Proyecciones Journal of Mathematics
Vol. 31, N^o 1, pp. 1-9, March 2012.
Universidad Católica del Norte
Antofagasta - Chile

COMPACT COMPOSITION OPERATORS ON BLOCH TYPE SPACES

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Received : July 2011. Accepted : October 2011

Abstract

In this paper we characterize continuity and compactness of composition operators C_ϕ mapping the α-Bloch space into the μ-Bloch space, where μ is a weight defined on the unit disk $\mathbf{D}$, *in term of certain expression that involve the n-power of the symbol ϕ.*

Keywords : *Bloch spaces, Composition operators.*

MSC 2010 : *30D45, 47B33.*

*Partially supported by the Comisión de Investigación, Universidad de Oriente, Project CI-2-010301-1678-10.

1. Introduction and preliminaries

Let $\mathbf{D}$ denote the unit disk in the complex plane $\mathbf{C}$ and let $H(\mathbf{D})$ be the space of all holomorphic functions on $\mathbf{D}$ with the topology of uniform convergence on compact subsets of $\mathbf{D}$. A weight μ is a bounded, continuous and strictly positive function defined on $\mathbf{D}$. The μ-Bloch space, denoted as $\mathcal{B}^{\mu}$, consists of all holomorphic functions f on $\mathbf{D}$ such that

$$\|f\|_{\mu} := \sup_{z\in\mathbf{D}} \mu(z)\,|f'(z)| < \infty.$$

When $\mu(z) = 1-|z|^2$, the μ-Bloch space becomes the classical Bloch space $\mathcal{B}$ and when $\mu(z) = \left(1-|z|^2\right)^{\alpha}$, with $\alpha > 0$, we get the α-Bloch spaces, denoted as $\mathcal{B}^{\alpha}$. The μ-Bloch space is a Banach space with the norm

$$\|f\|_{\mathcal{B}^{\mu}} := |f(0)| + \|f\|_{\mu}$$

and it appears in the literature in a natural way when one studies properties of some operators in certain spaces of holomorphic functions; for instance, if $\mu_1(z) = w(z)\log\frac{2}{w(z)}$, where $w(z) = 1-|z|^2$ and $z \in \mathbf{D}$, Attele in [1] proved that the Hankel operator induced by a function f in the Bergman space is bounded if and only if $f \in B^{\mu_1}$. The space B^{μ_1} is also known as the *Log-Bloch space* as well as *weighted Bloch space*. In the last decade, there is a big interest in the investigation of Bloch-type spaces and various concrete linear operators $L : X \to Y$ where at least one of spaces X and Y is Bloch. For some other recent results in the area, see, for example, [1,...,12] and a lot of references therein.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two linear subspaces of $H(\mathbf{D})$. If ϕ is a holomorphic self-map of $\mathbf{D}$, such that $f\circ\phi$ belongs to $\mathcal{H}_2$ for all $f \in \mathcal{H}_1$, then ϕ induces a linear operator $C_{\phi} : \mathcal{H}_1 \to \mathcal{H}_2$ defined as

$$C_{\phi}(f) := f\circ\phi,$$

called the *composition operator* with *symbol* ϕ. Composition operators had been studied by numerous authors on many subspaces of $H(\mathbf{D})$ and in particular on Bloch-type spaces.

In [4], Madigan and Matheson characterized continuity and compactness for composition operators on the classical Bloch space $\mathcal{B}$. In turn, their results have been extended by Xiao [9] to the α-Bloch spaces and by Yoneda [10] to the *Log*-Bloch space. Also, in [11], Zhang and Xiao have characterized boundedness and compactness of weighted composition operators that act between μ-Bloch spaces on the unit ball of $\mathbf{C}^n$. In this case it is required that μ be a normal function. The results of Zhang and Xiao have been extended by Chen and Gauthier [2] to the μ-Bloch spaces where μ is a positive and non-decreasing continuous function such that $\mu(t) \to 0$ as $t \to 0$ and $\mu(t)/t^\delta$ is decreasing for small t and for some $\delta > 0$. Giménez, Malavé and Ramos-Fernández [3] have extended the results of Madigan and Matheson in [4], to certain μ-Bloch spaces, where the weight μ can be extended to non vanishing, complex valued holomorphic functions, that satisfy a reasonable geometric condition on the Euclidean disk $D(1,1)$. Ramos-Fernández in [5] extended all the results mentioned above to the Bloch-Orlicz spaces. The results obtained by the authors mentioned above can be summarized as follows, where μ_1 and μ_2 are the weights considered by these authors.

- The operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is continuous if and only if

$$\sup_{z\in\mathbf{D}} \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| < \infty.$$

- The composition operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is compact if and only if $\phi \in \mathcal{B}^{\mu_2}$ and

$$\lim_{|\phi(z)|\to 1^-} \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| = 0.$$

Recently, Ramos Fernández in [6], showed that the previous results is also valid for very general weights μ_1 and μ_2.

Other compactness criteria for composition operators on Bloch spaces have been found by Tjani [7]. More recently, Wulan, Zheng and Zhu (see [8]) proved the following result.

Theorem 1.1. *[8] Let ϕ be an analytic self-map of $\mathbf{D}$. Then C_ϕ is compact on the Bloch space $\mathcal{B}$ if and only if*

$$\lim_{n\to\infty} \|\phi^n\|_{\mathcal{B}} = 0.$$

Giménez, Malavé and Ramos-Fernández [3] showed that the composition operator $C_\phi : \mathcal{B} \to \mathcal{B}^\mu$ is compact if and only if $\lim_{n\to\infty} \|\phi^n\|_\mu = 0$, where the weight μ can be extended to non vanishing, complex valued holomorphic functions, that satisfy a reasonable geometric condition on the Euclidean disk $D(1,1)$. Ramos-Fernández in [5] extended the previous result to the

Bloch-Orlicz space. Recently, Zhao in [12] gave a formula for the essential norm of the composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$; in particular the operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if

$$\lim_{n\to\infty} n^{\alpha-1} \|\phi^n\|_\beta = 0.$$

The goal of this paper is to extend the previous result, that is, to show that for $\alpha > 0$ and μ any function weight,

- The operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ is continuous if and only if

$$\sup_{n\in\mathbf{N}} n^{\alpha-1} \|\phi^n\|_\mu < \infty.$$

- The composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ is compact if and only if

$$\lim_{n\to\infty} n^{\alpha-1} \|\phi^n\|_\mu = 0.$$

This paper is organized as follow: In Section 2, we characterize continuity of composition operators mapping $\mathcal{B}^\alpha$ into $\mathcal{B}^\mu$ and in Section 3 we characterize the compactness of $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$. These results extend the results mentioned before.

2. Continuity

In this section we study continuity of composition operators mapping the α-Bloch space into the μ-Bloch space. Throughout this note, we fix a positive parameter α and a weight function μ. Let ϕ be a holomorphic self-map of $\mathbf{D}$, and C_ϕ be its associated composition operator.

Theorem 2.1. *The composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ is bounded if and only if*

$$\sup_{n\in\mathbf{N}} n^{\alpha-1} \|\phi^n\|_\mu < \infty. \tag{2.1}$$

Proof. Suppose first that the composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ is bounded, then there exists a constant $L > 0$ such that $\|C_\phi f\|_{\mathcal{B}^\mu} \le L\|f\|_{\mathcal{B}^\alpha}$ for all functions $f \in \mathcal{B}^\alpha$. In particular, for any $n \in \mathbf{N}$, the function $f(z) = n^{\alpha-1} z^n$ satisfies $\|f\|_{\mathcal{B}^\alpha} \le 1$ and therefore

$$n^{\alpha-1} \|\phi^n\|_\mu \le \|C_\phi f\|_{\mathcal{B}^\mu} \le L.$$

Suppose now that $M = \sup_{n\in\mathbf{N}} n^{\alpha-1} \|\phi^n\|_\mu < \infty$, then for any $f \in \mathcal{B}^\alpha$ we have

$$\begin{aligned} \|f\circ\phi\|_\mu &= \sup_{n\in\mathbf{N}}\sup_{z\in A_n} \frac{n^\alpha\left(1-|\phi(z)|^2\right)^\alpha |\phi(z)|^{n-1}}{n^\alpha\left(1-|\phi(z)|^2\right)^\alpha |\phi(z)|^{n-1}}\mu(z)\left|f'(\phi(z))\right||\phi'(z)| \\ &\le \|f\|_\alpha \sup_{n\in\mathbf{N}}\sup_{z\in A_n} \frac{\mu(z) n^\alpha |\phi(z)|^{n-1}|\phi'(z)|}{n^\alpha\left(1-|\phi(z)|^2\right)^\alpha |\phi(z)|^{n-1}}, \qquad (2.2) \end{aligned}$$

where $A_n = \{z \in \mathbf{D} : \mathbf{r_{n-1}} \le |\phi(\mathbf{z})| < \mathbf{r_n}\}$ and

$$r_n = \left(\frac{n-1}{n-1+2\alpha}\right)^{\frac{1}{2}}.$$

By elementary calculation it is not hard to show that the function

$$h_n(t) = n^\alpha\left(1-t^2\right)^\alpha t^{n-1},$$

with $n \in \mathbf{N}$ is fixed, is decreasing for $t \in [r_{n-1}, r_n]$. Hence, there exists a constant $K_\alpha > 0$, depending only on α, such that $h_n(t) \ge K_\alpha$ for all $t \in [r_{n-1}, r_n)$. Thus, substituting this observation into (2.2) we have

$$\|f\circ\phi\|_\mu \le \frac{1}{K_\alpha}\|f\|_\alpha \sup_{n\in\mathbf{N}}\sup_{z\in A_n} \mu(z) n^\alpha |\phi(z)|^{n-1}|\phi'(z)| \le \frac{M}{K_\alpha}\|f\|_\alpha.$$

Also, since f is analytic on $\mathbf{D}$, we can find a constant $C > 0$, depending only on α and $\phi(0)$, such that

$$|f(\phi(0))| \le |f(0)| + \int_0^{\phi(0)} \frac{(1-|s|^2)^\alpha}{(1-|s|^2)^\alpha}|f'(s)||ds| \le |f(0)| + C\|f\|_\alpha.$$

We conclude that

$$|f(\phi(0))| + \|f\circ\phi\|_\mu \le |f(0)| + \left(\frac{M}{K_\alpha} + C\right)\|f\|_\alpha$$

and the composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ is continuous.

3. Compactness

Now, we are going to characterize compactness of composition operators mapping $\mathcal{B}^\alpha$ into $\mathcal{B}^\mu$. Our goal is to obtain extensions of the results in [8, 12]. In [7], Tjani showed the following result.

Lemma 3.1. *Let X, Y be two Banach spaces of analytic functions on $\mathbf{D}$. Suppose that*

1. *The point evaluation functionals on X are continuous.*
2. *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*
3. *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

An immediate consequence of the previous lemma is the following criterion.

Lemma 3.2. *The composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ is compact if and only if given a bounded sequence $\{f_n\}$ in $\mathcal{B}^\alpha$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$, then $\|C_\phi(f_n)\|_\mu \to 0$ as $n \to \infty$.*

Next we establish our criterion for the compactness of the composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$.

Theorem 3.3. *The composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ is compact if and only if*

$$\lim_{n\to\infty} n^{\alpha-1} \|\phi^n\|_\mu = 0. \tag{3.1}$$

Proof. Let us suppose first that $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ is compact. Observe that the sequence $\{f_n\}$, given by $f_n(z) = n^{\alpha-1} z^n$ with $z \in \mathbf{D}$ is bounded in the α-Bloch space, in fact, we have

$$\|f_n\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbf{D}} \left(1 - |z|^2\right)^\alpha n^\alpha |z|^{n-1} \le 1$$

for any $n \in \mathbf{N}$. Furthermore, the sequence $\{f_n\}$ converges to 0 uniformly on compact subset of $\mathbf{D}$, then by [7, Lemma 2.10] we have

$$\lim_{n\to\infty} n^{\alpha-1} \|\phi^n\|_\mu = \lim_{n\to\infty} \|f_n \circ \phi\|_\mu = 0.$$

The proof of the converse is similar to the one given in [8, 12]. In this case, for $k \ge 1$ we set

$$r_k = \left(\frac{k-1}{k-1+2\alpha}\right)^{\frac{1}{2}}$$

and we consider the sets

$$A_k = \{z \in \mathbf{D} : \mathbf{r_{k-1}} \leq |\phi(\mathbf{z})| < \mathbf{r_k}\}.$$

Then there exists a constant $K_\alpha > 0$, depending only on α, such that

$$h_k(t) = k^\alpha \left(1 - t^2\right)^\alpha t^{k-1} \geq K_\alpha \tag{3.2}$$

for all $r_{k-1} \leq t < r_k$ and for all $k \geq 1$.

Let $\{f_n\}$ be a bounded sequence in $\mathcal{B}^\alpha$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$ and let us suppose that $\epsilon > 0$. By hypothesis, there exists a constant $M > 0$ such that

$$n^{\alpha-1} \|\phi^n\|_\mu \leq M$$

for all $n \in \mathbf{N}$. Also, there exists $N \in \mathbf{N}$ such that

$$n^{\alpha-1} \|\phi^n\|_\mu < \epsilon \tag{3.3}$$

whenever $n \geq N$. Furthermore, since $f_n' \to 0$ uniformly on compact subsets of $\mathbf{D}$, there exists $N_1 \in \mathbf{N}$ such that

$$\sup_{|w| \leq r_N} \left(1 - |w|^2\right)^\alpha |f_n'(w)| < \epsilon \tag{3.4}$$

whenever $n \geq N_1$. Thus, for $1 \leq k \leq N$ and $n \geq N_1$, we have

$$\begin{aligned} \sup_{z \in A_k} \mu(z) \left|(f_n \circ \phi)'\right| &\leq k^{\alpha-1} \left\|\phi^k\right\|_\mu \sup_{z \in A_k} \frac{\left(1 - |\phi(z)|^2\right)^\alpha |f_n'(\phi(z))|}{k^\alpha \left(1 - |\phi(z)|^2\right)^\alpha |\phi(z)|^{k-1}} \\ &\leq \frac{\epsilon}{K_\alpha} k^{\alpha-1} \left\|\phi^k\right\|_\mu \leq \frac{M}{K_\alpha} \epsilon. \end{aligned}$$

While if $k > N$, then

$$\begin{aligned} \sup_{z \in A_k} \mu(z) \left|(f_n \circ \phi)'\right| &\leq \|f_n\|_{\mathcal{B}^\alpha} \sup_{z \in A_k} \frac{\mu(z)\, k^\alpha |\phi(z)|^{k-1} |\phi'(z)|}{k^\alpha \left(1 - |\phi(z)|^2\right) |\phi(z)|^{k-1}} \\ &\leq \frac{\|f_n\|_{\mathcal{B}^\alpha}}{K_\alpha} k^{\alpha-1} \left\|\phi^k\right\|_\mu < \frac{\epsilon}{K_\alpha} \|f_n\|_{\mathcal{B}^\alpha}. \end{aligned}$$

Finally, we conclude that

$$\|f_n \circ \phi\|_\mu \leq \frac{\epsilon}{K_\alpha} \left(M + \|f_n\|_{\mathcal{B}^\alpha}\right)$$

whenever $n \geq N_1$ and $\lim_{n\to\infty} \|f_n \circ \phi\|_\mu = 0$ since $\{f_n\}$ is a bounded sequence in $\mathcal{B}^\alpha$. Finally, since $f_n \circ \phi(0) \to 0$ as $n \to \infty$, we conclude that

$$\|f_n \circ \phi\|_{\mathcal{B}^\mu} = |f_n \circ \phi(0)| + \|f_n \circ \phi\|_\mu \to 0$$

as $n \to \infty$, which means that $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\mu$ is a compact operator.

Acknowledgment

The authors wish to express their sincere gratitude to the anonymous referee for a thorough review and insightful suggestions.

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