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## Nonlinear Bessel potentials and generalizations of the Kato class

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### Abstract

*We study the scale of function spaces  $M_p$  introduced by Zamboni. For these spaces, we get a characterization in terms of nonlinear Bessel potentials. This result is based on a known characterization of the Kato class  $K_{n,s}$  of order  $s$  in terms of Bessel potentials and the space of bounded uniformly continuous functions.*

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## 1. Introduction

The Kato class  $K_n$  was introduced and studied by Aizenman and Simon (see [7] and [2]). For  $n \geq 3$ , it consists of locally integrable functions  $f$  on  $\mathbf{R}^n$  such that

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbf{R}^n} \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-2}} dy = 0.$$

For  $1 < p < n$ , the following classes were defined by Zamboni (see [9]): the class  $\tilde{M}_p$  of functions  $f$  such that

$$\sup_{x \in \mathbf{R}^n} \left\{ \int_{B(x,r)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(x,r)} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{\frac{1}{p-1}} dy \right\}^{p-1} < \infty,$$

and the class  $M_p$  of functions  $f$  such that  $f \in \tilde{M}_p$  and

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbf{R}^n} \left\{ \int_{B(x,r)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(x,r)} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{\frac{1}{p-1}} dy \right\}^{p-1} = 0.$$

In [3], Davies and Hinz introduced the scale  $K_{n,s}$  of the Kato classes of order  $s > 0$ . It was shown by Gulisashvili (see [4] Theorem 1) that for a locally integrable function  $f$  the following conditions are equivalent:

- (a)  $f \in K_{n,s}$  for  $s > 0$ ;
- (b)  $J^{-s}|f| \in L_\infty$  and  $\lim_{\alpha \rightarrow 0^+} \alpha^s \|J^{-s}(|f|)_\alpha\|_\infty = 0$ ;
- (c)  $J^{-s}|f| \in BUC$ .

In (a) and (c), the symbol  $J^{-s}$  stands for the Bessel potential of order  $s$ ,  $BUC$  denotes the space of bounded uniformly continuous functions on  $\mathbf{R}^n$ , and  $|f|_\alpha(x) = |f(\alpha x)|$ ,  $x \in \mathbf{R}^n$ ,  $\alpha > 0$ . Previously, this result was obtained for the Kato class  $K_n$  and the Kato class of measures  $\tilde{K}_n$ , in [6] and [5], respectively.

In the present paper, we generalize the theorem formulated above for the classes  $\tilde{M}_p$  and  $M_p$ , using the nonlinear Bessel potentials (see Theorems 1 and 2 below)

## 2. Definitions and Notation

In this section, we gather definitions and notation that will be used throughout the paper. We also include several simple lemmas. By  $L^1_{loc}(\mathbf{R}^n)$  we will denote the space of functions which are locally integrable on  $\mathbf{R}^n$ , and by  $L^1_{loc,u}$  the space of functions  $f$  such that

$$\sup_{x \in \mathbf{R}^n} \int_{B(x,1)} |f(y)| dy < \infty.$$

**DEFINITION 1.** Let  $f \in L^1_{loc}(\mathbf{R}^n)$ . For any  $1 < p < n$  and  $r > 0$ , we set

$$\Phi(r) = \sup_{x \in \mathbf{R}^n} \left( \int_{B(x,r)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(x,r)} \frac{|f(z)|dz}{|z-y|^{n-1}} \right)^{\frac{1}{p-1}} dy \right)^{p-1},$$

where  $B(x, r) = \{y : |x - y| < r\}$ .

We say that  $f$  belongs to the space  $\tilde{M}_p(\mathbf{R}^n)$ , if  $\Phi(r) < \infty$  for all  $r > 0$ .

**DEFINITION 2.** We say that a function  $f \in M_p(\mathbf{R}^n)$  if  $\lim_{r \rightarrow 0} \Phi(r) = 0$ .

We are now ready to formulate some simple properties of the classes  $M_p$  and  $\tilde{M}_p$ .

**LEMMA 1.** (See [9], p. 151) For  $1 < p < n$ , we have  
 (i)  $M_p(\mathbf{R}^n) \subset \tilde{M}_p(\mathbf{R}^n)$ , and  
 (ii)  $M_2(\mathbf{R}^n) = K_n$ .

From Lemma 1 we conclude that both  $M_p(\mathbf{R}^n)$  and  $\tilde{M}_p(\mathbf{R}^n)$  are generalizations of  $K_n$ .

**REMARK 1.** The following example shows that  $K_n$  is properly contained in  $M_p(\mathbf{R}^n)$  for  $p > 2$ . It is known that the function  $f(x) = |x|^{-2}$  is not in the Kato class  $K_n$ . However,  $f \in M_p$ . Indeed,

$$(2.1) \lim_{r \rightarrow 0} \sup_x \left\{ \int_{B(x,r)} \frac{1}{|x-y|^{n-2}} \left( \int_{B(x,r)} \frac{dz}{|z|^2 |z-y|^{n-1}} \right)^{\frac{1}{p-1}} dy \right\}^{p-1} = 0.$$

This can be shown by splitting the domain of integration in the interior integral into the following three parts  $B(x, r) \cap \{|z| < \frac{1}{2}|y|\}$ ,  $B(x, r) \cap \{\frac{1}{2}|y| \leq |z| \leq \frac{3}{2}|y|\}$  and  $B(x, r) \cap \{|z| > \frac{3}{2}|y|\}$ . After routine calculations we see that

$$\int_{B(x,r)} \frac{dz}{|z|^2 |z-y|^{n-1}}$$

is majorized by  $C|y|^{-1}$ . Finally we have

$$C \sup_x \left\{ \int_{B(x,r)} \frac{dy}{|y|^{\frac{1}{p-1}} |x-y|^{n-1}} \right\}^{p-1} \rightarrow 0 \text{ as } r \rightarrow 0,$$

this shows that (2.1) holds. Thus,  $f \in \bigcap_{p>2} M_p$ .

**REMARK 2.** (i) For  $0 < r < 1$ , it is not hard to check that for  $1 < p \leq 2$  the expression

$$\|f\|_{\tilde{M}_p(\mathbf{R}^n)} = \sup_{x \in \mathbf{R}^n} \left( \int_{B(x,1)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(x,1)} \frac{|f(z)|}{|z-y|^{n-1}} dz \right)^{\frac{1}{p-1}} dy \right)^{p-1}$$

(2.2)

defines a norm on  $\tilde{M}_p(\mathbf{R}^n)$ .

(ii) For  $p > 2$ , the expression (2.2) satisfies the following inequality.

$$(2.3) \quad \|f + g\|_{\tilde{M}_p(\mathbf{R}^n)} \leq 2^{p-2} \left( \|f\|_{\tilde{M}_p(\mathbf{R}^n)} + \|g\|_{\tilde{M}_p(\mathbf{R}^n)} \right),$$

for all  $f$  and  $g$  in  $\tilde{M}_p(\mathbf{R}^n)$ . If  $U$  is a neighborhood of 0, from (2.3) we have  $2^{p-1}U + 2^{p-1}U \subset U$ ,

then  $\tilde{M}_p(\mathbf{R}^n)$  is a topological vector space.

**LEMMA 2.**  $\tilde{M}_p(\mathbf{R}^n) \subset L^1_{loc,u}(\mathbf{R}^n)$  for  $1 < p < n$ .

**Proof.** Let  $f \in \tilde{M}_p(\mathbf{R}^n)$ , and fix  $r_0 > 0$ . Then there exists a positive constant  $C$  such that  $\Phi(r_0) \leq C$ . It follows that

$$\begin{aligned} & \sup_{x \in \mathbf{R}^n} \left( \int_{B(x,r_0)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(x,r_0)} \frac{f(z)}{|x-y|^{n-1}} dz \right)^{\frac{1}{p-1}} dy \right)^{p-1} \\ & \geq \sup_{x \in \mathbf{R}^n} \left( \int_{B(x,r_0)} \frac{dy}{r_0^{n-1}} \left( \int_{B(x,r_0)} \frac{f(z)}{(2r_0)^{n-1}} dz \right)^{\frac{1}{p-1}} \right)^{p-1} \\ & \geq \sup_{x \in \mathbf{R}^n} \left( \frac{1}{2r_0} \right)^{n-1} \left( \frac{m(B(x,r_0))}{r_0^{n-1}} \right)^{p-1} \int_{B(x,r_0)} f(z) dz. \end{aligned}$$

Therefore

$$\sup_{x \in \mathbf{R}^n} \int_{B(x,r_0)} f(z) dz < BC,$$

where

$$B = (2r_0)^{n-1} (r_0 m(B(0,1)))^{p-1}.$$

Finally, let  $B(x, 1) \subseteq \bigcup_{k=1}^n B(x_k, r_0)$ , then

$$\sup_{x \in \mathbf{R}^n} \int_{B(x,1)} f(z) dz \leq \sum_{k=1}^n \sup_{x \in \mathbf{R}^n} \int_{B(x_k,r_0)} f(z) dz,$$

so

$$\sup_{x \in \mathbf{R}^n} \int_{B(x,1)} f(z) dz < \infty$$

therefore

$$\tilde{M}_p(\mathbf{R}^n) \subset L^1_{loc,u}(\mathbf{R}^n). \quad \square$$

**LEMMA 3.** For  $1 < p < n$ ,  $\tilde{M}_p(\mathcal{R}^n)$  is a complete space.

**Proof.** Let  $\{f_n\}_{n \in \mathcal{N}}$  be a Cauchy sequence in  $\overline{B}(0, r) = \{f \in \tilde{M}_p(\mathcal{R}^n) : f|_{\tilde{M}_p(\mathcal{R}^n)} \leq r\}$ .

By Lemma 2,  $\{f_n\}_{n \in \mathcal{N}}$  is a Cauchy sequence in  $L^1_{loc,u}(\mathcal{R}^n)$ . Since this space is complete, there exists a function  $f \in L^1_{loc,u}(\mathcal{R}^n)$  such that  $f_n \rightarrow f$  in  $L^1_{loc,u}(\mathcal{R}^n)$ .

By Fatou's Lemma, we have  $f|_{\tilde{M}_p(\mathcal{R}^n)} \leq \liminf f_n|_{\tilde{M}_p(\mathcal{R}^n)} \leq r$ .

Thus  $f \in \overline{B}(0, r)$ , which means that  $\overline{B}(0, r)$  is complete with respect to the topology generated by  $L^1_{loc,u}(\mathcal{R}^n)$  - norm. By Corollary 2 of Proposition 9 in [4, Chapter III § 3, no.5] we obtain the assertion.  $\square$

**LEMMA 4.** If  $1 < p < n$ , then  $M_p(\mathbf{R}^n)$  is closed in  $\tilde{M}_p(\mathbf{R}^n)$ .

**Proof.** Let us define the map  $\varphi : \tilde{M}_p(\mathbf{R}^n) \rightarrow [0, \infty)$  by  $\varphi(f) = \lim_{r \rightarrow 0} \phi_f(r)$  (see definition 1). It is not hard to prove that the family  $\{\varphi_r\}_{r > 0}$  where  $\varphi_r(f) = \phi_f(r)$  is equicontinuous and  $\varphi_r \rightarrow \varphi$  pointwise as  $r \rightarrow 0$ . Since  $M_p(\mathbf{R}^n) = \varphi^{-1}(0)$ . We obtain the result.  $\square$

### Nonlinear Bessel Potentials

In this section, we gather some well-known results concerning Riesz and Bessel potentials (see, e.g., [8]). Let

$$G_\alpha(x) = \frac{1}{(2\pi)^{\frac{n}{2}} 2^{\frac{\alpha}{2}} \Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha-n}{2}} e^{-\frac{|x|^2}{2t} - \frac{t}{2}} \frac{dt}{t},$$

denote the Bessel kernel of order  $\alpha > 0$ . For more information on the Bessel kernel, we refer the reader to [8], Chapter 5.

**DEFINITION 3.** For any  $f \in L^1_{loc}(\mathbf{R}^n)$ , and  $\alpha > 0$ , the function

$$G_\alpha * (G_\alpha * f)^{\frac{1}{p-1}}$$

is called the nonlinear Bessel potential of  $f$ , (see [1], p. 21).

The symbol  $I_\alpha$  will stand for the Riesz potential kernel which is defined as follows:

$$(2.4) \quad I_\alpha(x) = \frac{\gamma_\alpha}{|x|^{n-\alpha}},$$

where  $\gamma_\alpha$  is a certain constant (see [8], section V.1). It is Known that

$$(2.5) \quad I_\alpha(x) = \frac{1}{(2\pi)^{\frac{n}{2}} 2^{\frac{\alpha}{2}} \Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha-n}{2}} e^{-\frac{\pi|x|^2}{2t}} \frac{dt}{t},$$

where  $0 < \alpha < n$ . We have from (2.3) and (2.4) that

$$(2.6) \quad 0 < G_\alpha(x) < I_\alpha(x) \text{ for } 0 < \alpha < n.$$

It is known that the local behavior of the Bessel potential kernel and the corresponding Riesz potential kernel is the same for  $0 < \alpha \leq n$ . It is also known that the Bessel potential kernels decay exponentially at infinity. More exactly, the following estimates holds: if  $0 < \alpha < n$ , then there exist  $C_\alpha > 0$  and  $\tilde{C}_\alpha > 0$  such that

$$(2.7) \quad \tilde{C}_\alpha |x|^{\alpha-n} \leq G_\alpha(x) \leq C_\alpha |x|^{\alpha-n},$$

for all  $x$  with  $0 < |x| < 1$ . On the other hand, for every  $\alpha > 0$  we have

$$(2.8) \quad G_\alpha(x) \leq C_\alpha e^{-c|x|},$$

for all  $x \in \mathbf{R}^n$  with  $|x| > 1$ . We have from (2.7) and (2.8) that for all  $x$  with  $0 < |x| < \infty$ ,

$$(2.9) \quad G_\alpha(x) \leq C_\alpha \left( \frac{\chi_{B(0,1)}(x)}{|x|^{n-\alpha}} + e^{-c|x|} \right).$$

### Main Results

In this section we will give a characterization of the classes  $\tilde{M}_p(\mathbf{R}^n)$  and  $M_p(\mathbf{R}^n)$  in terms of nonlinear Bessel potentials.

**REMARK 3.** *It is not hard to prove that the following are equivalent*

- (a)  $f \in M_p(\mathbf{R}^n)$ ,
- (b)  $\lim_{r \rightarrow 0^+} \sup_{x \in \mathbf{R}^n} \int_{|y-x| \leq r} \frac{1}{|x-y|^{n-1}} \left( \int_{|y-x| \leq 1} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{\frac{1}{p-1}} dy = 0$ ,
- (c)  $\lim_{r \rightarrow 0^+} \sup_{x \in \mathbf{R}^n} \int_{|y-x| \leq r} \frac{1}{|x-y|^{n-1}} \left( \int_{\mathbf{R}^n} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} = 0$ .

**THEOREM 1.** Let  $f \in L^1_{loc}(\mathbf{R}^n)$ , and  $1 < p < n$ . Then  $f \in \tilde{M}_p(\mathbf{R}^n)$  if and only if  $\sup_{x \in \mathbf{R}^n} \{G_1 * (G_1 * |f|)^{\frac{1}{p-1}}\} < \infty$ .

**Proof.** Let  $f \in \tilde{M}_p(\mathbf{R}^n)$ ,  $G_{in} = \chi_{B(0,1)}G_1$  and  $G_{out} = \chi_{\mathbf{R}^n \setminus B(0,1)}G_1$ . Since  $G_1 = G_{in} + G_{out}$  and using (2.9), we have  $\sup_{x \in \mathbf{R}^n} \{G_1 * (G_1 * |f|)^{\frac{1}{p-1}}(x)\}$

$$\begin{aligned}
 &= \sup_{x \in \mathbf{R}^n} \{(G_{in} + G_{out}) * (G_1 * |f|)^{\frac{1}{p-1}}(x)\} \\
 &\leq \sup_{x \in \mathbf{R}^n} \{G_{in} * (G_1 * |f|)^{\frac{1}{p-1}}(x)\} \\
 &\quad + \sup_{x \in \mathbf{R}^n} \{G_{out} * (G_1 * |f|)^{\frac{1}{p-1}}(x)\} \\
 &= \sup_{x \in \mathbf{R}^n} \{G_{in} * [(G_{in} + G_{out}) * |f|]^{\frac{1}{p-1}}(x)\} \\
 &\quad + \sup_{x \in \mathbf{R}^n} \{G_{out} * (G_1 * |f|)^{\frac{1}{p-1}}(x)\} \\
 &\leq \sup_{x \in \mathbf{R}^n} \{G_{in} * (G_{in} * |f|)^{\frac{1}{p-1}}(x)\} \\
 &\quad + \sup_{x \in \mathbf{R}^n} \{G_{out} * (G_1 * |f|)^{\frac{1}{p-1}}(x)\} \\
 &= \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} G_{in}(x-y) \left( \int_{\mathbf{R}^n} G_{in}(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &\quad + \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} G_{in}(x-y) \left( \int_{\mathbf{R}^n} G_{out}(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &\quad + \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} G_{out}(x-y) \left( \int_{\mathbf{R}^n} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &= \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} \chi_{B(0,1)} G_1(x-y) \left( \int_{\mathbf{R}^n} \chi_{B(0,1)} G_1(x-y) |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &\quad + \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} \chi_{B(0,1)} G_1(x-y) \left( \int_{\mathbf{R}^n} \chi_{\mathbf{R}^n \setminus B(0,1)} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &\quad + \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} \chi_{\mathbf{R}^n \setminus B(0,1)} G_1(x-y) \left( \int_{\mathbf{R}^n} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &\quad = \sup_{x \in \mathbf{R}^n} \int_{B(x,1)} G_1(x-y) \left( \int_{B(y,1)} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &\quad + \sup_{x \in \mathbf{R}^n} \int_{B(x,1)} G_1(x-y) \left( \int_{\mathbf{R}^n \setminus B(y,1)} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &\quad + \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n \setminus B(x,1)} G_1(x-y) \left( \int_{\mathbf{R}^n} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy
 \end{aligned}$$

by (2.9) we have

$$\begin{aligned}
 &\sup_{x \in \mathbf{R}^n} \{G_1 * (G_1 * |f|)^{\frac{1}{p-1}}(x)\} \\
 &\leq \sup_{x \in \mathbf{R}^n} \int_{B(x,1)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(y,1)} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{\frac{1}{p-1}} \\
 &\quad + \sup_{x \in \mathbf{R}^n} \int_{B(x,1)} \frac{1}{|x-y|^{n-1}} \left( \int_{\mathbf{R}^n \setminus B(y,1)} e^{-|y-z|} |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &\quad + \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n \setminus B(x,1)} e^{-|y-z|} \left( \int_{\mathbf{R}^n} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &\leq \sup_{x \in \mathbf{R}^n} \int_{B(x,1)} \frac{1}{|x-y|^{n-1}} \left( \int_{\mathbf{R}^n \setminus B(y,1)} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{\frac{1}{p-1}} dy
 \end{aligned}$$

$$\begin{aligned}
 &+ e^{-1} \sup_{x \in \mathbf{R}^n} \int_{B(x,1)} \frac{1}{|x-y|^{n-1}} \left( \int_{\mathbf{R}^n|B(y,1)} |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &+ \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n|B(x,1)} e^{-|x-y|} \left( \int_{\mathbf{R}^n} |f(z)| dz \right)^{\frac{1}{p-1}} dy \\
 &< \infty.
 \end{aligned}$$

To prove the sufficiency in Theorem 1, let us assume that  $\sup_{x \in \mathbf{R}^n} G_1 * (G_1 * |f|)^{\frac{1}{p-1}}(x) < \infty$  and  $0 < r < 1$ . Then (2.7) gives

$$\begin{aligned}
 &C_p \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} \frac{\chi_{B(0,r)}(x-y)}{|x-y|^{n-1}} \left( \int_{\mathbf{R}^n} \frac{\chi_{B(0,r)}(y-z) f(z)}{|y-z|^{n-1}} dz \right)^{\frac{1}{p-1}} dy \\
 &\leq \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} G_1(x-y) \left( \int_{\mathbf{R}^n} G_1(y-z) f(z) dz \right)^{\frac{1}{p-1}} dy < \infty,
 \end{aligned}$$

therefore,  $f \in \tilde{M}_p(\mathbf{R}^n)$ .

This completes the proof of Theorem 1.

□

**THEOREM 2.** For  $1 < p < n$ , then  $f \in M_p(\mathbf{R}^n)$  if and only if  $G_1 * (G_1 * |f|)^{\frac{1}{p-1}} \in BUC$ .

**Proof.** Let  $f \in M_p(\mathbf{R}^n)$ , and  $\varphi$  be any function in  $C_c(\mathbf{R}^n)$  such that

$$(2.10) \quad \varphi(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2, \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

with  $0 \leq \varphi(x) \leq 1$  for all  $x \in \mathbf{R}^n$  and  $\text{spt } \varphi \subseteq B(0, 1)$ .

Let us define  $G_{in,\alpha} = \varphi(\frac{1}{\alpha} \cdot) G_1$  and  $G_{out,\alpha} = (1 - \varphi(\frac{1}{\alpha} \cdot)) G_1$ . Observe that  $G_{out,\alpha}$  is a continuous function. We claim that  $G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}} \in BUC$  for  $1 < p < n$ , to prove this let us consider

$$\begin{aligned}
 &\sup_{x \in \mathbf{R}^n} \left| G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}}(x+h) - G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}}(x) \right| \\
 &= \sup_{x \in \mathbf{R}^n} \left| \int_{\mathbf{R}^n} [G_{out,\alpha}(x+h-y) - G_{out,\alpha}(x-y)] (G_1 * |f|)^{\frac{1}{p-1}}(y) dy \right| = I
 \end{aligned}$$

by Lemma 2 we obtain

$$\begin{aligned}
 I &\leq \sup_{x \in \mathbf{R}^n} \left| \int_{\mathbf{R}^n} (G_{out,\alpha}(x+h-y) - G_{out,\alpha}(x-y)) \right. \\
 &\quad \left. \left( \sup_{x \in \mathbf{R}^n} \int_{B(x,1)} |f(z)| \right)^{\frac{1}{p-1}} \right. \\
 &\leq \left[ \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} |G_1(x+h) - G_1(x)| + \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} |G_{in,\alpha}(x+h) \right. \\
 &\quad \left. - G_{in,\alpha}(x)| dx \right] \rightarrow 0
 \end{aligned}$$



as  $h \rightarrow 0$ , and the claim is proved.

Next we want to show that  $G_1 * (G_1 * |f|)^{\frac{1}{p-1}}$  can be approximate by  $G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}}$ . Since we get  $G_1 = G_{in,\alpha} + G_{out,\alpha}$  we have

$$\begin{aligned} & \sup_{x \in \mathbf{R}^n} \left| G_1 * (G_1 * |f|)^{\frac{1}{p-1}}(x) - G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}}(x) \right| \\ &= \sup_{x \in \mathbf{R}^n} \left| \int_{\mathbf{R}^n} G_{in,\alpha}(x-y) \left( \int_{\mathbf{R}^n} G_1(y-z) |f(x)| dz \right)^{\frac{1}{p-1}} dy \right| \\ &= \sup_{x \in \mathbf{R}^n} \int_{|x-y| \leq \alpha/2} G_1(x-y) \left( \int_{\mathbf{R}^n} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \end{aligned}$$

by hypothesis and Remark 3 we have

$$\leq \sup_x \int_{|x-y| \leq \alpha/2} \frac{1}{|x-y|^{n-1}} \left( \int_{\mathbf{R}^n} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \rightarrow 0$$

as  $\alpha \rightarrow 0$ .

Next, assume that  $G_1 * (G_1 * |f|)^{\frac{1}{p-1}} \in BUC$ . Then by theorem 1 in [5]

$\lim_{\alpha \rightarrow 0} \alpha \sup_{x \in \mathbf{R}^n} \left( \int_{\mathbf{R}^n} G_1(x-y) \left( \int_{\mathbf{R}^n} G_1(\alpha y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \right) = 0$ , using (2.7), we see that

$$\begin{aligned} & \alpha \sup_{x \in \mathbf{R}^n} \left( \int_{\mathbf{R}^n} G_1(x-\alpha y) \left( \int_{\mathbf{R}^n} G_1(\alpha y-z) |f(z)| dz \right)^{\frac{1}{p-1}} dy \right) \\ & \leq \alpha^{1-n} \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} \frac{\chi_{B(0,1)}\left(\frac{x-u}{\alpha}\right)}{\alpha^{1-n}|x-u|^{n-1}} \left( \int_{\mathbf{R}^n} \frac{\chi_{B(0,1)}(u-z)}{|u-z|^{n-1}} |f(z)| dz \right)^{\frac{1}{p-1}} du \\ & = \sup_{x \in \mathbf{R}^n} \int_{B(x,\alpha)} \frac{1}{|x-u|^{n-1}} \left( \int_{B(x,\alpha)} \frac{|f(z)|}{|u-z|^{n-1}} dz \right)^{\frac{1}{p-1}} du \rightarrow 0, \end{aligned}$$

as  $\alpha \rightarrow 0$ . Applying Theorem 1 in [5] again, we get  $f \in M_p(\mathbf{R}^n)$ .

This completes the proof of Theorem 2

□

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