

Schauder basis in a locally K -convex space and perfect sequence spaces

R. AMEZIANE

UNIVERSITÉ SIDI MOHAMED BEN ABDELLAH, MOROCCO

A. EL AMRANI

UNIVERSITÉ SIDI MOHAMED BEN ABDELLAH, MOROCCO

and

M. BABAHMED

UNIVERSITÉ MOULAY ISMAIL, MOROCCO

Received : July 2010. Accepted : September 2011

Abstract

In this work, we are dealing with the natural topology in a perfect sequence space and the transfert of topologies of a locally K -convex space E with a Schauder basis $(e_i)_i$ to such Space. We are also interested with the compatible topologies on E for which the basis $(e_i)_i$ is equicontinuous, and the weak basis problem. Finally, we give some applications to barrelled Spaces and G -Spaces.

Keywords : *non archimedean analysis, locally K -convex spaces, Schauder basis, the weak basis theorem, compatible topologies, perfect sequence spaces, K -barrelled spaces and G -spaces.*

AMS subject class. (2000): 46S10

1. Introduction

The perfect sequence spaces on a field K have been widely studied by several authors, either in the classical case ($K = \mathbb{R}$ or $K = \mathbb{C}$) Garling [18] and [19], Köthe [23], ..., or in the case of K is a non-archimedean valued field Monna [25], Dorleyn [14], De Grande-De Kimpe [8], The importance of the sequence spaces lies on the fact that each space which is locally convex and having a Schauder basis is isomorphic to the sequence space. Thus, instead of studying the spaces that are locally convex and having a Schauder basis one only has to study the sequence spaces.

In this work we are going to establish a way of transforming topologies between this space and a perfect sequence space Λ , where

$$\Lambda = \left\{ (\lambda_i)_i \in \omega : \sum_{i=1}^{\infty} \lambda_i e_i \text{ converges in } (E, \tau) \right\} \text{ and } (e_i)_i \text{ is a Schauder}$$

basis of a locally K -convex space (E, τ) in question (K is a non-archimedean valuated field complete with a non trivial valuation). This study will allow us to solve the following problem:

(1) if $(e_i)_i$ is a Schauder basis of a locally K -convex space (E, τ) ($lKcs$), determine the compatible topologies on E for which $(e_i)_i$ is an equicontinuous Schauder basis.

This problem was studied by many Mathematicians, in particular by De Grande-De Kimpe [10]. It is also proved in ([22], 3.2. see also [32], 2.1 and [12], 2.1) that in a $lKcs$ (E, τ) there exists the finest locally K -convex topology ν of countable type compatible with τ . The existence of this topology was also proved in ([21], proposition 2, p. 153). Thus, we are going to make, in the case of $lKcs$ (E, τ) such that $E_\sigma = (E, \sigma(E, E'))$ and $E'_\sigma = (E', \sigma(E', E))$ are sequentially complete, this topology in relation with the original topology of E , by distinguishing the following three cases: K is local, K is spherically complete and K is not spherically complete; which will give us a complete solution of this problem, we'll give a characterization of polar topologies for which the weak basis problem is true in the case when K is not spherically complete. We should remind that the problem of the weak basis was formulated by several ways and that's one of them ([21], p. 150)

(2) Is every weak Schauder basis for E a Schauder basis for E ?

We shall say that for a $lKcs$ E the weak basis theorem holds if every weak Schauder basis in E is a Schauder basis .

In archimedean analysis, the weak basis theorem was first given for Banach spaces in 1932 by Banach ([3], p. 238) and extended to (F) -spaces

by Bessaga and Pełczyński [5] ($a(F)$ -space is a complete, metrizable topological vector space). McArthur [24] proved an analogue for bases of subspaces in Fréchet spaces. Arsove and Edwards [1] proved that the answer is positive if E is a barrelled space. Singer shows by an example ([33], p. 153) that a weak basis need not be Schauder basis. Dubinsky and Retherford [15] observed that the answer is negative in general. Bennet and Cooper [4] proved it for strict (LF) -spaces and Floret [17] for sequentially retractive (LF) -spaces. M. De Wilde [13] obtained a rather general result for bornological, sequentially complete and webbed spaces. W. J. Stiles ([35], corollary 4.5, p. 413) showed that the theorem fails in non locally convex spaces l^p ($0 < p < 1$). N. J. Kalton [20] gave a class of spaces for which this theorem is true. Joel H. Shapiro ([29], theorem 1, p. 1294) gave the following generalization of Stiles' result

The weak basis theorem fails in every locally bounded non locally convex (F) -space which has a weak basis; he also gave a wide class of space which the weak basis theorem fails and proved the same for the space H^p ($H^p :=$ the linear space of functions f analytic in the open unit disc $|z| < 1$ such that $\|f\|_p^p = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})|^p dt < \infty$). Efimova [16] proved the weak basis theorem for regular inductive limits of a sequence of normed barrelled spaces. M. Valdivia has shown the result for metrizable barrelled spaces. J. Orihuela [27] gave a result which showed the linking between the weak basis theorem and the closed graph theorem. In [10] N. De Grande-De Kimpe solved completely the weak basis problem for locally convex space (lcs) E having a $\sigma(E', E)$ -sequentially complete topological dual E' .

In *n.a* analysis, the weak basis problem has the following simple solution [21]: For a $lKcs$ E with a weak Schauder basis the weak basis theorem holds iff E is an Orlicz-Pettis-space (a space, where every weakly convergent sequence is convergent). We know that, if K is spherically complete then every $lKcs$ E is an OP -space; so the weak basis theorem holds in this case. If K is not spherically complete J. Kąkol and T. Gilsdorf ([21], corollary 6, p. 155) proved that the weak basis theorem holds if E is a polarly barrelled polar space (a locally K -convex space is called polarly barrelled if every closed, polar and absorbing absolutely K -convex subset of E is a zero-neighbourhood), they provided a wide class of non-polar spaces E with a weak Schauder basis which is a basic sequence in the original topology of E ([21], example 7, p. 155 and 156). Finally J. Kąkol and T. Gilsdorf remark that they do not know if the following result is true: Let E be a Banach

space with a weak Schauder basis; then, E is a polar space iff every weak Schauder basis in E is a basic sequence ([21], remark 13, p. 160). This conjecture was set by P. Garcia and Schikhof ([28], p. 233) in the form of the following question: Does there exist a polar Banach space which is not OP -space, and does not contain l^∞ ? . The answer of this question is negative; in the other words, the conjecture is true (see proposition 33).

In §. 2 we'll give general results that are related to $lKcs$, to polar topologies of \mathcal{A} -convergence and to space of sequences. In §. 3 we study the perfect sequence spaces over K , we give a characterization of the natural topology noted Na with the sets absolutely K -convex, weakly bounded and compactoid if K is spherically complete, with the sets absolutely K -convex and compact if K is local and with the sets absolutely K -convex weakly bounded and Λ -closed (Λ is a perfect sequence space on K) when K is not spherically complete. We are interested in §. 4 in the study of transfer of topologies between a perfect sequence spaces and a $lKcs (E, \tau)$ that has a Schauder basis, using the two following algebraic isomorphisms

$$\Phi : E \longrightarrow \Lambda \quad x \longmapsto (\lambda_i)_i \text{ and } \Psi : E' \longrightarrow \Delta \quad f \longmapsto (\mu_i)_i \text{ for every } x = \sum_{i=1}^{\infty} \lambda_i e_i \text{ and } f = \sum_{i=1}^{\infty} \mu_i f_i ;$$

where $(f_i)_i$ is the weak Schauder basis of E' associated to the Schauder basis $(e_i)_i$ of E ([9], lemma 3, p. 402) and Λ and Δ are two sequence spaces which we'll define like in [9]. This study will allow us to solve the problem (1) by distinguishing the three cases K is local, K is spherically complete and K is not spherically complete. Some results that are related to problem (2) are given in the §. 5 by considering a polar $lKcs (E, \tau)$ which has a weak Schauder basis $(e_i)_i$ and as E_σ and E'_σ are sequentially complete, we characterize the finest compatible topology on E for which $(e_i)_i$ is a Schauder basis; this basis is necessary equicontinuous. Then, we give a necessary and sufficient condition which the topology τ must fulfill so as to admit $(e_i)_i$ as Schauder basis in the case when K is non spherically complete. Then we deduce a new characterization of OP -spaces.

Finally, in §. 6 we give applications to G -spaces and to K -barrelled spaces. We show that the result established by N. De Grande- De Kimpe in [10] in the classical case, for the barrelled spaces, is also true in the non archimedean case. For a G -space (E, τ) ; we show that τ is the only topology on E compatible with the duality $\langle E, E' \rangle$ for which $(e_i)_i$ is an equicontinuous Schauder basis.

2. Preliminaries

1. Throughout K is a non-archimedean (n.a) non trivially valued complete field with the valuation $|\cdot|$, and the valuation ring is $B(0, 1) = \{\lambda \in K : |\lambda| \leq 1\}$.

2. Let E be a K -vector space. A subset A of E is absolutely K -convex if it is $B(0, 1)$ -module. For a set $X \subset E$ its absolutely K -convex hull $\Gamma(X)$ is the smallest absolutely K -convex set containing X .

3. A topology on a vector space E over K is said to be locally K -convex if there exists in E a fundamental system of zero-neighbourhoods consisting of absolutely K -convex subsets of E .

In this paper the letter E will always stand for Hausdorff locally K -convex space over a field K .

4. A subset A of E is called compactoid if for every zero-neighbourhood U in E , there exists a finite set $F \subset E$ such that $A \subset \Gamma(F) + U$.

5. A subset A of E is called c -compact if every convex filter on A has a cluster point on A .

- An absolutely K -convex subset of a locally K -convex space E is called K -closed if for every $x \in E$ the set $\{|\lambda| / \lambda \in K : \lambda x \in A\}$ is closed in $|K|$; the K -closed hull of A is the smallest subset of E which is K -closed and contains A , it is denoted by $K_c(A)$.

6. A sequence $(e_i)_i$ is a Schauder basis of E if every $x \in E$ can be written uniquely as $x = \sum_{i=1}^{\infty} \lambda_i e_i$ where the coefficient functionals $f_n : x = \sum_{i=1}^{\infty} \lambda_i e_i \mapsto \lambda_n$ are continuous.

- The sequence $(f_n)_n$ is called the weak Schauder basis associated to basis $(e_i)_i$.

- For every $n \geq 1$, let p_n the map $x = \sum_{i=1}^{\infty} \lambda_i e_i \mapsto \lambda_n e_n$; the Schauder basis $(e_i)_i$ is called equicontinuous if the sequence $(p_n)_n$ is equicontinuous on E , this is equivalent to the equicontinuity of the sequence $(S_n)_n$ where $S_n : x = \sum_{i=1}^{\infty} \lambda_i e_i \mapsto \sum_{i=1}^n \lambda_i e_i$, for every $n \geq 1$.

7. Let $\langle \cdot, \cdot \rangle$ be a duality between E and F where E and F are two vectors spaces over K (see [2] for general results);

- If A is a subset of E , the polar of A is a subset of F defined by: $A^\circ = \{y \in F / |\langle x, y \rangle| \leq 1 \text{ for all } x \in A\}$.

We define also the polar of a subset B of F in the same way.

- The weak topology $\sigma(E, F)$ on E is noted simply σ and $\{A^\circ / A \in \mathcal{F}\}$ is a zero-neighbourhood base, where \mathcal{F} is the set of finite subset of F .

- A subset A of F is said to be E -closed if for every $y \in F \setminus A$, there exists $x \in E$ such that $|\langle x, y \rangle| > 1$ and $|\langle x, A \rangle| \leq 1$; the E -closed hull $E_c(A)$ of A is the smallest E -closed subset of F containing A .

Proposition 1. *Let A be an absolutely K -convex subset of F , then A is E -closed, if and only if, A is K -closed and $\sigma(F, E)$ -closed.*

Proof. By [2], theorem 4.2, p. 233, proposition 2.5, p. 224 and corollary 4.3, p. 233. ■

8. Let \mathcal{A} be a family of $\sigma(F, E)$ -bounded subsets of F such that

(a) \mathcal{A} is directed by inclusion,

(b) $F = \bigcup_{A \in \mathcal{A}} A$,

(c) there exists $\lambda_0 \in K, |\lambda_0| > 1$, such that $\lambda_0 A \in \mathcal{A}$, for all $A \in \mathcal{A}$.

A topology τ on E is called polar topology of \mathcal{A} -convergence, if τ has a fundamental system of zero-neighbourhoods consisting of $\{A^\circ / A \in \mathcal{A}\}$

- A vector topology τ on E is called polar topology if there exists a family \mathcal{A} of $\sigma(F, E)$ -bounded subsets of F which has the properties (a), (b) and (c), such that τ is a polar topology of \mathcal{A} -convergence.

- If τ is a polar topology of \mathcal{A} -convergence on E , it is determined by the family of n.a seminorms $(p_A)_{A \in \mathcal{A}}$, where $p_A(x) = \sup \{|\langle x, y \rangle| / y \in A\}$ ([10], p. 277).

- If \mathcal{A} is the family of all subsets of F that are:

1. Absolutely K -convex, weakly bounded and weakly c-compact, we have the c-compact topology $\tau_c(E, F) = \tau_c$,

2. Absolutely convex and $\sigma(F, E)$ -compact, we have the Mackey topology $\tau_m(E, F) = \tau_m$,

3. $\sigma(F, E)$ -bounded and E -closed, we have the E -closed topology $\tau_e(E, F) = \tau_e$.

9. A locally K -convex topology τ on E is called compatible with the duality $\langle E, F \rangle$ or (E, F) -compatible if, F is isomorphic to the topological dual of E provided with the topology τ . $\sigma(E, F)$ is the smallest of (E, F) -compatible topology.

- A sequence $(e_n, f_n)_n$ of $E \times F$ is called biorthogonal if $\langle e_n, f_n \rangle = \delta_{nm}$, for all n, m where δ_{nm} is the Kronecker delta.

10. The space of all sequences in K is denoted by ω , it is provided with the product topology τ_ω . A linear subspace of ω is called a sequence space.

φ , c_0 and l^∞ are respectively, the space of all sequences in K with only finitely many non-zero terms, the space of the sequences in K converging to zero and the space of the bounded sequences in K .

- for all $n \geq 1$, $e^n = (\delta_{nm})_m$.

- Let $A \subset \omega$, the β -dual of A is the subset A^β of ω defined as

$$A^\beta = \left\{ \lambda = (\lambda_i) \in \omega / \lim_i \lambda_i \alpha_i = 0, \text{ for all } \alpha = (\alpha_i)_i \in A \right\}.$$

- A is called β -perfect (or perfect) if $A = A^{\beta\beta}$.

- A is solid if whenever $(a_n)_n \in A$ and $(\lambda_n)_n \in \omega$ such that $|\lambda_n| \leq 1$ for each n , then $(\lambda_n a_n)_n \in A$. The spaces ω, φ and c_0 are solids.

- The smallest solid subset of ω containing A is called the solid hull of A , it is denoted by \hat{A} ; and we have

$$\hat{A} = \{ (\lambda_n a_n)_n / (a_n)_n \in A \text{ and } (\lambda_n)_n \in \omega : |\lambda_n| \leq 1 \text{ for all } n \geq 1 \}.$$

- Let X be a sequence space in K ; $A \subset X$ is called solid in X if $\hat{A} \cap X = A$. $\hat{A} \cap X$ is called the solid hull of A in X .

- A topology on a vector space X is called solid if there exists in X a fundamental system of zero neighbourhoods consisting of solids subsets in X .

11. A G-space is a locally K -convex space (E, τ) such that E' is $\sigma(E', E)$ -sequentially complete and $\tau = \tau_c$ (resp. τ_e, τ_m) if K is spherically complete, (resp. not spherically complete, local). In the last case, we find the notion of G-space given and studied by N. De Grande-De Kimpe in [11] in the classical case ($K = \mathbb{R}$ or $K = \mathbb{C}$).

3. The natural topology in a perfect sequence spaces

Let Λ be a sequence space over K containing φ , we consider the duality $\langle \Lambda, \Lambda^\beta \rangle$ defined by: $((\lambda_n)_n, (\mu_n)_n) \mapsto \langle (\lambda_n)_n, (\mu_n)_n \rangle = \sum_{n=1}^{\infty} \lambda_n \mu_n$ for

every $(\lambda_n)_n \in \Lambda$ and $(\mu_n)_n \in \Lambda^\beta$.

For every $\mu = (\mu_n)_n \in \Lambda^\beta$, let \hat{p}_μ the n.a seminorm defined as $\hat{p}_\mu(\lambda) = \sup_n |\lambda_n \mu_n|$, For every $\lambda = (\lambda_n)_n \in \Lambda$.

We call the locally K -convex topology on Λ determined by the family of seminorms $(\hat{p}_\mu)_{\mu \in \Lambda^\beta}$ the natural topology; it will be denoted by Na .

Remark 1. The weak topology σ on Λ is weaker than the natural topology Na .

Proposition 2. If Λ is perfect, then it is weakly sequentially complete.

Proof. ([25], 5.2, p. 1550). ■

Proposition 3. *If Λ is perfect then every σ -bounded subset of Λ is τ_b -bounded, where τ_b is the strong topology $\tau_b(\Lambda, \Lambda^\beta)$ on Λ .*

Proof. ([8], proposition 8, p. 476). ■

Corollary 1. *If Λ is perfect, all polars topologies on Λ yield the same bounded sets.*

Lemma 1. *The solid hull of a finite subset of Λ^β is σ -bounded.*

Proof. Obvious. ■

Lemma 2. *Let A be a σ -bounded and solid subset of Λ^β , then the polar of A in the duality $\langle \Lambda, \Lambda^\beta \rangle$ is given by $A^\circ = \{\lambda \in \Lambda / \hat{p}_\mu(\lambda) \leq 1, \text{ for all } \mu \in A\}$.*

Proof. ([8], proposition 1, p. 472). ■

Proposition 4. *The natural topology on Λ is a polar topology.*

Proof. Obvious. ■

Remark 2. *The natural topology is a solid topology; in fact it is the coarsest of the polar and solid topologies on Λ .*

Proposition 5. *If Λ is perfect, the natural topology N_a is compatible with the duality $\langle \Lambda, \Lambda^\beta \rangle$.*

Proof. The N_a topology is polar and for every $\mu \in \Lambda^\beta$, $(\{\hat{\mu}\})^\circ = \left[K_c \left(\overline{\{\hat{\mu}\}}^{\sigma(\Lambda^\beta, \Lambda)} \right) \right]^\circ$ [2], corollary 4.3, p. 233. Then, if we take $\mathcal{A} = \left\{ K_c \left(\overline{A}^{\sigma(\Lambda^\beta, \Lambda)} \right) / A \subset \Lambda^\beta \text{ and } A \text{ is finite} \right\}$ so N_a is a polar topology of \mathcal{A} -convergence, where \mathcal{A} is formed by a $\sigma(\Lambda^\beta, \Lambda)$ -bounded and Λ -closed subsets of Λ^β (proposition 1). Then by [2], theorem 4.3, p. 233 the natural topology N_a is compatible. ■

Remark 3. *For every $\mu \in \Lambda^\beta$, $\overline{\Gamma(\{\hat{\mu}\})}$ is weakly-c-compact if K is spherically complete and weakly compact if K is local.*

Proposition 6. *If Λ is perfect, then it is complete under any polar solid topology.*

Proof. Let τ be a solid polar topology of \mathcal{A} -convergence on Λ ; we consider $(\lambda^i)_{i \in I}$ as a Cauchy-net in (Λ, τ) .

Let $A \in \mathcal{A}$, there exists $i_0 \in I$ such that $\lambda^i - \lambda^j \in A^\circ$, for all $i, j \geq i_0$; so we have (1) $\sup_{\alpha = (\alpha_n)_n \in A} \sup_n |\alpha_n (\lambda_n^i - \lambda_n^j)| \leq 1$, for all $i, j \geq i_0$ (lemma 2).

Then for every n , $(\lambda_n^i)_{i \in I}$ is a Cauchy-net in K , so there exists $\lambda_n \in K$ such that $\lambda_n = \lim_i \lambda_n^i$. Therefore, from (1) we obtain:

$$(2) \quad \sup_{\alpha = (\alpha_n)_n \in A} \sup_n |\alpha_n (\lambda_n^i - \lambda_n)| \leq 1, \text{ for all } i \geq i_0.$$

Let $\alpha = (\alpha_n)_n \in \Lambda^\beta$, there exists $A \in \mathcal{A}$ such that $\alpha \in A$ and we have for all $n \geq 1$, $|\alpha_n \lambda_n| \leq \max(|\lambda_n^{i_0} \alpha_n|, |(\lambda_n^{i_0} - \lambda_n) \alpha_n|)$.

Hence $\lambda = (\lambda_n)_n \in \Lambda^{\beta\beta} = \Lambda$ (Λ is perfect) and by (2) we have $\lambda = \lim_i \lambda^i$ in (Λ, τ) . ■

Proposition 7. *Let A be a subset of Λ ; if A is Na -bounded then \hat{A} is Na -bounded.*

Proof. Obvious. ■

Proposition 8. *Suppose that Λ is perfect and let τ be a polar topology on Λ and A be a subset of Λ . If A is τ -bounded, then \hat{A} is τ -bounded.*

Proof. A is τ -bounded $\Rightarrow A$ is Na -bounded (corollary 1 and proposition 4) $\Rightarrow \hat{A}$ is Na -bounded (proposition 7) $\Rightarrow \hat{A}$ is τ -bounded (corollary 1 and proposition 4). ■

Corollary 2. τ_b is a solid topology.

Proof. Λ^β is perfect, then for every $A \subset \Lambda^\beta$, A is σ -bounded $\iff \hat{A}$ is σ -bounded. ■

Lemma 3. *Let E and F be a locally K -convex spaces and A a compactoid subset of E . If $(f_n)_{n \geq 1}$ is an equicontinuous sequence of linear mappings from E to F pointwise converging to a mapping f , then $(f_n)_{n \geq 1}$ converges to f uniformly on A .*

Proof. ([8], proposition 13, p. 477). ■

Remark 4. *The lemma before is true if we replace compactoid by precompact (if K is local) or bounded and c -compact (if K is spherically complete).*

Proposition 9. *Let A be a compactoid subset of (Λ, Na) . Then for every $\alpha = (\alpha_n)_n \in \Lambda^\beta, \lim_k |\alpha_k| \sup_{\lambda \in A} |\lambda_k| = 0$.*

Proof. Let $\alpha = (\alpha_n)_n \in \Lambda^\beta$; for every $n \in \mathbb{N}$, we consider $\alpha^n = \alpha_n e^n$ with $e^n = (\delta_{nm})_m$. Then for every $\mu = (\mu_n)_n \in \Lambda, \hat{p}_\mu(\alpha^n) = |\mu_n \alpha_n| \xrightarrow{n \rightarrow \infty} 0$, so $\lim_{n \rightarrow \infty} \alpha^n = 0$ in (Λ^β, Na) . On the other hand $(\alpha^n)_n$ is Na -equicontinuous ([8], proposition 3, p. 474); then according to lemma before $(\alpha^n)_n$ converges to 0 uniformly on A . ■

Remark 5. *Suppose that K is spherically complete and let τ be a locally K -convex topology compatible with the duality $\langle \Lambda, \Lambda^\beta \rangle$ on Λ and A an absolutely K -convex bounded and c -compact subset of Λ in (Λ, τ) . Then for every $\alpha \in \Lambda^\beta, \lim_k |\alpha_k| \sup_{\lambda \in A} |\lambda_k| = 0$.*

Proof. Remark 4, proposition 5, [36] theorem 4.21 and [7] proposition 3. ■

Proposition 10. *The sequence $(e^n)_n$ is a Schauder basis of (Λ, Na) .*

Proof. Let $\lambda = (\lambda_n)_n \in \Lambda$, then for every $\mu = (\mu_n)_n \in \Lambda^\beta, \hat{p}_\mu(\lambda_i e^i) = |\lambda_i \mu_i| \xrightarrow{i \rightarrow \infty} 0$. Therefore $\sum_i \lambda_i e^i$ converges in (Λ, Na) and so every element

$\lambda = (\lambda_n)_n \in \Lambda$ can be written uniquely as $\lambda = \sum_{n=1}^\infty \lambda_n e^n$. On the other hand, for every $n \in \mathbb{N}, e^n \in \Lambda^\beta$ and we have $\hat{p}_{e^n}(\lambda) = |\lambda_n|$ for all $\lambda = (\lambda_n)_n \in \Lambda$, hence the maps $\lambda = \sum_{i=1}^\infty \lambda_i e^i \mapsto \lambda_n$ is Na -continuous. ■

Theorem 1. *Suppose that K is spherically complete and Λ is perfect. A subset A of Λ is compactoid in Λ_{Na} if, and only if, it is a subset of the solid hull of a singleton of Λ .*

Proof. \implies] Let A be a compactoid subset of Λ_{Na} .

Let $\varrho > 1$ and $\lambda = (\lambda_n)_n \in \omega$ such that $|\lambda_n| = \varrho^n$ for all $n \geq 1$.

A is compactoid in Λ_{Na} , so it is Na -bounded, and therefore $\sup_{(\alpha_i)_i \in A} |\alpha_i| < +\infty$, for all $i \geq 1$, there exists $n_i \geq 1$ such that $\varrho^{n_i-1} \leq \sup_{(\alpha_i)_i \in A} |\alpha_i| \leq \varrho^{n_i}$.

Let $\mu = (\mu_i)_i$ the element of ω given by $\mu_i = \lambda_{n_i}$ for all $i \geq 1$, then for every $\alpha = (\alpha_i)_i \in \Lambda^\beta$ we have: for all $i \geq 1$,

$$|\mu_i \alpha_i| \leq |\alpha_i| \left(\varrho^{n_i} - \sup_{(\gamma_i) \in A} |\gamma_i| \right) + |\alpha_i| \sup_{(\gamma_i) \in A} |\gamma_i| \leq \varrho |\alpha_i| \sup_{(\gamma_i) \in A} |\gamma_i|.$$

Now, $\lim_i |\alpha_i| \sup_{(\gamma_i) \in A} |\gamma_i| = 0$ (proposition 9), hence $\lim_i \mu_i \alpha_i = 0$. Then

$\mu \in \Lambda^{\beta\beta} = \Lambda$. On the other hand, if $\alpha = (\alpha_i)_i \in A$, we have $|\alpha_i| \leq |\mu_i|$, for every $i \geq 1$, hence $\alpha \in \{\hat{\mu}\}$.

\Leftarrow] It suffices to prove that $\{\hat{\lambda}\}$ is compactoid in Λ_{Na} , for every $\lambda \in \Lambda$.

Let $\lambda = (\lambda_n)_n$ an element of Λ , then for every $\alpha = (\alpha_n)_n$ in Λ^β , there exists $n_0 \in \mathbb{N}$ such that $|\lambda_n \alpha_n| \leq 1$ for all $n > n_0$. We put $\lambda^i = \lambda_i e^i$, for all $i, 1 \leq i \leq n_0$.

If $\mu = (\mu_i \lambda_i)_i$ is an element of $\{\hat{\lambda}\}$,
$$\mu = \sum_{i=1}^{\infty} \mu_i \lambda_i e^i = \sum_{i=1}^{n_0} \mu_i \lambda^i + \sum_{i>n_0} \mu_i \lambda_i e^i.$$

Now,
$$\hat{p}_\alpha \left(\sum_{i>n_0} \mu_i \lambda_i e^i \right) = \sup_{i>n_0} |\mu_i \lambda_i \alpha_i| \leq 1$$

So $\mu \in \Gamma(\lambda^1, \lambda^2, \dots, \lambda^{n_0}) + B_{\hat{p}_\alpha}(0, 1)$, where $B_{\hat{p}_\alpha}(0, 1) = \{\lambda / \hat{p}_\alpha(\lambda) \leq 1\}$.

Then $\{\hat{\lambda}\} \subset \Gamma(\lambda^1, \lambda^2, \dots, \lambda^{n_0}) + B_{\hat{p}_\alpha}(0, 1)$. ■

Remarks 1. 1. *N. De Grande-De Kimpe gave an analogue proposition of theorem in which she characterize the weakly c-compact subsets ([8], proposition 15, p. 478). This proposition is in fact true for all compatible topologies with the duality $\langle \Lambda, \Lambda^\beta \rangle$ in particular for the natural topology Na .*

2. $K_c \left(\overline{\{\hat{\mu}\}}^{\sigma(\Lambda, \Lambda^\beta)} \right)$ is compactoid for every field K .

Consequently the solid hull of a bounded subset which is absolutely K -convex and c-compact of Λ is also bounded and c-compact for every compatible topology.

Corollary 3. *i. For every $\lambda \in \Lambda^\beta, \{\hat{\lambda}\}$ is compactoid in (Λ^β, Na) for every field K .*

ii. Suppose that Λ is perfect and K is spherically complete and let A be a subset of Λ . Then A is compactoid in Λ_{Na} if, and only if, there exists a sequence $(\alpha^n)_n$ converging to zero in Λ_{Na} such that $A \subset \overline{\Gamma(\alpha^1, \alpha^2, \dots, \alpha^n, \dots)}$.

Proof. i. Λ^β is perfect; it suffices to use the theorem 1.

ii. Let $\lambda = (\lambda_n)_n \in \Lambda$ such that $A \subset \{\hat{\lambda}\}$ (theorem 1). We put $\alpha^n = \lambda_n e^n$ for all $n \geq 1$, then for every $\mu = (\mu_n)_n \in \Lambda^\beta$ we have $\hat{p}_\mu(\alpha^n) = |\mu_n \lambda_n| \xrightarrow{n \rightarrow \infty} 0$, so the sequence $(\alpha^n)_n$ is converging to zero in Λ_{Na} . On the other hand for every $a = (a_n)_n \in A$, $a \in \{\hat{\lambda}\}$, therefore there exists $(\mu_n)_n \in \omega$ such that $|\mu_n| \leq 1$ for all n and $a = (\mu_n \lambda_n)_n$; so $a = \sum_{n=1}^{\infty} \mu_n \alpha^n$ (proposition 10). Then $a \in \overline{\Gamma(\alpha^1, \alpha^2, \dots, \alpha^n, \dots)}$.

Conversely, suppose that $A \subset \overline{\Gamma(\alpha^1, \alpha^2, \dots, \alpha^n, \dots)}$ where $(\alpha^n)_n$ converges to zero in Λ_{Na} . Let U be a zero-neighbourhood in Λ_{Na} , then there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\alpha^n \in U$. So $\Gamma(\alpha^1, \alpha^2, \dots, \alpha^n, \dots) \subset \Gamma(\alpha^1, \alpha^2, \dots, \alpha^{n_0}) + U$, (we can choose U absolutely K -convex and open). Then $\overline{\Gamma(\alpha^1, \alpha^2, \dots, \alpha^n, \dots)} \subset \Gamma(\alpha^1, \alpha^2, \dots, \alpha^{n_0}) + U$, because $\Gamma(\alpha^1, \alpha^2, \dots, \alpha^{n_0}) + U$ is Na -closed.

Characterization of the natural topology a- K is spherically complete

Theorem 2. If Λ is perfect, then the natural topology on Λ is the polar topology of \mathcal{A} -convergence, where \mathcal{A} is the family of all compactoid subsets of (Λ^β, Na) .

Proof. Let \mathcal{A} be the family of all compactoid subsets of (Λ^β, Na) then for every $A \in \mathcal{A}$, A is σ -bounded and \mathcal{A} satisfies the conditions (a), (b) and (c) of 8. Let τ be the polar topology of \mathcal{A} -convergence on Λ ; then $\tau = Na$, (theorem 1). ■

Theorem 3. If Λ is perfect, then the natural topology on Λ is the polar topology of \mathcal{A} -convergence where \mathcal{A} is the family of all absolutely K -convex, bounded and c -compact subsets of (Λ^β, Na) .

Proof. The same as theorem 2 and apply remarks 1. ■

Remark 6. If K is spherically complete and Λ is perfect then $Na = \tau_c$; where τ_c is the c -compact topology on Λ . And for every topology τ on Λ , τ is compatible with the duality $\langle \Lambda, \Lambda^\beta \rangle$ if, and only if, $\sigma \leq \tau \leq Na$.

b- K is local

If K is local, then [31], proposition 1 and [7], proposition 2, p. 177 induce that all results before still hold when the word absolutely K -convex, bounded and c -compact (or compactoid) is replaced by absolutely K -convex and compact; and the characterization of the natural topology became:

Theorem 4. *If Λ is perfect, then the natural topology on Λ is the polar topology of \mathcal{A} -convergence where \mathcal{A} is the family of all absolutely K -convex and compact subsets of (Λ^β, Na) .*

c- K is not spherically complete

Theorem 5. *If Λ is perfect, then the natural topology on Λ is the polar topology of \mathcal{A} -convergence where*

$$\mathcal{A} = \left\{ K_c \left(\overline{A}^{\sigma(\Lambda^\beta, \Lambda)} \right) / A \subset \Lambda^\beta \text{ and } A \text{ is finite} \right\}.$$

Proof. By proposition 5. ■

4. Locally K -convex spaces with a Schauder basis and perfect sequence spaces

Let (E, τ) be a locally K -convex space where τ is a polar topology of \mathcal{A} -convergence, $(e_i)_i$ be a Schauder basis of (E, τ) and $(f_i)_i$ the associated weak Schauder basis. If $S_n(x) = \sum_{i=1}^n \lambda_i e_i$ and $T_n(f) = \sum_{i=1}^n \mu_i f_i$ for all $x \in E$, all $f \in E'$ (the topological dual of E) and all $n \geq 1$, $\langle S_n(x), f \rangle = \langle x, T_n(f) \rangle$, for all $n \geq 1$, $x \in E$ and $f \in E'$. For every $A \subset E'$, we put $\tilde{A} = \{T_n(a) / n \in \mathbb{N}; a \in A\}$ with $T_0 = id_{E'}$ and for every $A \subset E$ we put $S(A) = \{x \in A / S_n(x) \in A, \text{ for all } n \geq 1\}$. We define also \tilde{A} for $A \subset E$ and $S(A)$ for $A \subset E'$. N. De Grande-De Kimpe has defined the topology $\tilde{\tau}$ of $\tilde{\mathcal{A}}$ -convergence where $\tilde{\mathcal{A}} = \{\tilde{A} / A \in \mathcal{A}\}$, and she gave a characterization of this topology ([10], proposition 1.2, p. 278). We enhance this result in theorem 6, p. 19.

Remarks 2. 1. $\tilde{p}_A(x) = \sup_n p_A \left(\sum_{i=1}^n \langle x, f_i \rangle e_i \right)$ for all $A \in \mathcal{A}$ for all $x \in E$; in the case where $F = E'$, $p_{\tilde{A}}(x) = \tilde{p}_A(x) = \sup_n p_A(S_n(x)) = \sup_n p_A(f_n(x) . x_n)$ ([10], proposition 1.1, p. 278).

2. The $\tilde{\sigma}$ - topology associated to the weak topology $\sigma = \sigma(E, E')$ on E is defined by the family of seminorms n.a $(p_f)_{f \in E'}$, where $p_f(x) = \sup_n |\langle S_n(x), f \rangle|$, for every $x \in E$ and $f \in E'$; and we have $\tilde{\sigma} \leq \tilde{\tau}$.

Example 1. Let Λ be a perfect sequence space over K . The topology $\tilde{\sigma} = \tilde{\sigma}(\Lambda, \Lambda^\beta)$ associated to $\sigma = \sigma(\Lambda, \Lambda^\beta)$ is defined by the family of seminorms n.a $(p_\mu)_{\mu \in \Lambda^\beta}$, where $p_\mu(\lambda) = \sup_n \left| \sum_{i=1}^n \lambda_i \mu_i \right| = \sup_n |\lambda_n \mu_n|$, for every $\lambda = (\lambda_i)_i \in \Lambda$ and $\mu = (\mu_i)_i \in \Lambda^\beta$. Then the topology $\tilde{\sigma}$ is exactly the natural topology studied in §. 3.

We Consider the two linear mappings $\Phi : E \longrightarrow \Lambda, x = \sum_{i=1}^\infty \lambda_i e_i \longmapsto (\lambda_i)_i$ and $\Psi : E' \longrightarrow \Delta, f = \sum_{i=1}^\infty \mu_i f_i \longmapsto (\mu_i)_i$; where Λ and Δ are the sequence spaces defined as $\Lambda = \left\{ (\lambda_i)_i \in \omega / \sum_{i=1}^\infty \lambda_i e_i \text{ converges in } (E, \tau) \right\}$ and $\Delta = \left\{ (\mu_i)_i \in \omega / \sum_{i=1}^\infty \mu_i f_i \text{ converges in } E'_\sigma \right\}$. Φ and Ψ are algebraic isomorphisms.

Proposition 11. $\Lambda \subset \Delta^\beta$ and $\Delta \subset \Lambda^\beta$.

Proposition 12. i. Φ is $(\sigma(E, E'), \sigma(\Lambda, \Delta))$ – continuous;
 ii. Ψ is $(\sigma(E', E), \sigma(\Delta, \Lambda))$ – continuous.

Proof. i. Let $\mu \in \Delta$, we consider $V = \{ \lambda = (\lambda_i)_i \in \Lambda / |\langle \lambda, \mu \rangle| \leq 1 \}$. We put $f = \psi^{-1}(\mu)$, then $f \in E'$. We Consider $U = \{ x \in E / |\langle x, f \rangle| \leq 1 \}$; U is a zero neighbourhood in $(E, \sigma(E, E'))$, and we have $\Phi(U) = \{ \lambda = \Phi(x) \in \Lambda / |\langle x, f \rangle| \leq 1 \} = V$.

If $x = \sum_{i=1}^\infty \lambda_i e_i$ and $\mu = (\mu_i)_i$, then $\Phi(x) = (\lambda_i)_i$ and $f = \sum_{i=1}^\infty \mu_i f_i$.

Therefore $\langle x, f \rangle = \sum_{i=1}^\infty \lambda_i \mu_i = \langle \Phi(x), \mu \rangle$.

ii. Same proof as for i. ■

Proposition 13. *If Φ^* and Ψ^* are the algebraic adjoints of Φ and Ψ respectively, then $\Phi^* = \Psi^{-1}$ and $\Psi^* = \Phi^{-1}$.*

Proof. Φ^* take his values in E' ([30], p. 128). For every $x \in E$ and $\mu \in \Delta$ we have $\langle x, \Phi^*(\mu) \rangle = \langle \Phi(x), \mu \rangle = \sum_{i=1}^{\infty} \lambda_i \mu_i$, where $x = \sum_{i=1}^{\infty} \lambda_i e_i$ and $\mu = (\mu_i)_i$. So $\langle x, \Phi^*(\mu) \rangle = \langle x, \Psi^{-1}(\mu) \rangle$. Then $\Phi^* = \Psi^{-1}$.
The same for $\Psi^* = \Phi^{-1}$. ■

Proposition 14. a. For every $A \subset E, (\Phi(A))^\circ = \Psi(A^\circ)$;
b. For every $B \subset E', (\Psi(B))^\circ = \Phi(B^\circ)$.

Proof. a. Let $A \subset E$, then $(\Phi(A))^\circ = (\Phi^*)^{-1}(A^\circ)$ [[2], proposition 2.8, p. 225].

Now $\Phi^* = \Psi^{-1}$ (proposition 13), so $(\Phi(A))^\circ = (\Psi^{-1})^{-1}(A^\circ) = \Psi(A^\circ)$.

b. The same proof. ■

The topology τ_Φ defined on Λ by Φ has a zero-neighbourhood base consisting of the family $(\Phi(A^\circ))_{A \in \mathcal{A}}$ ([6], II. 29), τ_Φ is a polar topology of $\Psi(\mathcal{A})$ -convergence, where $\Psi(\mathcal{A}) = \{\Psi(A) / A \in \mathcal{A}\}$ (proposition 14).

Examples 1. 1. If we consider the space E_σ , then the topology σ_Φ has a zero-neighbourhood base the set $\{(\psi(A))^\circ / A \subset E' \text{ and } A \text{ is finite}\}$.

For $A = \{(f^i)_{1 \leq i \leq n} / f^i \in E'\}$, put $f^i = \sum_{j=1}^{\infty} \mu_j^i f_j$, for every $i, 1 \leq i \leq n$;
 $(\Psi(A))^\circ = \left(\left\{ (\mu_j^i)_{j \geq 1}, 1 \leq i \leq n \right\} \right)^\circ$. Hence σ_Φ is exactly the weak topology.

2. Let τ be a polar topology of \mathcal{A} -convergence on E and $\tilde{\tau}$ the associated polar topology, then $\tilde{\tau}_\Phi = (\tilde{\tau})_\Phi$ has $\left[(\Psi(\tilde{A}))^\circ \right]_{A \in \mathcal{A}}$ as a zero-neighbourhood base. For every $A \in \mathcal{A}, \Psi(\tilde{A}) = \Psi(\widetilde{A})$, then $\tilde{\tau}_\Phi$ is defined by the family of n.a seminorms $(\tilde{p}_{\Psi(A)})_{A \in \mathcal{A}}$, where :

$$\tilde{p}_{\Psi(A)} \left((\lambda_i)_{i \geq 1} \right) = \sup_n p_{\psi(A)}(\lambda_1, \dots, \lambda_n, 0, \dots) = \sup_n \sup_{\mu = (\mu_i) \in A} \left| \sum_{i=1}^n \lambda_i \mu_i \right| = \sup_{\mu = (\mu_i) \in A} \tilde{p}_\mu \left((\lambda_i)_{i \geq 1} \right).$$

3. The direct image topology of $\tilde{\sigma}$ with Φ on Λ , noted $\tilde{\sigma}_\Phi$, is defined with the family of semi-norms n.a $(\tilde{p}_{\Psi(f)})_{f \in E'}$, where $\tilde{p}_{\Psi(f)}((\lambda_i)_i) = \tilde{p}_\mu((\lambda_i)_i)$, $(\lambda_i)_i \in \Lambda$ and $\mu = \Psi(f)$. Then $\tilde{\sigma}_\Phi$ is exactly the natural topology on Λ .

Some properties of the topology $\tilde{\sigma}$

- Lemma 4.** i. If E_σ is sequentially complete then $\Lambda = \Delta^\beta$;
 ii. If E'_σ is sequentially complete then $\Delta = \Lambda^\beta$;
 iii. If E_σ and E'_σ are sequentially complete then Λ is perfect.

Proof. i. $\Lambda \subset \Delta^\beta$ (proposition 11).

Let $\lambda = (\lambda_i)_i$ an element of Δ^β , so $\lambda = \lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i e^i$ in $(\Delta^\beta, \sigma(\Delta^\beta, \Delta^\beta)) = \Delta^\beta_\sigma$ where $e^i = (\delta_{ij})_j$ for all $i \geq 1$, (Δ^β is perfect). Then $(e^i)_i$ is a Schauder basis of Δ^β_σ (propositions 5, 10 and remark 1), so $\lambda = \lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i e^i$ in $(\Delta^\beta, \sigma(\Delta^\beta, \Delta))$. $(\sum_{i=1}^n \lambda_i e^i)_n$ is a Cauchy-sequence in $(\Lambda, \sigma(\Lambda, \Delta))$ which is sequentially-complete (examples 1. 1), then $\lambda = \lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i e^i$ in $(\Lambda, \sigma(\Lambda, \Delta))$ and so $\lambda \in \Lambda$.

ii. $\Delta \subset \Lambda^\beta$ (proposition 11). Let $\lambda = (\lambda_i)_i \in \Lambda^\beta$, for every $x \in E$, there exists $\alpha = (\alpha_i)_i \in \Lambda$ such that $x = \sum_{i=1}^\infty \alpha_i e_i$. for all $n \geq 1$, $\lambda_n f_n(x) = \sum_{j \geq 1} \lambda_j f_j(\sum_{i=1}^n \alpha_i e_i) = \sum_{j \geq 1} \lambda_j \alpha_j$ is convergent in E'_σ ; and so $\lambda = (\lambda_i)_i \in \Delta$. ■

- Proposition 15.** i. If E_σ is sequentially complete then $E_{\tilde{\sigma}}$ is complete;
 ii. If E'_σ is sequentially complete then $E'_{\tilde{\sigma}}$ is complete;
 where $E_{\tilde{\sigma}} = (E, \tilde{\sigma}(E, E'))$ and $E'_{\tilde{\sigma}} = (E', \tilde{\sigma}(E', E))$.

Proof. i. Suppose that E_σ is sequentially complete, then by lemma 4 $\Lambda = \Delta^\beta$ and $(\Lambda, \tilde{\sigma}(\Lambda, \Delta^\beta))$ is complete (proposition 6 and remark 2), then $(\Lambda, \tilde{\sigma}(\Lambda, \Delta))$ is complete and $E_{\tilde{\sigma}}$ is also complete (examples 1. 3).
 ii. Same proof as for i. ■

Proposition 16. If E_σ is sequentially complete and has an equicontinuous Schauder basis, then E is isomorphic to a closed subspace of some power of K .

Proof. Suppose that E_σ is sequentially complete and admits an equicontinuous Schauder basis, then the topologies $\sigma(E, E')$ and $\tilde{\sigma}(E, E')$ coincides on E , then by proposition 15 $(E, \sigma(E, E'))$ is complete; and the proposition follows ([7], proposition 7, p. 179). ■

Proposition 17. *If E_σ and E'_σ are sequentially complete and E has a weak Schauder basis $(e_i)_{i \geq 1}$, then $\tilde{\sigma}$ is the smallest compatible topology on E for which $(e_i)_i$ is an equicontinuous Schauder basis.*

Proof. E'_σ is sequentially complete, then $\Delta = \Lambda^\beta$ (lemma 3.1) and so $(\Lambda, \tilde{\sigma}(\Lambda, \Lambda^\beta))' = \Delta$, since $(\Lambda, \tilde{\sigma}(\Lambda, \Lambda^\beta))' = \Lambda^\beta$. Consequently $(E, \tilde{\sigma}(E, E'))' = E'$ (example 1. 3) and the result follows from propositions 15 and [10], proposition 1.2, p. 278. ■

Compatibility of the $\tilde{\tau}$ -topology

We establish the compatibility and the completeness of $\tilde{\tau}$ by distinguishing the three cases: K is local, K is spherically complete and K is non spherically complete.

a. K is local

Lemma 5. *Let E be a topological vector space with an equicontinuous Schauder basis $(e_i)_i$; then for every $A \subset E$ the following are equivalent:*

- a. A is precompact;
- b. (i). for all $i \geq 1$, $p_i(A)$ is precompact and (ii). $\left(\sum_{i=1}^n p_i\right)_n$ converges uniformly on A . Where the p_n are defined in §. 2.

Proof. a \implies b For every $i \geq 1$ $p_i(A)$ is precompact (p_i is continuous). On the other hand, the sequence of linear mappings $\left(\sum_{i=1}^n p_i\right)_n$ is equicontinuous and converging pointwise to a mapping id_E and A is compactoid, then $\left(\sum_{i=1}^n p_i\right)_n$ converges uniformly on A (§. 3, lemma 3).

b \implies a Let U be a zero-neighbourhood, then there exists V a neighbourhood of zero and $n_0 \in \mathbb{N}^*$ such that $V + V \subset U$ and $\sum_{n_0 < i} p_i(a) \in V$

, for all $a \in A$. On the other hand $B = \sum_{i=1}^{n_0} p_i(A)$ is precompact, then

there exist $b_1, b_2, \dots, b_p \in E$ such that $B \subset \bigcup_{i=1}^p (b_i + V)$. Then $A \subset B + V \subset \bigcup_{i=1}^p (b_i + V) + V \subset \bigcup_{i=1}^p (b_i + U)$. ■

Lemma 6. For every $n \geq 1$, the mapping $p_n : E'_\sigma \longrightarrow E'_{\tilde{\sigma}}$, $\sum_{i=1}^\infty \mu_i f_i \longmapsto \mu_n f_n$, is continuous; where $(f_i)_i$ is the weak Schauder basis associated to $(e_i)_i$.

Proof. Let $n \geq 1$ and $x \in E$, then for every $f = \sum_{i=1}^\infty \mu_i f_i = \sum_{i=1}^\infty \langle e_i, f \rangle f_i$ we have: $\tilde{p}_x(p_n(f)) = \sup_m |\langle x, T_m(p_n(f)) \rangle| = |\langle x, p_n(f) \rangle| = |\langle x, f_n \rangle| |\langle x_n, f \rangle|$.
 Take $y = \langle x, f_n \rangle x_n$, then $y \in E$ and we have $p_y(f) = \tilde{p}_x(p_n(f))$. ■

Remark 7. Lemma 6 is true for every K .

Lemma 7. Let $A \in \mathcal{A}$, then the statements **a.** and **b.** are equivalente

- a.** A is precompact in E'_σ ;
- b. (i).** A is precompact in E'_σ and **(ii).** for all $x \in E$, $\lim_n \tilde{p}_A(x - S_n(x)) = 0$.

Proof. We consider $A \in \mathcal{A}$ such that A is precompact in E'_σ , then A is precompact in E'_σ ($\sigma \leq \tilde{\sigma}$). On the other hand, for every $x \in E$, we have :
 $\tilde{p}_A(x - S_n(x)) = \sup_{k \in \mathbb{N}} p_A(S_k(x - S_n(x))) = \sup_{f \in A} \tilde{p}_x(f - T_n(f))$.
 Since $(e_i)_i$ is a Schauder basis of (E, τ) , the sequence $(f_i)_i$ is an equicontinuous Schauder basis of E'_σ [9], lemma 3, p. 402 and [10], proposition 1.2, p. 278.

Furthermore A is precompact in E'_σ , so $(T_n)_n$ converges to $id_{E'}$ uniformly on A in E'_σ (§. 3, lemma 3); then $\lim_{n \rightarrow \infty} \sup_{f \in A} \tilde{p}_x(f - T_n(f)) = 0$ for all $x \in E$, and so $\lim_{n \rightarrow \infty} \tilde{p}_A(x - S_n(x)) = 0$.

Conversely A is precompact in $E'_\sigma \implies$ for all $i \geq 1$ $p_i(A)$ is precompact in E'_σ (lemma 6). On the other hand we have for all $x \in E$ and all $n \geq 1$ $\tilde{p}_A(x - S_n(x)) = \sup_{f \in A} \tilde{p}_x(f - T_n(f)) = \sup_{f \in A} \tilde{p}_x\left(f - \sum_{i=1}^n p_i(f)\right)$.
 So $\lim_{n \rightarrow \infty} \tilde{p}_x\left(f - \sum_{i=1}^n p_i(f)\right) = 0$ for all $x \in E$, this means that $\left(\sum_{i=1}^n p_i\right)_n$

converges uniformly to $id_{E'}$ on A in E'_σ ; then by lemma 5, A is precompact in E'_σ . ■

Lemma 8. *If E' is $\sigma(E', E)$ -sequentially complete, then for every $A \in \mathcal{A}$, the following are equivalent:*

- a. A is $\tilde{\sigma}$ -relatively compact;
- b. (i). A is relatively compact in E'_σ and (ii). for all $x \in E$, $\lim_{n \rightarrow \infty} \tilde{p}_A(x - S_n(x)) = 0$.

Proof. Suppose that A is $\tilde{\sigma}$ -relatively compact in E'_σ ; $\overline{A}^{\tilde{\sigma}}$ is compact in E'_σ ($\sigma \leq \tilde{\sigma}$). Since $A \subset \overline{A}^{\tilde{\sigma}}$ and $\overline{A}^{\tilde{\sigma}}$ is closed in E'_σ then $\overline{A}^\sigma \subset \overline{A}^{\tilde{\sigma}}$ and so \overline{A}^σ is compact in E'_σ . Furthermore by lemma 7 we have (ii).

Conversely, take A such that (i) and (ii) of **b** holds, then A and so $\overline{A}^{\tilde{\sigma}}$ are precompacts in E'_σ (lemma 7). Consequently $\overline{A}^{\tilde{\sigma}}$ is compact in E'_σ (E'_σ is complete: proposition 15.ii). ■

Proposition 18. *Let $A \in \mathcal{A}$.*

1. \tilde{A} is $\tilde{\sigma}(E', E)$ -precompact;
2. If E' is $\sigma(E', E)$ -sequentially complete, then
 - i. \tilde{A} is $\tilde{\sigma}(E', E)$ -relatively compact;
 - ii. $\Gamma(\tilde{A})$ is $\sigma(E', E)$ -relatively compact.

Proof. Let $A \in \mathcal{A}$.

1. \tilde{A} is $\sigma(E', E)$ -bounded [10], lemma 1.2, p. 277, then it is $\sigma(E', E)$ -relatively compact ([2], proposition 2.3, p. 223) and so \tilde{A} is precompact in $(E', \sigma(E', E))$. On the other hand for every $x \in E$ $\lim_{n \rightarrow \infty} \tilde{p}_A(x - S_n(x)) = 0$ ($(e_i)_i$ is a Schauder basis of $(E, \tilde{\tau})$). Therefore, by lemma 7 and remarks 2, \tilde{A} is precompact in E'_σ .

2. \tilde{A} is $\sigma(E', E)$ -relatively compact and $\lim_{n \rightarrow \infty} \tilde{p}_A(x - S_n(x)) = 0$ for every $x \in E$, then \tilde{A} is $\tilde{\sigma}(E', E)$ -relatively compact (lemma 8).

3. \tilde{A} is $\tilde{\sigma}(E', E)$ -relatively compact, so $B = \overline{\Gamma(\tilde{A})}^{\tilde{\sigma}}$ is $\tilde{\sigma}(E', E)$ -compact because B is a closed in a complete space E'_σ (proposition 15 and [30], p. 26). Hence $\Gamma(\tilde{A})$ is $\sigma(E', E)$ -relatively compact in E'_σ (lemma 8). ■

Proposition 19. *If (E, τ) has a Schauder basis and E' is $\sigma(E', E)$ – sequentially complete, then $\tilde{\tau}$ is compatible with the duality $\langle E, E' \rangle$.*

Proof. We have $\sigma \leq \tilde{\tau}$. On the other hand, $\tilde{\tau}$ is generated by the family $\left(\overline{\Gamma(\tilde{A})}^{\sigma(E', E)} \right)_{A \in \mathcal{A}}$ ([2], §. 3, proposition 3.4, p. 228) and $\overline{\Gamma(\tilde{A})}^{\sigma(E', E)}$ is $\sigma(E', E)$ – compact for every $A \in \mathcal{A}$, so $\tilde{\tau} \leq \tau_m$, where τ_m is the Mackey topology on E . ■

b. K is spherically complete

Lemma 9. *Let E be a topological K –vector space with an equicontinuous Schauder basis $(e_i)_i$; then for every $A \subset E$ the statements **a** and **b** are equivalente*

a. *A is compactoid;*

b. (i). *for all $i \geq 1$ $p_i(A)$ is compactoid and (ii). $\left(\sum_{i=1}^n p_i \right)_n$ converges uniformly on A .*

Proof. Suppose that A is compactoid; then for every $i \geq 1$ $p_i(A)$ is compactoid. On the other hand, $(e_i)_i$ is an equicontinuous Schauder basis, so $\left(\sum_{i=1}^n p_i \right)_n$ converges pointwise to the mapping id_E ; since A is compactoid, this convergence is uniform on A (§. 3, lemma 3).

Conversely let U and V are two zero-neighbourhoods such that $V + V \subset U$, then the convergence of $\left(\sum_{i=1}^n p_i \right)_n$ on A implies the existence of $n_0 \in \mathbb{N}$ such that $\sum_{i=n_0+1}^{\infty} p_i(x) \in V$ for all $x \in A$. On the other hand, **(i)** of lemma

induces the existance of $x_1, \dots, x_n \in E$ such that $\sum_{i=1}^{n_0} p_i(A) \subset V + \Gamma(B)$,

where $B = \{x_1, \dots, x_n\} \left(\sum_{i=1}^{n_0} p_i(A) \text{ is compactoid} \right)$.

Then for every $x \in A$, $x = \sum_{i=1}^{n_0} \lambda_i e_i + \sum_{i=n_0+1}^{\infty} \lambda_i e_i$

$$= \sum_{i=1}^{n_0} p_i(x) + \sum_{i=n_0+1}^{\infty} p_i(x) \in V + \Gamma(B) + V,$$

so $A \subset U + \Gamma(B)$. ■

Lemma 10. *Let $A \in \mathcal{A}$, then the following are equivalent*

- a. A is compactoid in E'_σ ;
- b. (i). A is compactoid in E'_σ and (ii). for all $x \in E \lim_{n \rightarrow \infty} \tilde{p}_A(x - S_n(x)) = 0$.

Proof. Same proof as for lemma 7 using lemma 9 and remark 7. ■

Proposition 20. *Let τ be a polar topology of \mathcal{A} -convergence on E and $(e_i)_i$ be a Schauder basis of (E, τ) , then for every $A \in \mathcal{A}$*

- i. \tilde{A} is $\tilde{\sigma}$ -compactoid;
- ii. If E' is $\sigma(E', E)$ -sequentially complete then
 - a. \tilde{A} is $\tilde{\sigma}$ -relatively-c-compact;
 - b. $\Gamma(\tilde{A})$ is σ -relatively-c-compact.

Proof. i. Let $A \in \mathcal{A}$, then \tilde{A} is σ -bounded [10], lemma 1.2, p. 277 and, since K is spherically complete, \tilde{A} is compactoid in E'_σ ([31], proposition 18.ii, p. 145). On the other hand for all $x \in E \lim_{n \rightarrow \infty} \tilde{p}_A(x - S_n(x)) = 0$ since $(e_i)_i$ is a Schauder basis of $(E, \tilde{\tau})$. Then \tilde{A} is compactoid in E'_σ (lemma 10).

ii. Let $A \in \mathcal{A}$; then

a. $\overline{\tilde{A}}^{\tilde{\sigma}}$ is compactoid in E'_σ (by i) $\implies E'_\sigma$ is complete, because E' is $\sigma(E', E)$ -sequentially complete (proposition 15), then $\overline{\tilde{A}}^{\tilde{\sigma}}$ is also complete and so it is c-compact in E'_σ [31], theorem 9, p. 141.

b. $B = \overline{\Gamma(\tilde{A})}^{\tilde{\sigma}}$ is c-compact in E'_σ , then it is $\sigma(E', E)$ -c-compact and $\sigma(E', E)$ -closed ($\sigma \leq \tilde{\sigma}$), therefore $\overline{\Gamma(\tilde{A})}^{\sigma(E', E)}$ is $\sigma(E', E)$ -c-compact.

Proposition 21. *If (E, τ) has a Schauder basis and E' is $\sigma(E', E)$ -sequentially complete, then $\tilde{\tau}$ is compatible with the duality $\langle E, E' \rangle$.*

Proof. The topology $\tilde{\tau}$ is a polar topology of \mathcal{B} -convergence; where $\mathcal{B} = \left(\overline{\Gamma(\tilde{A})}^{\sigma(E', E)} \right)_{A \in \mathcal{A}}$ and \mathcal{A} is the family which defines the topology τ ; for every $A \in \mathcal{A}$, $\overline{\Gamma(\tilde{A})}^{\sigma(E', E)}$ is $\sigma(E', E)$ - c -compact in E'_σ (proposition 20), then $\tilde{\tau}$ is compatible with the duality $\langle E, E' \rangle$ [2], theorem 4.4, p. 234. ■

c. K is not spherically complete

K is not spherically complete $\implies K$ is dense \implies For every absolutely K -convex A in E' , $K_c(A) = \bigcap_{|\lambda| > 1} \lambda A \implies$ for all $|\mu| > 1$ $\mu K_c(A) = K_c(\mu A)$. Then we have the following proposition :

Proposition 22. *If (E, τ) has a Schauder basis, then $\tilde{\tau}$ is compatible.*

Proof. Let \mathcal{A} be a family which defines the topology τ , such that for all $A \in \mathcal{A}$ A is absolutely K -convex; then $K_c \left(\overline{A}^{\sigma(E', E)} \right)^\circ = (\tilde{A})^\circ$ for all $A \in \mathcal{A}$ [2], corollary 4.3, p. 233, so if we take $\beta = \left(K_c \left(\overline{A}^{\sigma(E', E)} \right) \right)_{A \in \mathcal{A}}$, then β verify the conditions (a), (b) and (c) of 8. Therefore $\tilde{\tau}$ is a polar topology of β -convergence and its elements are E -closed. Then $\tilde{\tau}$ is compatible [2], theorem 4.3, p. 233. ■

Completeness of the topology $\tilde{\tau}$

Proposition 23. *Let (E, τ) be a locally K -convex space and $(e_i)_i$ be a Schauder basis of (E, τ) . If E and E' are weakly-sequentially complete, then $(E, \tilde{\tau})$ is complete.*

Proof. The space $(E, \tilde{\sigma}(E, E'))$ is complete (proposition 15), then by remarks 2 and ([2], theorem 3.2, p. 230) $(E, \tilde{\tau})$ is complete. ■

The following theorem is a consequence for previous results

Theorem 6. *Let (E, τ) be a locally K -convex space with a Schauder basis $(e_i)_i$ such that E_σ and E'_σ are sequentially complete, then $\tilde{\tau}$ is complete and it is the coarsest compatible topology on E finer than τ for which $(e_i)_i$ is an equicontinuous Schauder basis.*

5. The weak basis Problem

Throughout this section we shall assume that (E, τ) has a weak Schauder basis $(e_i)_i$ and the spaces E'_σ and E_σ are sequentially complete. We then characterize the finest (E, E') -compatible topology on E for which $(e_i)_i$ is a Schauder basis; according to theorem 6, $(e_i)_i$ is equicontinuous for that topology. We shall distinguish three cases: K is local, K is spherically complete or K is not spherically complete.

a. K is local

Let $\mathcal{B} = \{B \subset E' / B = \tilde{B} \text{ and } B \text{ is } \tilde{\sigma}\text{-precompact}\}$; it is obviously that \mathcal{B} is not empty and verifies the properties (a), (b) and (c) of 8. Let \mathcal{U} be the polar topology of \mathcal{B} -convergence on E ; we have the following propositions

Proposition 24. \mathcal{U} is compatible with the duality $\langle E, E' \rangle$ and $(e_i)_i$ is an equicontinuous Schauder basis of (E, \mathcal{U}) .

Proof. E' is $\sigma(E', E)$ -sequentially complete \implies for all $B \in \mathcal{B}$, $\overline{\Gamma(B)}^{\sigma(E', E)}$ is $\sigma(E', E)$ -compact (proposition 18) $\implies \mathcal{U}$ is compatible with the duality $\langle E, E' \rangle$ [2], theorem 4.5, p. 235.

We'll prove that $(e_i)_i$ is a Schauder basis of (E, \mathcal{U}) . $(e_i)_i$ is a weak Schauder basis $\implies (f_i)_i$ is a Schauder basis of $(E', \sigma(E', E))$, then $(f_i)_{i \geq 1}$ is an equicontinuous Schauder basis of $(E', \tilde{\sigma}(E', E))$; therefore $(T_n)_n$ is equicontinuous in $(E', \tilde{\sigma}(E', E))$ and converges pointwise to the mapping $id_{E'}$, then the convergence is uniformly on every $B \in \mathcal{B}$, this means that for all $x \in E$ $\limsup_n \sup_{f \in B} \tilde{p}_x \left(f - \sum_{i=1}^n p_i(f) \right) = 0$.

For every $x \in E$ and for every $B \in \mathcal{B}$ we have $\sup_{f \in B} \tilde{p}_x \left(f - \sum_{i=1}^n p_i(f) \right) = \tilde{p}_B(x - S_n(x))$ (lemma 7). Then $\lim_n \tilde{p}_B(x - S_n(x)) = 0$ for all $x \in E$ and for all $B \in \mathcal{B}$, and so $(S_n(x))_n$ converges to x in (E, \mathcal{U}) for every $x \in E$.

Moreover the associated sequence $(f_i)_i$ of $(e_i)_i$ verifies for all $i \geq 1$ $f_i \in (E, \mathcal{U})'$ (\mathcal{U} is compatible). ■

Proposition 25. \mathcal{U} is the finest compatible topology on E for which $(e_i)_i$ is a Schauder basis.

Proof. Let τ be a compatible topology on E such that $(e_i)_i$ be a Schauder basis of (E, τ) . Then $B = \overline{\Gamma(\tilde{A})}^{\sigma(E', E)}$ is a $\sigma(E', E)$ – compact of E' (proposition 18) and since $B = \tilde{B}$, B° is a zero-neighbourhood in (E, \mathcal{U}) ; now $B^\circ = (\tilde{A})^\circ$, for every $A \in \mathcal{A}$, where \mathcal{A} is a family which defines the topology τ , then $\tilde{\tau} \leq \mathcal{U}$ and so $\tau \leq \mathcal{U}$. ■

Proposition 26. *A weak Schauder basis $(e_i)_i$ is a Schauder basis for a polar compatible topology on E if, and only if, for every $A \in \mathcal{A}$, \tilde{A} is $\tilde{\sigma}(E', E)$ – relatively compact.*

Proof. \implies] By proposition 18.

\impliedby] Suppose that for every $A \in \mathcal{A}$, \tilde{A} is $\tilde{\sigma}(E', E)$ –relatively compact, then $\overline{\Gamma(\tilde{A})}^{\sigma(E', E)}$ is $\sigma(E', E)$ – compact (proposition 18), so $\tau \leq U$ and $(e_i)_i$ is a Schauder basis of (E, τ) . ■

Corollary 4. *A weak Schauder basis $(e_i)_i$ is a Schauder basis for a polar compatible topology τ on E if, and only if, for every $A \in \mathcal{A}$, \tilde{A} is $\tilde{\sigma}(E', E)$ – relatively compact, where \mathcal{A} is a family that define the topology τ .*

Proof. τ_m is compatible, then it suffices to use the proposition 26. ■

Remark 8. *If τ is a polar topology of \mathcal{A} – convergence on E having a weak Schauder basis $(e_i)_i$, then for every $A \in \mathcal{A}$, \tilde{A} is $\tilde{\sigma}(E', E)$ – relatively compact.*

Proof. E is an OP – space, so $(e_i)_i$ is a Schauder basis of (E, τ) , then according to proposition 18 we have the conclusion. ■

b. K is spherically complete

Let $\mathcal{N} = \{N \subset E' / N = \tilde{N} \text{ and } N \text{ is } \tilde{\sigma}(E', E) \text{ – compactoid}\}$; it is obviously that \mathcal{N} is not empty and verifies the properties (a), (b) and (c) of 8. Let \mathcal{V} the polar topology of \mathcal{N} –convergence on E , then we have the following propositions

Proposition 27. *The topology \mathcal{V} is compatible with the duality $\langle E, E' \rangle$ and $(e_i)_i$ is an equicontinuous Schauder basis of (E, \mathcal{V}) .*

Proof. Same proof as for proposition 24 using the proposition 20 and [2], theorem 4.4, p. 234. ■

Proposition 28. \mathcal{V} is the finest compatible topology on E for which $(e_i)_i$ is a Schauder basis.

Proof. Same proof as for proposition 25 using the proposition 20. ■

Proposition 29. A weak Schauder basis $(e_i)_i$ is a Schauder basis for a polar compatible topology τ on E if, and only if, for every $A \in \mathcal{A}$, \tilde{A} is $\tilde{\sigma}(E', E)$ -relatively c -compact, where \mathcal{A} is a family that defines the topology τ .

Proof. Same proof as for proposition 26 using the proposition 20. ■

Remark 9. if τ is a polar topology of \mathcal{A} -convergence on E that having a weak Schauder basis $(e_i)_i$, then for every $A \in \mathcal{A}$, \tilde{A} is $\tilde{\sigma}(E', E)$ -relatively c -compact.

Corollary 5. A weak Schauder basis $(e_i)_i$ is a Schauder basis for a topology τ_c on E if, and only if, for every absolutely K -convex, $\sigma(E', E)$ -bounded and $\sigma(E', E)$ - c -compact A of E' , \tilde{A} is $\tilde{\sigma}(E', E)$ -relatively c -compact.

Proof. It is sufficient to take $\tau = \tau_c$ in proposition 29. ■

c. K is not spherically complete

Let \mathcal{M} the family of all $M \subset E'$ such that $M = \tilde{M}$, M is $\sigma(E', E)$ -bounded, E -closed and $(T_n)_n$ converges uniformly on M in E'_σ , where $E'_\sigma = (E', \tilde{\sigma}(E', E))$. Let ϑ be the polar topology of \mathcal{M} -convergence.

Theorem 7. ϑ is the finest compatible topology on E for which $(e_i)_i$ is an equicontinuous Schauder basis.

Proof. ϑ is compatible [2], theorem 4.3, p. 233.

Let $M \in \mathcal{M}$, then $(T_n)_n$ converges uniformly on M in $E'_\sigma \implies \lim_n \sup_{f \in M} \tilde{p}_x(f - T_n(f)) = 0$, so $\lim_n p_M(x - S_n(x)) = 0$. Then $(e_i)_i$ is a Schauder basis of ϑ .

Let τ be a polar and compatible topology of \mathcal{A} -convergence such that $(e_i)_i$ be an equicontinuous Schauder basis, then $\tau = \tilde{\tau}$. Therefore

$$\mathcal{A} = \left\{ A \subset E' / A = \tilde{A} \text{ and } A \text{ is } \sigma(E', E) \text{ - bounded and } E \text{ - closed} \right\}.$$

Let $A \in \mathcal{A}$, so for every $x \in E \lim_n p_A(x - S_n(x)) = 0$, then $\lim_n \sup_{f \in A} \tilde{p}_x(f - T_n(f)) = 0$, therefore $(T_n)_n$ converges uniformly on A in E'_σ . Then $\tau \leq \vartheta$. ■

Theorem 8. *Let τ be a polar topology of \mathcal{A} - convergence on E such that E_σ and E'_σ are sequentially complete and $(e_i)_i$ be a weak Schauder basis of E , then $(e_i)_i$ is a Schauder basis of (E, τ) if, and only if, for all $A \in \mathcal{A}$ the sequence $(T_n)_n$ converges uniformly on A in E'_σ .*

Proof. If $(e_i)_i$ is a Schauder basis of (E, τ) , then $(e_i)_i$ is an equicontinuous Schauder basis of $\tilde{\tau}$; therefore for all $A \in \mathcal{A}$ the sequence $(T_n)_{n \geq 1}$ converges uniformly on \tilde{A} in E'_σ , so for all $A \in \mathcal{A}$, $(T_n)_n$ converges uniformly on A in E'_σ .

Conversely, let $A \in \mathcal{A}$, then for all $x \in E \tilde{p}_A(x - S_n(x)) = \sup_{f \in A} \tilde{p}_x(f - T_n(f))$.

But $\lim_n \sup_{f \in A} \tilde{p}_x(f - T_n(f)) = 0$, then $\lim_n \tilde{p}_A(x - S_n(x)) = 0$ and so $\lim_n p_A(x - S_n(x)) = 0$. ■

Theorem 9. *Under the conditions of theorem 8, E is an OP - space if, and only if, for all $A \in \mathcal{A}$ the sequence $(T_n)_n$ converges uniformly on A in E'_σ .*

Proof. ([21], proposition 1) and theorem 8. ■

Proposition 30. *Let τ be a polar topology of \mathcal{A} - convergence on E and $(e_i)_i$ be a weak Schauder basis of E , then if every $\tilde{\sigma}$ - equicontinuous sequence of E' that converging pointwise to zero converges uniformly on every $A \in \mathcal{A}$ in E'_σ , then $(e_i)_i$ is a Schauder basis of (E, τ) .*

Proof. $(f_i)_i$ is an equicontinuous Schauder basis of $(E', \tilde{\sigma}(E', E)) = E'_\sigma \implies (T_n)_n$ is an equicontinuous sequence of $E'_\sigma \implies$ the sequence $(id_{E'} - T_n)_n$ is pointwise converging to zero in E'_σ and this convergence is uniformly on every $A \in \mathcal{A}$ in $E'_\sigma \implies \lim_n \sup_{f \in A} \tilde{p}_x(f - T_n(f)) = 0$ for all $x \in E \implies \lim_n \tilde{p}_A(x - S_n(x)) = 0$ for all $x \in E \implies \lim_n p_A(x - S_n(x)) = 0$ for all $x \in E$. ■

6. Application to barrelled spaces and G -spaces

Barrelled spaces

a. K is spherically complete

Proposition 31. *If (E, τ) is a barrelled locally K -convex space which is $\sigma(E, E')$ -sequentially complete and having a weak Schauder basis, then (E, τ) is complete and every weak Schauder basis is an equicontinuous Schauder basis of (E, τ) .*

Lemma 11. *If (E, τ) is barrelled and having a Schauder basis then $\tau = \tilde{\tau}$.*

Proof. Let $A \in \mathcal{A}$, where \mathcal{A} is a family that defines the topology τ ; $(\tilde{A})^\circ$ is a barrel, so it is a zero-neighbourhood in (E, τ) , then $\tilde{\tau} \leq \tau$. ■

Proof of proposition Let $(e_i)_i$ be a weak Schauder basis of E , then $(e_i)_i$ is a Schauder basis of (E, τ) (E is an OP -space) $\implies (e_i)_i$ is an equicontinuous Schauder basis of $\tilde{\tau} \implies (e_i)_i$ is an equicontinuous Schauder basis of (E, τ) ($\tau = \tilde{\tau}$). ■

b. K is not spherically complete

Proposition 32. *Every weak Schauder basis in a polarly barrelled polar locally K -convex space is an orthogonal basic sequence.*

Proof. ([21], corollary 6, p. 155). ■

Proposition 33. *Let E be a Banach space with a weak Schauder basis; then E is a polar space if and only if, every weak Schauder basis in E is a basic sequence.*

Proof. For the sufficient condition, one only has to use theorem 3.2 ($\alpha \implies \beta$) of [28]. The necessary condition is a particular case of proposition 32. ■

G -spaces

Proposition 34. *If (E, τ) is a weakly-sequentially complete G -space that having a Schauder basis, then (E, τ) is complete and this basis is equicontinuous.*

Proof. By theorem 6. ■

Proposition 35. *If (E, τ) is a G -space with a Schauder basis $(e_i)_i$, there is no strictly finer locally K -convex topology on E for which $(e_i)_i$ is still a Schauder basis.*

Proof. By proposition 19 if K is local, proposition 21 if K is spherically complete and by proposition 22 if K is not spherically complete. ■

Proposition 36. *Suppose that K is spherically complete; if (E, τ) is a weakly-sequentially complete G -space that having a weak Schauder basis $(e_i)_i$, then (E, τ) is complete and $(e_i)_i$ is an equicontinuous Schauder basis of (E, τ) .*

Proof. Let $(e_i)_i$ be a weak Schauder basis of E , then E is an OP -space $\implies (e_i)_i$ is a Schauder basis of (E, τ) . So the proposition is an immediate consequence of proposition 34. ■

References

- [1] M. G. Arsove, and R. E. Edwards, generalized bases in topological linear spaces. *Studia math.*19, pp. 95-113, (1960).
- [2] R. Ameziane Hassani and M. Babahmed, Topologies polaires compatibles avec une dualité séparante sur un corps valué non-archimédien, *Proyecciones*. Vol. 20, No. 2, pp. 217-241, (2001).
- [3] S. Banach, *Théorie des opérateurs linéaires*, Chelsea, New York (1955).
- [4] S. Bennet and J. B. Cooper, Weak basis in F and (LF) -spaces. *J. London Math. Soc.* 44, pp. 505-508, (1969).
- [5] G. Bessaga and A. Pełczyński, Properties of bases in spaces of type B_0 . *Prace Math.* 3, pp. 123-142, (1959).
- [6] N. Bourbaki, *Espaces vectoriels topologiques*, Chap.1 à 5, Paris, (1981).
- [7] N. De Grande-De Kimpe, C -compactness in locally K -convex spaces, *Indag. Math.* 33, pp. 176-180, (1971).
- [8] N. De Grande-De Kimpe, Perfect locally K -convex sequence spaces, *Indag. Math.* 33, pp. 471-482, (1971).

- [9] N. De Grande-De Kimpe, On the structure of locally K -convex spaces with a Schauder basis, *Indag. Math.* 34, pp. 396-406, (1972).
- [10] N. De Grande-De Kimpe, Equicontinuous Schauder basis and compatible locally convex topologies, *Proc. Kond. Ned. Akad. V. Wet.* A77(3), pp. 276-283 (1973).
- [11] N. De Grande-De Kimpe, On a class of locally convex spaces with a Schauder basis, *Proc. Kond. Ned. Akad. V. Wet.* pp. 307-312, (1976).
- [12] N. De Grande-De Kimpe, Structure theorems for locally K -convex spaces. *Proc. Kond. Ned. Akad. Wet.* 80: pp. 11-22, (1977).
- [13] M. De Wilde, *Reseaux dans les espaces linéaires a semi-normes.* Mem. Soc. R. Liège. (1969).
- [14] Dorleyn, M, *Beschouwingen over coördinatenruimten, oneindige matrices en determinanten in een niet-archmedisch gewaardeerd lichaam.* Thesis, Amsterdam, (1951).
- [15] E. Dubinsky, JR. Retherford, Schauder bases in compatible topologies. *Stud. Math.* 28: pp. 221-226, (1967).
- [16] T. A. Efimova, On weak basis in the inductive limits of barrelled normed spaces. *Vestnik. Leningrad Uni. Math. Meb. Astronom.* 119, pp. 21-26, (1981).
- [17] K. Floret, Bases in sequentially retractive limits spaces. *Proc. Int. Coll. on Nuclear Spaces and Ideals in operators Algebras, Warsaw 1969, Studia Math.* 38, pp. 221-226, (1970).
- [18] D. J. H. Garling, On topological sequence spaces, *Proc. Camb. Phil. Soc.* 63, pp. 997-1019, (1967).
- [19] D. J. H. Garling, The β -and γ -duality of sequence spaces, *Proc. Camb. Phil. Soc.* 63, pp. 963-981, (1967).
- [20] N. J. Kalton, On the weak-basis theorem. *Compositio Mathematica*, Vol. 27, Fasc. 2, pp. 213-215, (1973).
- [21] J. Kąkol and T. Gilsdorf, On the weak basis theorems for p -adic locally convex spaces, p -adic functional analysis edited by J. Kąkol, N. De Grande-De Kimpe and C. Perez-Garcia. Marcel Dekker, Ink. New York, (1999).

- [22] J. Kąkol, C. Perez-Garcia and W. H. Schikhof, Cardinality and Mackey topologies of non-archimedean Banach and Fréchet spaces. *Bull PolAcad Sci Math* 44: pp. 131-141, (1996).
- [23] G. Köthe, *Topologische lineare Räume*; Springer Verlag, (1960).
- [24] C. W. McArthur, The weak basis theorem, *Colloq. Math.* 17, pp. 71-76, (1967).
- [25] A. F. Monna, Espaces linéaires à une infinité dé nombrable de coordonnées. *Proc. Ned. Akad. V. Wetensch.* 53, pp. 1548-1559, (1950).
- [26] A. F. Monna, Sur le théorème de Banach-Steinhaus, *Proc. Kond. Ned. Akad. V. Wetensch.* A66, pp. 121-31 (1963).
- [27] J. Orihuela, On the equivalence of weak and Schauder bases, *Arch. Math.* Vol. 46, pp. 447-452, (1986).
- [28] C. Perez-Garcia and W. H. Schikhof, The Orlicz-Pettis property in p-adic analysis, *collect. Math.* 43, 3, pp. 225-233, (1992).
- [29] J. H. Shapiro, On the weak basis theorem in F-Spaces, *can. J. Math.* Vol. XXVI, No. 6, pp. 1294-1300, (1974).
- [30] H. H. Schaefer, *Topological vector spaces*, Springer-Verlag New-York, herdlberg Berlin, (1971).
- [31] W. H. Schikhof, Compact-like sets in non-archimedean functional analysis, *Proc. of the conférence on p-adic analysis henglehoef, Belgium*, pp. 137-147 (1986).
- [32] W. H. Schikhof, The continuous linear image of p-adic compactoid. *Proc Kon Ned Akad Wet* 92: pp. 119-123, (1989).
- [33] I. Singer, Weak*-bases in conjugate Banach spaces, *Stud. Math.* 21, (1961).
- [34] T. A. Springer, Une notion de compacité dans la théorie des espaces vectoriels topologiques, *Indag. Math.* 27, pp. 182-189 (1965).
- [35] W. J. Stiles, On properties of subspaces of l_p , $0 < p < 1$, *Trans. mer. Math. Soc.* 149, pp. 405-415 (1970).

- [36] J. Van-tiel, Espaces localement K -convexes, I-III. Proc. Kon. Ned. Akad. van Wetensch. A68, pp. 249-289 (1965).

R. Ameziane Hassani

Department of mathematics and computer science
Faculty of Sciences Dhar Mahraz
B. P. 1796 Atlas Fès,
Morocco
e-mail : ra.ameziane@yahoo.fr

A. El Amrani

Department of mathematics and computer science
Faculty of Sciences Dhar Mahraz
B. P. 1796 Atlas Fès,
Morocco
e-mail : abdelkhalek-amrani@hotmail.com

and

M. Babahmed

Department of mathematics and computer science
Faculty of Sciences Meknes,
Morocco
e-mail : babahmed@hotmail.com