences in X endowed with the product topology $\tau_\omega (X)$ which is generated by the family of n.a seminorms $(p_n)_{n\in I\!N,\ p\in(\mathcal{P})}$, $p_n (\overline{x}) = p (x_n)$ for all $\overline{x} = (x_n)_n \in \omega (X)$ and all $p \in (\mathcal{P})$, if X is a locally K-convex space and $(\mathcal{P})$ is a family of $n.a$ semi-norms which define his topology; this space is noted $\omega (K)$ (or ω, for short) in case when $X = K$. A sequence space over X is a subspace of $\omega (X)$.

We define the following sequence spaces over X

$c_0 (X) = \{(x_k)_k \in \omega (X) : (x_k)_k\ converges\ to\ zero\}$

$c (X) = \{(x_k)_k \in \omega (X) : (x_k)_k\ converges\ in\ X\}$,

$\varphi (X) = \{(x_k)_k \in \omega (X) :\ there\ exists\ k_0 \in I\!N\ : x_k = 0\ for\ all\ k \geq k_0\}$,

$m (X) = \{(x_k)_k \in \omega (X) : (x_k)_k\ is\ bounded\ in\ X\}$.

Over $m(X)$ we define the sequence of n.a semi-norms $(\overline{p})_{p\in(\mathcal{P})}$ by: $\overline{p}(\overline{x}) = \sup_k p(x_k)$ for all $\overline{x} = (x_k)_k \in m(X)$.

Let $\tau_\infty(X)$ be the topology on $m(X)$ defined with the sequence of n.a semi-norms $(\overline{p})_{p\in(\mathcal{P})}$.

3. Polar topologies

Let X and Y two $K-$vector spaces placed in separating duality $\langle X, Y\rangle$. If A is a subset of X, we denote by $A^\circ = \{y \in Y / |\langle x, y\rangle| \leq 1 \quad for\ all\ x \in A\}$ the polar of A and $A^{\circ\circ} = \{x \in X / |\langle x, y\rangle| \leq 1 \ \ for\ all\ y \in A^\circ\}$ the bipolar of A.

A° is absolutely $K-$convex and $\sigma(Y, X)-$bounded.

For each absolutely $K-$convex subset A of Y, $K_c\left(\overline{A}^{\sigma(Y,X)}\right) = A^{\circ\circ}$ ([1], corollary 4.3, p. 233). A subset A of Y is said to be $X-$closed if for every $y \in Y \setminus A$, there exits $x \in X$ such that $|\langle x, y\rangle| \succ 1$ and $|\langle x, A\rangle| \leq 1$. Intersections of $X-$closed sets are $X-$closed. For a subset A of Y the $X-$closed hull $X_c(A)$ of A is the smallest $X-$closed subset of Y that contains A. For each subset A of Y, $X_c(A) = A^{\circ\circ}$([1], proposition 2.5, p. 224). Using these two results and by [1], theorem 4.2, p. 233 we have: for all absolutely $K-$convex subset A of Y, A is $X-$closed, if and only if, A is $K-$closed and $\sigma(Y, X)-$closed.

Let $\mathcal{A}$ be a family of $\sigma(Y, X)-$bounded subsets of Y such that

(a) $\mathcal{A}$ is directed by inclusion,

(b) $Y = \bigcup_{A\in\mathcal{A}} A$,

(c) there exists $\lambda_0 \in K$, $|\lambda_0| > 1$ such that $\lambda_0 A \in \mathcal{A}$, for all $A \in \mathcal{A}$.

A topology τ on X is called polar topology of $\mathcal{A}-$convergence, if τ has a fundamental system of zero$-$neighbourhood $(F.S.N)$ consisting of $\{A^\circ / A \in \mathcal{A}\}$.

A vector topology τ on X is called polar topology if there exists a family $\mathcal{A}$ of $\sigma(Y, X)-$bounded subsets of Y which has the properties (a), (b) and (c), such that τ is a polar topology of $\mathcal{A}-$convergence. it is defined by the family of n.a. semi-norms $(P_A)_{A\in\mathcal{A}}$, where $P_A(x) = \sup\{|\langle x, y\rangle| / y \in A\}$.

If $\mathcal{A}$ is the family of all subsets of Y that are:

1. Absolutely $K-$convex, weakly bounded and weakly $C-$compacts, we have the $C-$compact topology $\tau_c(X, Y) = \tau_c$,

2. Absolutely convex and $\sigma(Y, X)-$compact, we have the Mackey topology $\tau_m(X, Y) = \tau_m$,

3. $\sigma(Y,X)-$bounded and $X-$closed, we have the $X-$closed topology $\tau_e(X,Y)=\tau_e$.

4. $\sigma(Y,X)-$bounded, we have the strong topology $\tau_b(X,Y)$.

A locally $K-$convex topology τ on X is called compatible with the duality $\langle X,Y\rangle$ or $(X,Y)-$compatible if Y is isomorphic to the topological dual of X provided with the topology τ. The weak topology $\sigma(X,Y)$ is the coarsest topology among all topologies $(X,Y)-$compatible, and the upper bound topology of all topologies $(X,Y)-$compatible topology is the finest among all the topologies $(X,Y)-$compatible.

We say that X is semi-reflexive if X is isomorphic to the strong topological dual of Y and if τ is a locally $K-$convex topology on X we say that X is $\tau-$reflexive if X is semi-reflexive and $\tau=\tau_b(X,X')$.

For further information about polar topology of $\mathcal{A}-$convergence and general properties of locally $K-$convex spaces we refer to [1], [11] and [12].

If $A\subset\omega(X)$, the $\beta-$dual of A is the subspace of $\omega(Y)$ which is define by $A^\beta=\left\{(y_n)_n\in\omega(Y):\lim\limits_n\langle x_n,y_n\rangle=0\ for\ all\,(x_n)_n\in A\right\}$. A is called perfect if $A^{\beta\beta}=A$. If A is perfect then $\varphi(X)\subset A$. For all $A\subset\omega(X)$, A^β is perfect. We define B^β if $B\subset\omega(Y)$ on the same way.

A subset D of $\omega(X)$ is said to be solid if for every $\overline{x}=(x_k)_k\in D$ and $\alpha=(\alpha_k)_k\in\omega$ such that $|\alpha_k|\leq 1$ for all k, we have $\alpha\overline{x}=(\alpha_k x_k)_k\in D$. The solid hull $S(D)$ of D is the smallest solid set of sequence containing D.

A topology on $E(X)$, with respect the duality $\left\langle E(X),E(X)^\beta\right\rangle$, will be called solid if the elements of the determining family of weakly bounded subsets of $E(X)^\beta$ are solids sets.

Let $E(X)$ and $E(Y)$ be two sequence spaces on X and Y respectively such that $E(Y)\subset E(X)^\beta$, we define on the pair $(E(X),E(Y))$ the following duality $\langle(x_n)_n,(y_n)_n\rangle=\sum\limits_{n=1}^{\infty}\langle x_n,y_n\rangle$ for all $(x_n)_n\in E(X)$ and all $(y_n)_n\in E(Y)$.

If $\varphi(X)\subset E(X)$ and $\varphi(Y)\subset E(Y)$, the duality $\langle E(X),E(Y)\rangle$ is separate.

In the sequel $\langle E(X),E(Y)\rangle$ denotes a duality of this type.

$S(E(Y))\subset[S(E(X))]^\beta$ and $\langle S(E(X)),S(F(Y))\rangle$ is a separating duality extending the separating duality $\langle E(X),F(Y)\rangle$, therefore, we can assume that $E(X)$ and $F(Y)$ are solid.

For all $j\geq 1$, we consider the following linear mappings:

$$\begin{array}{ll} \pi_j^X : E(X) \longrightarrow X & \delta_j^X : X \longrightarrow E(X) \\ (x_n) \longrightarrow x_j & a \longrightarrow \delta_j(a) \end{array}$$

where $\delta_j(a)$ is the sequence with a in the j-th place and 0's elsewhere.

We define also π_j^Y and δ_j^Y.

Let $x = (x_k) \in \omega(X)$, for all $n \geq 1$ $\quad x^{[n]} = \sum_{j=1}^{n} \delta_j(x_j)$ is called the n^{ith} section of x.

We have: $\pi_j^X o \delta_j^X = id_X$, $\pi_j^Y o \delta_j^Y = id_Y$, $(\pi_j^X)^*/Y = \delta_j^Y$ and $(\delta_j^X)^*/F(Y) = \pi_j^Y$ where u^* is the algebraic adjoint of the linear map u.

Proposition 5. *Let A be a subset of $E(X)$ if A is solid, A° is solid and we have: $A^\circ = [A \cap \varphi(X)]^\circ$.*

Definition 1. *Let A a subset of $\omega(X)$.*

a. Is said that A is δ_j^X-saturated if for all $(x_n) \in A, \delta_j^X(x_j) \in A$.

b. It is said that A is δ^X-saturated if A is δ_j^X-saturated for all $j \geq 1$.

c. It is said that A is π^X-saturated if: $x_j \in \pi_j^X(A)$ for all $j \geq 1 \Rightarrow (x_n) \in A$.

If A is solid, A is δ^X-saturated.

$\varphi(X)$ is δ^X-saturated and not π^X-saturated.

If p is a n.a. semi-norm on X, $\left\{(x_n) \in \omega(X)/ \sup_n p(x_n) \leq 1\right\}$ is π^X-saturated.

The following results are demonstrated in a direct:

Proposition 6. *Let A be a subset of $E(X)$.*

1. If A is π^X-saturated, $S(A)$ is π^X-saturated.

2. If A is δ^X-saturated, $S(A)$ and $c_0(A)$ are δ^X-saturated, and A° is δ^Y- saturated and π^Y-saturated.

3. $\left[\pi_j^X(A)\right]^\circ \subset \pi_j^Y(A^\circ)$ for all $j \geq 1$.

4. If A is δ_j^X-saturated, $\left[\pi_j^X(A)\right]^\circ = \pi_j^Y(A^\circ)$.

5. If A is δ^X-saturated,

$$A^\circ = \pi_j^X\left[\pi_j^Y(A^\circ)\right] = \left\{(y_k) \in F(Y)/ \sup_k |\langle x_k, y_k\rangle| \leq 1 \quad for\ all\ (x_k) \in A\right\}.$$

6. $S(A)^\circ \subset S(A^\circ)$; and if A is δ^X-saturated, $A^\circ = S(A)^\circ = S(A^\circ)$.

7. If A is δ^X-saturated and $F(Y)-$closed, $\pi_j^X(A)$ is $Y-$closed for all $j \geq 1$.

8. If A is π^X-saturated and $\pi_j^X(A)$ is $Y-$closed for all $j \geq 1$, A is $F(Y)-$ closed.

Corollary 1. *Let A be a subset of $E(X)$ δ^X-saturated and π^X-saturated.*

For A is $F(Y)-$closed, it is necessary and enough that $\pi_j^X(A)$ be $Y-$closed for all $j \geq 1$.

Proposition 7. *Let A be an absolutely $K-$convex subset of $E(X)$.*

1. If A is $K-$closed and δ_j^X-saturated, $\pi_j^X(A)$ is $K-$closed.

2. If A is π^X-saturated and $\pi_j^X(A)$ is $K-$closed for all $j \geq 1$, A is $K-$closed.

Proposition 8. *Let τ be a topology on $E(X)$ and τ_j the topology image reciprocal of τ by the linear map δ_j^X on X. If τ admits as S.F.N of 0 $\{A^\circ / A \in \mathcal{A}\}$, then $\left\{\left[\pi_j^Y(A)\right]^\circ / A \in \mathcal{A}\right\}$ is a F.S.N. of 0 for τ_j.*

Proof. ([1], *proposition* 2.9). ∎

Proposition 9. *For all $j \geq 1$, π_j^X is $(\sigma(E(X), F(Y)), \sigma(X,Y))-$continuous and δ_j^X is $(\sigma(X,Y), \sigma(E(X), F(Y)))-$continuous.*

Proof. $(\pi_j^X)^*(Y) \subset F(Y)$ and $(\delta_j^X)^*(F(Y)) \subset Y$, and the result follows from ([9], *p.* 128). ∎

Proposition 10. *1. $\left[\pi_j^X(A)\right]^\circ = (\delta_j^Y)^{-1}(A^\circ)$ for all $A \subset E(X)$.*

2. $\left[\delta_j^X(B)\right]^\circ = (\pi_j^Y)^{-1}(B^\circ)$ for all $B \subset X$.

3. $\pi_j^X(A) \subset B \Rightarrow \delta_j^Y(B^\circ) \subset A^\circ$ for all $A \subset E(X)$ and for all $B \subset X$.

4. $\delta_j^X(B) \subset A \Rightarrow \pi_j^Y(A^\circ) \subset B^\circ$ for all $A \subset E(X)$ and for all $B \subset X$.

5. $(\pi_j^X)^{-1}(D^\circ) = \left[\delta_j^Y(D)\right]^\circ$ for all $D \subset Y$.

6. $(\delta_j^X)^{-1}(C^\circ) = \left[\pi_j^Y(C)\right]^\circ$ for all $C \subset F(Y)$.

7. $(\pi_j^X)^*(D) \subset C \Rightarrow \pi_j^X(C^\circ) \subset D^\circ$ for all $D \subset Y$ and for all $C \subset E(Y)$.

8. $(\delta_j^X)^*(C) \subset D \Rightarrow \delta_j^X(D^\circ) \subset C^\circ$ for all $D \subset Y$ and for all $C \subset E(Y)$.

Proof. ([1], *proposition* 2.8). ∎

A polar topology of $\mathcal{A}-$convergence on $E(X)$ is said solid, if all $A \in \mathcal{A}$ is solid. Thus, any polar, solid topology admits a *F.S.N* from 0 consisting of solid subsets .

If τ is the polar topology of $\mathcal{A}-$convergence on $E(X)$ such that A is δ^Y-saturated for all $A \in \mathcal{A}$, τ coincides with the polar topology of $S(\mathcal{A})-$convergence (proposition 6), and then τ is a polar and solid topology .

Proposition 11. *Let τ be a polar topology of $\mathcal{A}-$convergence over $E(X)$ and τ_j the topology image reciprocal of τ by the linear map δ_j^X on X.*

1. *τ_j is the polar topology of $\pi_j^Y(\mathcal{A})-$convergence.*
2. *π_j^X is $(\tau, \tau_j)-$continuous if and only if $\delta_j^Y o \pi_j^Y(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$.*

Proof. ([1], *proposition* 3.8). ∎

Proposition 12. *If τ is the weak topology (resp. Mackey, resp. $C-$compact, resp.*
$E(X)-$closed; resp. strong) of $E(X)$ for all $j \geq 1, \tau_j$ is the weak topology (resp. Mackey, resp. $C-$compact, resp. $X-$closed; resp. strong) on X

Proof. ([1], *proposition* 3.9). ∎

Proposition 13. *Let τ a polar topology of $\mathcal{A}-$convergence on $E(X)$, for all $j \geq 1$, we have:*

1. *δ_j^X is $(\tau_j, \tau)-$continuous;*
2. *If τ is solid, π_j^X is $(\tau, \tau_j)-$continuous;*
3. *If π_j^X is $(\tau, \tau_j)-$continuous, δ_j^X is $(\tau_j, \tau)-$closed.*

Proof. 1. τ_j is a polar topology of $\pi_j^Y(\mathcal{A})-$convergence, and we have:
$\delta_j^X\left(\left[\pi_j^Y(A)\right]^\circ\right) \subset A^\circ$ for all $A \in \mathcal{A}$.

2. If τ is solid, we have : $\pi_j^X(A^\circ) \subset \left[\pi_j^Y(A)\right]^\circ$ for all $A \in \mathcal{A}$.

3. Let M a closed in (X, τ_j), there exists $A \in \mathcal{A}$ such that $\left[\pi_j^Y(A)\right]^\circ \subset M^\circ$, therefore $A^\circ \subset \delta_j^X(M^\circ) = \left[\delta_j^X(M)\right]^\circ$. ∎

Let τ be a locally $K-$convex topology on $E(X)$ such that $E(X)$ be $\tau-$polar; if τ is $(E(X)\,F(Y))-$compatible, τ is a polar topology of $\mathcal{A}-$convergence, where $\mathcal{A}$ is constituted of $\sigma(F(Y), E(X))-$bounded and

$E(X)-$closed subsets of $F(Y)$, ([1], theorem 4.3). For all $j \geq 1$, τ_j is the polar topology of $\pi_j^Y(\mathcal{A})-$convergence on X and X is τ_j-polar if all $A \in \mathcal{A}$ is δ^Y-saturated, $\pi_j^X(A)$ is $\sigma(Y,X)-$bounded and $X-$closed (Proposition 6), and then τ_j is $(X,Y)-$compatible.

If K is spherically complete, we have the following theorem:

Theorem 1. *Suppose that K be spherically complete, and let τ a locally $K-$convex topology on $E(X)$; if τ is $(E(X),F(Y))-$compatible, τ_j is $(X,Y)-$compatible, for all $j \geq 1$.*

Proof. τ is a polar topology of $\mathcal{A}$ convergence, where $\mathcal{A}$ consists of absolutely K convex , $\sigma(E(Y),E(X))-$bounded and $\sigma(E(Y),E(X))-C-$compact subsets of $F(Y)$ ([1], *theorem* 4.4). For all $j \geq 1$, π_j^Y is $(\sigma(F(Y),E(X)),\ \sigma(Y,X))-$continuous, then $\pi_j^Y(A)$ is absolutely $K-$convex, $\sigma(Y,X)-$ bounded and $\sigma(Y,X)-C-$compact for all $A \in \mathcal{A}$ and then τ_j is $(X,Y)-$compatible. ∎

Theorem 2. *Let τ a solid and polar topology on $E(X)$; if $E(X)$ is $\tau-$barreled, X is τ_j-barreled for all $j \geq 1$.*

Proof. Let B a τ_j-barrel in X; δ_j^X is $(\tau_j,\tau)-$closed, then $\delta_j^X(B)$ is a $\tau-$ barrel into $E(X)$ and then $(\delta_j^X)^{-1}(\delta_j^X(B))$ is a neighborhood of 0 in (X,τ_j) then B is a neighborhood of 0 for τ_j. ∎

Remark 1. *Instead of assuming that τ is solid, we can assume only that π_j^X be $(\tau,\tau_j)-$continuous for all $j \geq 1$.*

A subset A of $E(X)$ said to be δ^X-stable if for all $x=(x_k) \in E(X)$ such that there exists $j \geq 1$ satisfying $\delta_j^X(x_j) \in A$, then $x \in A$.

Let $A \subset E(X)$ such that $A \cap \left\{\delta_j^X(a)/a \in X \ \ and \ \ j \geq 1\right\} = \phi$, A is δ^X stable.

Definition 2. *Let τ a vector topology on $E(X)$; we say that $E(X)$ is $\delta^X\tau-$barreled if every $\tau-$barrel δ^X-stable, is a neighborhood of 0.*

If $E(X)$ is $\tau-$barreled, it is $\delta^X\tau-$barreled.

Theorem 3. *Let τ a polar and solid topology on $E(X)$; if there exists $j \geq 1$ such that X is τ_j-barreled, $E(X)$ is $\delta^X\tau-$barreled*

Proof. Let B a $\tau-$barrel δ^X-stable in $E(X)$; δ_j^X is $(\tau_j,\tau)-$continuous, so $(\delta_j^X)^{-1}(B)$ is a τ_j-barrel, and then $(\delta_j^X)^{-1}(B)$ is a neighborhood of 0 in (X,τ_j) and hence $(\pi_j^X)^{-1}\left[(\delta_j^X)^{-1}(B)\right]$ is a neighborhood of 0 in $(E(X),\tau)$. B is δ^X- stable, then $(\pi_j^X)^{-1}\left[(\delta_j^X)^{-1}(B)\right]\subset B$ and then B is a neighborhood of 0 in $(E(X),\tau)$. ∎

Theorem 4. *Suppose that X and Y are semi-reflexive, and let τ a topology on $E(X)$ which is $(E(X),F(Y))-$compatible. If $E(X)$ is $\tau-$reflexive, X is τ_j-reflexive for every $j\geq 1$.*

Proof. $\tau=\tau_b(E(X),E(X)^{'})=\tau_b(E(X),F(Y))$; so for all $j\geq 1$ $\tau_j=\tau_b(X,Y)$ (Proposition 12). Y is semi-reflexive, then τ_j is $(X,Y)-$compatible ([1], proposition 5.9) and then $\tau_j=\tau_b(X,(X,\tau_j)^{'})$. ∎

Corollary 2. *If K is spherically complete and τ is a topology on $E(X)$ which is $(E(X),F(Y))-$compatible and solid such that $E(X)$ is $\tau-$barreled, then X is τ_j reflexive for any $j\geq 1$.*

Proof. For all $j\geq 1,\tau_j$ is $(X,Y)-$compatible (theorem 1) and X is τ_j-barreled for all $j\geq 1$, then X is τ_j-reflexive ([1], *theorem* 5.2). ∎

4. Compactness and $C-$compactness

Let τ a polar topology on $E(X)$ such that π_j^X be $(\tau,\tau_j)-$continuous for all $j\geq 1$. If M is a compact subset of $(E(X),\tau)$; $\pi_j^X(M)$ is a compact subset of (X,τ_j) for all $j\geq 1$.

In order to study the converse, we introduce the notion of $TK-$convergent net.

Definition 3. *A net $(x^i)_{i\in I}$ in $E(X)$ is called $TK-$convergent if for all $j\geq 1$, $(x_j^i)_{i\in I}$ is convergent in (X,τ_j).*

Theorem 5. *Let M a subset of $E(X)$; M is relatively compact in $(E(X),\tau)$ if and only if:*

(i.) $\pi_j^X(M)$ is relatively compact in (X,τ_j) for all $j\geq 1$;
(ii.) All $TK-$convergent net in M, converges in $(E(X),\tau)$.

Proof. N.C.] π_j^X is $(\tau, \tau_j)-$continuous for all $j \geq 1$, then $\pi_j^X(M)$ is relatively compact in (X, τ_j). Let $(x^i)_{i\in I}$ a $TK-$convergent net in M. For all $j \geq 1$ let $x_j \in X$ such that $(x_j^i)_{i\in I}$ converges to x_j in (X, τ_j). $(x^i)_{i\in I}$ has a cluster point $z = (z_n)$ in $(E(X), \tau)$. For all $j \geq 1$, z_j is a cluster point of $(x_j^i)_{i\in I}$ in (X, τ_j); then $z_j = x_j$. (x_n) is the unique cluster point of $(x^i)_{i\in I}$, therefore $(x^i)_{i\in I}$ converges to (x_n) in $(E(X), \tau)$.

S.C.] Let $(x^i)_{i\in I}$ a net in M, and let $\mathcal{A}$ the family of $\sigma(F(Y), E(X))-$bounded subset of $F(Y)$ which defines the topology τ. For any $j \geq 1$, τ_j is the polar topology of $\pi_j^Y(\mathcal{A})-$convergence on X.

Let x_1 a cluster point of $(x_1^i)_{i\in I}$ in (X, τ_1). For all $A \in \mathcal{A}$ and for all $i \in I$, there exists $i_A > i$ such that $x_1^{i_A} \in \left[\pi_1^Y(A)\right]^\circ$. Consider the sub family $(i_A)_{A\in\mathcal{A}}$ of I, it is ordered by: $i_A \leq i_B \Leftrightarrow A \subset B$ for all $A, B \in \mathcal{A}$. $(i_A)_{A\in\mathcal{A}}$ is a filter on the right family. Let $A_0 \in \mathcal{A}$; $i_A \geq i_{A_0} \Rightarrow A_0 \subset A \Rightarrow \left[\pi_1^Y(A)\right]^\circ \subset \left[\pi_1^Y(A_0)\right]^\circ \Rightarrow x_1^{i_A} - x_1 \in \left[\pi_1^Y(A_0)\right]^\circ$. Therefore $(x_1^{i_A})_{A\in\mathcal{A}}$ converges to x_1 in (X, τ_1).
Let x_2 a cluster point of $(x_2^{i_A})_{A\in\mathcal{A}}$ in (X, τ_2). for all $A \in \mathcal{A}$, there exists $l_1(i_A) > i_A$ such that $x_2^{l_1(i_A)} - x_2 \in \left[\pi_2^Y(A)\right]^\circ$.

Let $A_0 \in \mathcal{A}$; $i_A \geq i_{A_0} \Rightarrow A \supset A_0 \Rightarrow \left[\pi_2^Y(A)\right]^\circ \subset \left[\pi_2^Y(A_0)\right]^\circ \Rightarrow x_2^{l_1(i_A)} - x_2 \in \left[\pi_2^Y(A_0)\right]^\circ$. Therefore $(x_2^{l_1(i_A)})_{A\in\mathcal{A}}$ converges to x_2 in (X, τ_2). Let x_3 a cluster point of $(x_3^{l_1(i_A)})_{A\in\mathcal{A}}$ in (X, τ_3). For all $A \in \mathcal{A}$, there exists $l_2(l_1(i_A)) > l_1(i_A)$ such that $x_3^{l_2 o l_1(i_A)} - x_3 \in \left[\pi_3^Y(A)\right]^\circ$. $(x_3^{l_2 o l_1(i_A)})_{A\in\mathcal{A}}$ converges to x_3 in (X, τ_3).

Inductively, for all $j \geq 3$ and for all $A \in \mathcal{A}$, there exists $l_j o l_{j-1} o....l_1(i_A) > l_{j-1} o.....o l_1(i_A)$ such that $(x_{j+1}^{l_j o...o l_1(i_A)})_{A\in\mathcal{A}}$ converges to x_{j+1} in (X, τ_{j+1}).

Put $y = (x^{i_A}, x^{l_1(i_A)}, x^{l_2 o l_1(i_A)}, ..., x^{l_k o....o l_1(i_A)}, ...)_{A\in\mathcal{A}}$.

For all $j \geq 1$, $(x_j^{i_A}, x_j^{l_1(i_A)}, x_j^{l_2 o l_1(i_A)}, ..., x_j^{l_k o....o l_1(i_A)}, ...)_{A\in\mathcal{A}}$ converges to x_j in (X, τ_j); therefore y is $TK-$convergent, and hence it converges to x in $(E(X), \tau)$. Hence x is a cluster point of $(x^i)_{i\in I}$, and then M is relatively compact. ∎

Corollary 3. *Let M a subset of $E(X)$, M is compact in $(E(X), \tau)$ if and only if:*

(i.) $\pi_j^X(M)$ is compact in (X, τ_j) for all $j \geq 1$,

(ii.) Any $TK-$convergent net in M converges to an element of M in $(E(X), \tau)$.

To give version of theorem 5 using the filters, we need introduce the following definition:

Definition 4. *Let M a subset of $E(X)$ and $\mathcal{F}$ a filter on M; we say that $\mathcal{F}$ is $TK-$ convergent if for all $j \geq 1$ the filter generated by $\pi_j^X(\mathcal{F})$ converges in (X, τ_j).*

Every convergent filter is $TK-$convergent, and if $\mathcal{F}$ is a $TK-$convergent filter and $\mathcal{F}'$ is a filter finer than $\mathcal{F}$, $\mathcal{F}'$ is $TK-$convergent.

Proposition 14. *Let M a subset of $E(X)$.*

1. If $\mathcal{F} = (F_i)_{i \in I}$ is a $TK-$convergent filter on M, any net associated to $\mathcal{F}$ is $TK-$convergent.

2. If $(x^i)_{i \in I}$ is a $TK-$convergent net, the $K-$convex filter associated to $(x^i)_{i \in I}$ is $TK-$convergent.

Theorem 6. *Let M a subset of $E(X)$; M is compact in $(E(X), \tau)$ if and only if:*

(i.) $\pi_j^X(M)$ is compact in (X, τ_j) for all $j \geq 1$;

(ii.) Any $TK-$convergent filter on M converges to an element of M.

Proof. N.C.] Let $\mathcal{F}$ a $TK-$convergent filter on M. For any $j \geq 1$ let $x_j \in X$ such that $\pi_j^X(\mathcal{F})$ converges to x_j in (X, τ_j). $\mathcal{F}$ has at least one cluster point $z = (z_n)$ in M. For all $j \geq 1$, z_j is a cluster point of $\pi_j^X(\mathcal{F})$, therefore $z_j = x_j$; then (x_n) is the unique cluster point of $\mathcal{F}$ in M, so $\mathcal{F}$ converges to (x_n) in (M, τ).

S.C.] Let $\mathcal{F}$ a maximal filter on M; for all $j \geq 1$ $\pi_j^X(\mathcal{F})$ is a maximal filter on $\pi_j^X(M)$, therefore it converges to x_j in (X, τ_j), and then $\mathcal{F}$ is $TK-$convergent, therefore it converges to an element of M. ∎

Definition 5. *Let M a subset of $E(X)$, we say that M is an AK–complete subset of $(E(X), \tau)$ if every $x = (x_n)$ element of $E(X)$ such that $(x^{[n]})$ is a Cauchy sequence in (M, τ); $x \in M$ and $(x^{[n]})$ converges to x in $(E(X), \tau)$.*

We say that M is relatively $AK-$complete if its closure $\overline{M}$ in $(E(X), \tau)$ is $AK-$ complete.

If M is complete, it is $AK-$complete.
Any closed subset of a set $AK-$complete is $AK-$complete.

In the following result, we characterize the subsets solid and relatively compact of $(E(X), \tau)$.

Theorem 7. *Let M a solid subset of $E(X)$, M is relatively compact in $(E(X),\tau)$ if and only if:*

(i.) $\pi_j^X(M)$ is relatively compact in (X,τ_j) for all $j\geq 1$,

(ii.) $x^{[i]} \stackrel{i\to\infty}{\longrightarrow} x$ uniformly on M in $(E(X),\tau)$,

(iii.) M is relatively $AK-$complete in $(E(X),\tau)$.

Proof. N.C.] If M is relatively compact, M is relatively complete, and then it is relatively $AK-$complete.

Suppose we did not $(ii.)$ there exists $A\in\mathcal{A}$ a sequence $({}^ix)_i$ in M and a strictly increasing sequence of integers $(j_i)_i$ such that ${}^ix^{[j_i]} -{}^i x\notin A^\circ$ for all $i\geq 1$. The sequence $({}^ix^{[j_i]} -{}^i x)_i$ is $TK-$convergent to 0, so it converges to 0 in $(E(X),\tau)$ which is absurd.

S.C.] Let $({}^\alpha x)_{\alpha\in D}$ a net in M such that for all $j\geq 1$ $({}^\alpha x_j)_{\alpha\in D}$ converges to x_j in (X,τ_j). Let $A\in\mathcal{A}$ for all $i\geq 1$ ${}^\alpha x^{[i]}-x^{[i]}=\sum_{n=1}^{i}\delta_n^X({}^\alpha x_n-x_n)\in A^\circ$ for α sufficiently large. So for all $i\geq 1$ ${}^\alpha x^{[i]} \stackrel{\alpha}{\longrightarrow} x^{[i]}$ in $(E(X),\tau)$ in particular $x^{[i]}\in\overline{M}$ for all $i\geq 1$. Using this convergence and (ii), we can choose α as $x^{[i]}-x^{[j]}=(x^{[i]}-{}^\alpha x^{[i]})+({}^\alpha x^{[i]}-{}^\alpha x)+({}^\alpha x-{}^\alpha x^{[j]})+({}^\alpha x^{[j]}-x^{[j]})\in A^\circ$ for i,j sufficiently great. Therefore $(x^{[i]})$ is a Cauchy net in $\overline{M}$ and then $x^{[i]} \stackrel{i\to+\infty}{\longrightarrow} x$ in $(E(X),\tau)$. From this convergence and (ii), we can choose i such that ${}^\alpha x-x=({}^\alpha x-{}^\alpha x^{[i]})+({}^\alpha x^{[i]}-x^{[i]})+(x^{[i]}-x)\in A^\circ$ for α Large enough, so $({}^\alpha x)_{\alpha\in D}$ converges to x in $(E(X),\tau)$ and hence M is relatively compact (theorem 5). ∎

Corollary 4. *Let M a solid subset of $E(X)$; M is compact in $(E(X),\tau)$ if and only if:*

(i.) $\pi_j^X(M)$ is compact in (X,τ_j) for all $j\geq 1$,

(ii.) $x^{[i]} \stackrel{i\to\infty}{\longrightarrow} x$ uniformly on M in $(E(X),\tau)$

(iii.) M is $AK-$complete in $(E(X),\tau)$.

Corollary 5. *The envelope solid of a relatively compact subset of $(E(X),\tau)$ is not necessarily relatively compact.*

Proof. Let $x=(x_n)\in E(X)$ such that $(x^{[i]})_i$ does not converge to x in $(E(X),\tau)$ so $(z^{[i]})_i$ does not converge to z uniformly on $S(x)$ and then $S(x)$ is not relatively compact. ∎

Proposition 15. *1. Let $(x^i)_{i\in I}$ a net in $E(X)$; if $\mathcal{F}$ is a $K-$convex filter associated with $(x^i)_{i\in I}$, $\pi_j^X(\mathcal{F})$ is a $K-$convex filter associated with a net $(x_j^i)_{i\in I}$ for all $j\geq 1$.*

2. Let $\mathcal{F}$ a $K-$convex filter on $E(X)$; if $(x^i)_{i\in I}$ is a net associated to $\mathcal{F}$, $(x^i_j)_{i\in I}$ is a net associated to $\pi^X_j(\mathcal{F})$ for all $j \geq 1$.

Theorem 8. *Let M a $K-$convex subset of $E(X)$; M is $C-$compact in $(E(X),\tau)$ if and only if:*

(i.) $\pi^X_j(M)$ is $C-$compact in (X,τ_j) for all $j \geq 1$,

(ii.) Any $K-$convex and $TK-$convergent filter on M admits a cluster point in M.

Proof. N.C.] Obvious.

S.C.] Let $\mathcal{F}$ a maximum $K-$convex filter of M. For any $j \geq 1$, $\pi^X_j(\mathcal{F})$ is a maximum $K-$convex filter of $\pi^X_j(M)$ (proposition 2), so $\pi^X_j(\mathcal{F})$ converges to x_j in (X,τ_j). $\mathcal{F}$ is then $TK-$convergent, so it admits a cluster point in M, and hence $\mathcal{F}$ converges in $(E(X),\ \tau)$ (Proposition 1). ∎

Proposition 16. *Let M a $K-$convex subset of $E(X)$; if M is $C-$compact, any $K-$convex and $TK-$convergent filter on M has a unique cluster point in M.*

Proof. Let $\mathcal{F}$ a $K-$convex and $TK-$convergent filter on M. For all $j \geq 1$ let $x_j \in X$ such that $\pi^X_j(\mathcal{F})$ converges to x_j in (X,τ_j). $\mathcal{F}$ admits at least one cluster point (z_n) in M. For all $j \geq 1$, z_j is a cluster point of $\pi^X_j(\mathcal{F})$ in (X,τ_j), and then $x_j = z_j$. So (x_j) is the only cluster point of $\mathcal{F}$ in M. ∎

5. $AK-$completion and completion

Let M a subset of $E(X)$ and τ a topology on $E(X)$, we put:

$$S_M = \left\{x \in M / x^{[n]} \overset{n\to\infty}{\longrightarrow} x \quad in \quad (E(X),\tau)\right\}.$$

If M is a subspace of $E(X)$, we say that M is an $AK-$space if $S_M = M$.

Proposition 17. *Let τ a polar topology of $\mathcal{A}$ convergence on $E(X)$; $(E(X),\tau)$ is $AK-$complete.*

Proof. Let $x = (x_n) \in E(X)$ such that $(x^{[n]})$ is a Cauchy sequence in $(E(X),\tau)$. For all $A \in \mathcal{A}$ there exists $n_0 \geq 1$ such that $x^{[n]} - x^{[m]} \in A^\circ$ for all $n \geq m \geq n_0$, and then $x^{[n]} - x \in A^\circ$ for all $n \geq n_0$, then $x^{[n]} \overset{n\to\infty}{\longrightarrow} x$ in $(E(X),\tau)$. ∎

Corollary 6. *Let M a subset of $E(X)$. M is $AK-$complete if and only if M contains every element x of $E(X)$ such that $(x^{[n]})$ is the Cauchy sequence in M.*

Corollary 7. *Let τ' a locally $K-$convex topology on $E(X)$ coarser than τ; any $AK-$complete subset of $(E(X),\tau')$ is complete in $(E(X),\tau)$.*

Proof. Let M an $AK-$complete subset of $(E(X),\tau')$, and either $x \in E(X)$ such that $(x^{[n]})$ is a Cauchy sequence in (M,τ), $(x^{[n]})$ is a Cauchy sequence in (M,τ'), so $x \in M$ and hence M is $AK-$complete in $(E(X),\tau)$, (Corollary 6). ∎

For all $x = (x_n) \in E(X)$, we put $\begin{array}{ll} \psi_x : & E(Y) \longrightarrow c_0(K) \\ & (y_n) \longrightarrow (\langle x_n, y_n\rangle)_n \end{array}$

ψ_x is a linear map. ∎

Lemma 1. *For any $x \in E(X)$, ψ_x is $(\sigma(E(Y),E(X)),\sigma(c_0(K),m(K)))-$continuous.*

Proof. $c_0(K)^\beta = m(K)$ and $\langle c_0(K),m(K)\rangle$ is a separating duality. Let $(\alpha_n) \in m(K)$; $E(X)$ is solid, then $(\alpha_n x_n) \in E(X)$, and we have $\psi_x(\{(\alpha_n\, x_n)\}^\circ) \subset \{(\alpha_n)\}^\circ$. ∎

Proposition 18. *$(E(X),\sigma(E(X),E(Y)))$ is an $AK-$space.*

Proof. Let $x = (x_n) \in E(X)$. For all $y = (y_n) \in E(Y)$, $(\langle x_n, y_n\rangle) \in c_0(K)$; there exists $i_0 \geq 1$ such that $\sup\limits_{n\geq i_0} |\langle x_n, y_n\rangle| \leq 1$, then $x^{[i]} - x \in \{y\}^\circ$ for all $i \geq i_0$, and then $x^{[i]} \overset{i\to\infty}{\longrightarrow} x$ in $(E(X),\sigma(E(X),E(Y)))$. ∎

Proposition 19. *Suppose that K be local, and let τ a $(E(X),F(Y))-$compatible topology on $E(X)$; if τ is solid, $(E(X),\tau)$ is an $AK-$space.*

Proof. Let $\mathcal{A}$ a family of $\sigma(F(Y),E(X))-$compacts and absolutely $K-$convex subsets of $F(Y)$ such that τ be a polar topology of $\mathcal{A}-$convergence ([1] , theorem 4.5.) Let $x = (x_n) \in E(X)$; for all $A \in \mathcal{A}$, $\psi_x(A)$ is solid and $\sigma(c_0(K),\ m(K))-$compact in $c_0(K)$. Then $z^{[i]} \overset{i\to\infty}{\longrightarrow} z$ uniformly on $z \in \psi_x(A)$ in $(c_0(K),\ \sigma(c_0(K),m(K)))$ (theorem 7); there exists $i_0 \geq 1$ such that $\left|\left\langle z^{[i]} - z, e\right\rangle\right| \leq 1$ for all $i \geq i_0$ and for all $z \in \psi_x(A)$, then $x^{[i]} - x \in A^\circ$ for all $i \geq i_0$, and so $x^{[i]} \overset{i\to\infty}{\longrightarrow} x$ *in* $(E(X),\tau)$. ∎

We have the following result which is a kind of reciprocal of theorem 1:

Theorem 9. *Suppose that K be local, and let τ a polar and solid topology on $E(X)$ for separating duality $\left\langle E(X), E(X)^{\beta}\right\rangle$. If τ_j is $(X,Y)-$compatible for all $j \geq 1$, τ is $(E(X), E(X)^{\beta})-$compatible.*

Proof. $E(X)^{\beta} = (E(X), \sigma(E(X), E(X)^{\beta}))' \subset (E(X), \tau)'$. Let $f \in (E(X), \tau)'$ and $x = (x_n) \in E(X)$. $(E(X), \tau)$ is an $AK-$space (proposition 19), therefore $x^{[i]} \overset{i\to\infty}{\longrightarrow} x$ in $(E(X), \tau)$, and then $f(x) = \lim_i f(x^{[i]}) = \sum_j fo\delta_j^X(x_j)$. For all $j \geq 1$, $fo\delta_j^X \in (X, \tau_j)' = Y$; therefore $f(x) = \sum_j \langle x_j, y_j \rangle$, with $y_j = fo\delta_j^X$ for all $j \geq 1$. Hence $(y_j) \in E(X)^{\beta}$, and so $(E(X), \tau)' \subset E(X)^{\beta}$. ∎

Let $\mathcal{C}$ a family of subsets of $F(Y)$ such that:

1. $\mathcal{C}$ is the right filtering for inclusion;
2. There exist $\lambda_0 \in K$, $|\lambda_0| > 1$ such that $\lambda_0 A \in \mathcal{C}$ for all $A \in \mathcal{C}$;
3. $\pi_j^Y(A)$ is $\sigma(Y,X)-$bounded for all $j \geq 1$ and for all $A \in \mathcal{C}$
4. The subspace of $E(Y)$ generated by $\cup\{A/A \in \mathcal{C}\}$ contains $\varphi(Y)$.

We put: $\begin{cases} \mathcal{C}(X) = \left\{(x_n) \in \omega(X)/ \sup\limits_{(y_n)\in A} \left|\sum\limits_n \langle x_n, y_n\rangle\right| < \infty \ \ for\ all\ A \in \mathcal{C}\right\} \\ \mathcal{C}(Y) = subspace\ generated\ by\ \cup\{A/A \in \mathcal{C}\}. \end{cases}$

If $\mathcal{C}$ is the family of all finite subsets of $F(Y)$, $\mathcal{C}(X) = F(Y)^{\beta}$.

$\varphi(X) \subset \mathcal{C}(X)$ and $\langle \mathcal{C}(X), \mathcal{C}(Y)\rangle$ is a separating duality defined by the bilinear form:

$\langle (x_n), (y_n)\rangle = \sum\limits_n \langle x_n, y_n\rangle$ for all $(x_n) \in \mathcal{C}(X)$ and for all $(y_n) \in \mathcal{C}(Y)$.

If τ is the polar topology of $\mathcal{A}-$convergence of $E(X)$, $(\mathcal{A}(X), \tau_{\mathcal{A}})$ is defined, where $\tau_{\mathcal{A}}$ is the polar topology defined on $\mathcal{A}(X)$ by the family $\mathcal{A}$, and we have:

1. $E(X) \subset \mathcal{A}(X) \subset F(Y)^{\beta}$;
2. $\tau_{\mathcal{A}/E(X)} = \tau$.

Proposition 20. *Let τ a polar topology of $\mathcal{A}-$convergence on $E(X)$.*

1. *$S_{(\mathcal{A}(X), \tau_{\mathcal{A}})} \subset E(X)$,*
2. *$(\mathcal{A}(X), \tau_{\mathcal{A}})$ is $AK-$complete.*

Proof. 1. Let $x = (x_n) \in S_{(\mathcal{A}(X), \tau_{\mathcal{A}})}$; $x^{[i]} \overset{i\to\infty}{\longrightarrow} x\ (\tau_{\mathcal{A}})$, therefore $(x^{[i]})$ is Cauchy sequence in $(E(X), \tau)$ ($\tau = \tau_{\mathcal{A}/E(X)}$), and then $x \in E(X)$ (proposition 17).

2. Let $(x^{[i]})$ a Cauchy sequence in $(\mathcal{A}(X),\tau_{\mathcal{A}})$; for all $A\in\mathcal{A}$, there exists $i_0\geq 1$ such that for all $i,j\geq i_0$ $\sup\left\{\left|\sum_{n=i+1}^{j}\langle x_n,y_n\rangle\right|/(y_n)\in A\right\}\leq 1$. We have on the one hand, $\sup\left\{\left|\sum_{n>i_0}\langle x_n,y_n\rangle\right|/(y_n)\in A\right\}\leq 1$, therefore $\sup\left\{\left|\sum_{n}\langle x_n,y_n\rangle\right|/(y_n)\in A\right\}<\infty$ ($\varphi(X)\subset\mathcal{A}\ (X)$), and then $x\in\mathcal{A}\ (X)$; on the other hand, for all $i\geq i_0$ $\sup\left\{\left|\sum_{n=i+1}^{\infty}\langle x_n,y_n\rangle\right|/(y_n)\in A\right\}\leq 1$, therefore
$\sup\left\{\left|\left\langle x^{[i]}-x,(y_n)\right\rangle\right|/(y_n)\in A\right\}\leq 1$, and then $x^{[i]}\overset{i\to\infty}{\longrightarrow}x\ (\tau_{\mathcal{A}})$. ∎

Theorem 10. *Let τ a solid and polar topology of $\mathcal{A}-$convergence on $E(X)$. For $E(X)$ is a closed subspace of $(\mathcal{A}(X),\tau_{\mathcal{A}})$ it is necessary and sufficient that any Cauchy net $TK-$convergent of $E(X)$ converges in $(E(X),\tau)$.*

Proof. N.C.] A is solid for all $A\in\mathcal{A}$, therefore $A^\circ=[A\cap\varphi(X)]^\circ$.

Let $(x^i)_{i\in I}$ a Cauchy and $TK-$convergent net in $(E(X),\tau)$. For all $j\geq 1$, let $x_j\in X$ such that $(x^i_j)_{i\in I}$ converges in (X,τ_j) to x_j. τ_j is the polar topology of $\pi_j^Y(\mathcal{A})-$convergence on X. Let $A\in\mathcal{A}$, there exists $k_0\in I$ such that for all $r,s\geq k_0$ $\left|\sum_{j=1}^{N}\left\langle x^r_j-x^s_j,y_j\right\rangle\right|\leq 1$ for all $N\geq 1$ and for all $y\in A$. There exists $k_j\in I$ such that for all $r\geq k_j$, $\left|\left\langle x^r_j-x_j,y_j\right\rangle\right|\leq 1$ for all $(y_n)\in A$. Let $r_0=\max\{k_0,k_1,...,k_N\}$ for all $r\geq r_0$ we have: $\left|\sum_{j=1}^{N}\left\langle x^r_j-x_j,y_j\right\rangle\right|\leq\max_{1\leq j\leq N}\left|\left\langle x^r_j-x_j,y_j\right\rangle\right|\leq 1$ for all $(y_n)\in A$.

$\left|\sum_{j=1}^{N}\left\langle x^s_j-x_j,y_j\right\rangle\right|\leq 1$ for all $(y_n)\in A$ and for all $s\geq r_0$; therefore $x^s-x\in[A\cap\varphi(X)]^\circ$ for all $s\geq r_0$. Furthermore, $x=x^s-(x^s-x)\in\mathcal{A}(X)$. Therefore $(x^i)_{i\in I}$ converges to x in $(\mathcal{A}(X),\tau_{\mathcal{A}})$, and then $x\in E(X)$ and $(x^i)_{i\in I}$ converges to x in $(E(X),\tau)$.

S.C.] Let $(x^i)_{i\in I}$ a net in $E(X)$ which converges to x in $(\mathcal{A}(X),\tau_{\mathcal{A}})$. $(x^i)_{i\in I}$ is a Cauchy and $TK-$convergent net in $(E(X),\tau)$ $(\tau=\tau_{\mathcal{A}/E(X)})$, therefore $(x^i)_{i\in I}$ converges to x in $(E(X),\tau)$. ∎

Lemma 2. *Let L and M two $K-$ vector spaces, τ a topology on L, $L \stackrel{\pi}{\longrightarrow} M \stackrel{\delta}{\longrightarrow} L$ two linear maps such as $\pi o \delta = id_M$, and τ_δ the inverse image topology of τ by δ on M.*
The application $\psi : (M, \tau_\delta) \longrightarrow (\delta(M), \tau),\ x \longrightarrow \delta(x)$, is an homeomorphism.

Proof. If $\mathcal{U}$ is a F.S.N of 0 for τ; a F.S.N of 0 for τ_δ is $\delta^{-1}(\mathcal{U}) = \left\{\delta^{-1}(U)/U \in \mathcal{U}\right\}$, and we have: $\psi^{-1}(U \cap \delta(M)) = \delta^{-1}(U)$ for all $U \in \mathcal{U}$. ∎

Theorem 11. *Let τ a polar and solid topology of $\mathcal{A}-$convergence on $E(X)$; $(E(X), \tau)$ is complete if and only if:*
(i.) (X, τ_j) is complete for all $j \geq 1$;
(ii.) $E(X)$ is a closed subspace of $(\mathcal{A}(X), \tau_{\mathcal{A}})$.

Proof. N.C.] δ_j^X is $(\tau, \tau_j)-$closed for all $j \geq 1$ (proposition 13), therefore $\delta_j^X(X)$ is a closed subspace of $(E(X), \tau)$, hence $\left(\delta_j^X(X), \tau\right)$ is complete. Now $(\delta_j^X(X), \tau) \simeq (X, \tau_j)$ (lemma 2), therefore (X, τ_j) is complete. Furthermore $E(X)$ is a closed subspace of $(\mathcal{A}(X), \tau_{\mathcal{A}})$ (theorem 10).

S.C.] Let $(x^i)_{i\in I}$ a Cauchy net in $(E(X), \tau)$. For $j \geq 1$, $(x_j^i)_{i\in I}$ is Cauchy in (X, τ_j) so it converges, and then $(x^i)_{i\in I}$ is $TK-$convergent in $(E(X), \tau)$ so it converges in $(E(X), \tau)$, (theorem 10). ∎

Remark 2. *We can replace (ii) of theorem 11 by:*
(ii) Any Cauchy $TK-$convergent net in $(E(X), \tau)$ converges in $(E(X), \tau)$.

Corollary 8. *Let τ a polar and solid topology of $\mathcal{A}-$convergence on $E(X)$. If $E(X)$ is a closed subspace of $(\mathcal{A}(X), \tau_{\mathcal{A}})$; $(E(X), \tau)$ is sequentially complete if and only if (X, τ_j) is sequentially complete for all $j \geq 1$.*

Lemma 3. *Let τ a vector topology on $E(X)$; if τ is solid, $S_{E(X)}$ is the closure of $\varphi(X)$ in $(E(X), \tau)$.*

Proof. $S_{E(X)} \subset \overline{\varphi(X)}$. Let $x = (x_n) \in \overline{\varphi(X)}$ and U a solid neighborhood of 0, it is $z = (z_n) \in \varphi(X)$ as $x - z \in U$. Since U is solid $x^{[i]} - x \in U$ for i large enough, then $x^{[i]} \stackrel{i\to\infty}{\longrightarrow} x$ in $(E(X), \tau)$ and hence $x \in S_{E(X)}$. ∎

Proposition 21. *Let τ a solid and polar topology of $\mathcal{A}-$convergence on $E(X)$; if (X, τ_j) is complete for all $j \geq 1$, $(S_{E(X)}, \tau)$ is complete.*

Proof. $S_{E(X)} = \overline{\varphi(X)}$ (lemma 3), therefore $(S_{E(X)}, \tau)$ is a closed subspace of $(\mathcal{A}(X), \tau_{\mathcal{A}})$, and then $(S_{E(X)}, \tau)$ is complete. ∎

Application: Let $(X, \|.\|)$ a $n.a$ Banach space, we consider $m(X)$ endowed with the n.a. norm $\|.\|_\infty$. We have $c_0(X) = S_{m(X)}$, and $\|.\|_\infty$ defines a polar and solid topology on $m(X)$, therefore $(c_0(X),\ \|.\|_\infty)$ is complete.

Theorem 12. *Let τ a solid and polar topology of $\mathcal{A}-$convergence on $E(X)$; if $E(X)$ is an $AK-$space, $(E(X), \tau)$ is complete if and only if (X, τ_j) is complete for all $j \geq 1$.*

Proof. N.C.] Obvious.

S.C.] $E(X)$ is an $AK-$space, therefore $E(X) = S_{(E(X),\tau)}$. Now $S_{(\mathcal{A}(X),\tau_{\mathcal{A}})} \subset E(X)$ (proposition 20) and $S_{(E(X),\tau)} \subset S_{(\mathcal{A}(X),\tau_{\mathcal{A}})}$, therefore $E(X) = S_{(E(X),\tau)} = S_{(\mathcal{A}(X),\tau_{\mathcal{A}})}$, and then $E(X)$ is a closed subspace of $(\mathcal{A}(X), \tau_{\mathcal{A}})$. Hence $(E(X), \tau)$ is complete (theorem 11). ∎

References

[1] R. Ameziane Hassani, M. Babahmed, Topologies polaires compatibles avec une dualité séparante sur un corps valué non-Archimédien, Proyecciones Vol. 20, Núm. 2, pp. 217-240, (2001).

[2] H.R. Chillingworth, Generalised "dual" sequence spaces, Ned. Akad. Proc. Ser. A. 61, pp. 307-515, (1958).

[3] A. El amrani, R. Ameziane Hassani and M. Babahmed, Topologies on sequence spaces in non-archimedean analysis, J. of Mathematical Sciences: Advances and Applications Vol. 6, Núm. 2, pp. 193-214, (2010).

[4] T. Komura; Y. Komura, sur les espaces parfaits de suites et leurs généralisations, J. Math. Soc. Japon. 15, pp. 319-338, (1963).

[5] G. Köthe, Topological vector spaces, Springer-Verlag Berlin Heidlberg New york, (1969).

[6] -----------, Neubegründung der theorie der vollkommen Räume, Math. Nach. 4, pp. 70-80, (1951).

[7] -----------; O. Toeplitz, Lineare Raüme mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen, J. reine angew. Math. 171, pp. 193-226, (1934).

[8] G. Matthews, Generalised Rings of infinite matrices, Ned. Akad. Wet. Proc. 61, pp. 298-306 (1958).

[9] A.F.Monna, Analyse non-archimédienne, Springer-Verlag Berlin New York Heidelberg (1970).

[10] H.H. Schaefer, Topological vector spaces, Springer-Verlag Berlin New york Heidlberg, (1971).

[11] W. H. Schikhof, Locally convex spaces over nonspherically complete valued field I, II. Bull. Soc. Math. Belg. Sér. B. 38, pp. 187-224, (1986).

[12] J. Van Tiel, Espaces localement K-convexes I-III, Indag. Math. 27, pp. 249-289 (1965).

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