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GENERALIZED ULAM-HYERS STABILITIES OF QUARTIC DERIVATIONS ON BANACH ALGEBRAS

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Abstract

Let \mathcal{A} , \mathcal{B} be two rings. A mapping $\delta : \mathcal{A} \rightarrow \mathcal{B}$ is called quartic derivation, if δ is a quartic function satisfies $\delta(ab) = a^4\delta(b) + \delta(a)b^4$ for all $a, b \in \mathcal{A}$. The main purpose of this paper to prove the generalized Hyers–Ulam–Rassias stability of the quartic derivations on Banach algebras.

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1. Introduction

The study of stability problems as just mentioned originated from a famous talk given by S.M. Ulam [65] in 1940: *Under what condition does there exist a homomorphism near an approximate homomorphism?* In 1941, D. H. Hyers [28] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \rightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed $x \in E$, then T is linear. Finally in 1978, Th. M. Rassias [60] proved the following theorem.

Theorem 1.1. *Let $f : E \rightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \quad (1.1)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p \quad (1.2)$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from \mathbf{R} into E' is continuous in real t for each fixed $x \in E$, then T is linear.

In 1991, Z. Gajda [20] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers–Ulam–Rassias stability of functional equations.

In 1982–1994, J.M. Rassias (see [46]–[53]) solved the Ulam problem for different mappings and for many Euler–Lagrange type quadratic mappings, by involving a product of different powers of norms. In addition, J.M. Rassias considered the mixed product–sum of powers of norms control function

[61]. This concept is known as Ulam–Gavruta–Rassias stability of functional equations. For more details about the results concerning such problems and mixed product-sum stability (JMRassias Stability) the reader is referred to [1, 5, 6, 7, 8, 17, 19, 22, 24, 25, 26, 27, 30, 32, 34, 36, 37, 43, 45, 54, 55] and [56].

In 1994, a generalization of the Rassias, theorem was obtained by Găvruta as follows [21] (see also [23], [31]).

Suppose $(G, +)$ is an abelian group, E is a Banach space, and that the so-called admissible control function $\varphi : G \times G \rightarrow \mathbf{R}$ satisfies

$$\tilde{\varphi}(x, y) := 2^{-1} \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. If $f : G \rightarrow E$ is a mapping with

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$, then there exists a unique mapping $T : G \rightarrow E$ such that $T(x + y) = T(x) + T(y)$ and $\|f(x) - T(x)\| \leq \tilde{\varphi}(x, x)$ for all $x, y \in G$.

In [40], Won-Gil Park and Jea Hyeong Bae, considered the following functional equation:

$$f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) + 24f(x) - 6f(y). \quad (1.3)$$

In fact they proved that a function f between real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique symmetric multi-additive function $B : X \times X \times X \times X \rightarrow Y$ such that $f(x) = B(x, x, x, x)$ for all $x \in X$. It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.3), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function (see also [2]).

Let \mathcal{A} be an algebra over the real or complex field \mathbf{F} and \mathcal{X} a left \mathcal{A} -module (respectively \mathcal{A} -bimodule). An additive map $\delta : \mathcal{A} \rightarrow \mathcal{X}$ said to be a module left derivation (respectively module derivation) if $\delta(xy) = x.\delta(y) + y.\delta(x)$ (respectively $\delta(xy) = x.\delta(y) + \delta(x).y$) holds for all $x, y \in \mathcal{A}$ where $.$ denotes the module multiplication on \mathcal{X} . Since \mathcal{A} is a left \mathcal{A} -module (respectively \mathcal{A} -bimodule) with the product of \mathcal{A} giving the module multiplication (respectively two module multiplications), the module left derivation (respectively module derivation) $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is said to be a ring left derivation (respectively ring derivation) on \mathcal{A} . Furthermore, if the identity $\delta(kx) = k\delta(x)$ holds for all $k \in \mathbf{F}$ and all $x \in \mathcal{A}$, then δ is a linear left derivation (respectively linear derivation).

Let us introduce the background of our investigation.

Recently, T. Miura et al. [35] considered the stability of ring derivations on Banach algebras: Under suitable conditions, every approximate ring derivation f on a Banach algebra \mathcal{A} is an exact ring derivation. In particular, if \mathcal{A} is a commutative semisimple Banach algebra with the maximal ideal space without isolated points, then f is identically zero. The first stability result concerning derivations between operator algebras was obtained by P. Šemrl [62] (see also [4]–[18] and [38]–[44]).

In this paper, we investigate the generalized Hyers–Ulam–Rassias stability of quartic derivations from a Banach algebra into its Banach modules.

2. Main result

In this section, we assume that \mathcal{A} is a commutative Banach algebra and \mathcal{X} a Banach \mathcal{A} -module.

Definition 2.1. A mapping $\delta : \mathcal{A} \rightarrow \mathcal{X}$ is called a quartic derivation if δ is a quartic function satisfies $\delta(ab) = \delta(a)b^4 + a^4\delta(b)$ for all $a, b \in \mathcal{A}$.

Example 2.2. We take

$$\mathcal{T} = \begin{bmatrix} 0 & \mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A} \\ 0 & 0 & \mathcal{A} & \mathcal{A} & \mathcal{A} \\ 0 & 0 & 0 & \mathcal{A} & \mathcal{A} \\ 0 & 0 & 0 & 0 & \mathcal{A} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then \mathcal{T} is a Banach algebra equipped with the usual matrix-like operations and the following norm:

$$\left\| \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\| = \sum_{i=1}^{10} \|a_i\| \quad (a_i \in \mathcal{A}).$$

It is known that

$$\mathcal{T}^* = \begin{bmatrix} 0 & \mathcal{A}^* & \mathcal{A}^* & \mathcal{A}^* & \mathcal{A}^* \\ 0 & 0 & \mathcal{A}^* & \mathcal{A}^* & \mathcal{A}^* \\ 0 & 0 & 0 & \mathcal{A}^* & \mathcal{A}^* \\ 0 & 0 & 0 & 0 & \mathcal{A}^* \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is the dual of \mathcal{T} under the following norm

$$\left\| \begin{bmatrix} 0 & f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & f_5 & f_6 & f_7 \\ 0 & 0 & 0 & f_8 & f_9 \\ 0 & 0 & 0 & 0 & f_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\| = \max\{\|f_i\|; \quad f_i \in \mathcal{T}^*(i = 1, 2, \dots, 10)\}.$$

Let the left module action of \mathcal{T} on \mathcal{T}^* be trivial and let the right module action of \mathcal{T} on \mathcal{T}^* is defined as follows:

$$\left\langle \begin{bmatrix} 0 & f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & f_5 & f_6 & f_7 \\ 0 & 0 & 0 & f_8 & f_9 \\ 0 & 0 & 0 & 0 & f_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 & b_9 \\ 0 & 0 & 0 & 0 & b_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle$$

$$= \sum_{i=1}^{10} f_i(a_i b_i)$$

for all $f_i \in \mathcal{A}^*$, $a_i, b_i \in \mathcal{A}(i = 1, \dots, 10)$. Then \mathcal{T}^* is a Banach \mathcal{T} -module. Let

$$\begin{bmatrix} 0 & f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & f_5 & f_6 & f_7 \\ 0 & 0 & 0 & f_8 & f_9 \\ 0 & 0 & 0 & 0 & f_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}^*.$$

We define $\delta : \mathcal{T} \rightarrow \mathcal{T}^*$ by

$$\delta \left(\begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & f_5 & f_6 & f_7 \\ 0 & 0 & 0 & f_8 & f_9 \\ 0 & 0 & 0 & 0 & f_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & a_1 a_2 & a_3 a_4 & a_5 a_6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we can see that δ is a quartic derivation from \mathcal{T} into \mathcal{T}^* .

Now, we investigate the generalized Hyers–Ulam–Rassias stability of quartic derivations from \mathcal{A} into \mathcal{X} . For convenience, we use the following abbreviation for a given mapping $f : \mathcal{A} \rightarrow \mathcal{X}$;

$$D_f(x, y) = f(2x + y) + f(2x - y) - 4[f(x + y) + f(x - y)] - 24f(x) + 6f(y)$$

for all $x, y \in \mathcal{A}$.

Theorem 2.3. Let $f : \mathcal{A} \rightarrow \mathcal{X}$ with $f(0) = 0$ be a mapping for which there exists function $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ such that

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\| \leq \varphi(x, y, z, t), \quad (2.1)$$

$$\tilde{\varphi}(x) := \sum_{i=0}^{\infty} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) < \infty, \quad (2.2)$$

$$\lim_{i \rightarrow \infty} \frac{1}{16^i} \varphi(2^i x, 2^i y, 2^i z, 2^i t) = 0 \quad (2.3)$$

for all $x, y, z, t \in \mathcal{A}$. Then there exists a unique quartic derivation $\delta : \mathcal{A} \rightarrow \mathcal{X}$ such that

$$\|\delta(x) - f(x)\| \leq \frac{1}{32} \tilde{\varphi}(x) \quad (2.4)$$

for all $x \in \mathcal{A}$.

Proof. Letting $z = t = y = 0$ in (2.1), we get

$$\left\| \frac{1}{16} f(2x) - f(x) \right\| \leq \frac{1}{32} \varphi(x, 0, 0, 0) \quad (2.5)$$

for all $x \in \mathcal{A}$. By induction, we have

$$\left\| \frac{1}{16^n} f(2^n x) - f(x) \right\| \leq \frac{1}{32} \sum_{i=0}^{n-1} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) \quad (2.6)$$

for all $x \in \mathcal{A}$. In order to show that functions $\delta_n(x) = \frac{1}{16^n} f(2^n x)$ form a Convergent sequence, we used Cauchy convergence criterion. In deed, replace x by $2^m x$ in (2.6) and result divide by 16^m , where m is an arbitrary positive integer, we find that

$$\left\| \frac{1}{16^{n+m}} f(2^{n+m} x) - \frac{1}{16^m} f(2^m x) \right\| \leq \frac{1}{32} \sum_{i=m}^{m+n-1} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) \quad (2.7)$$

for all $x \in \mathcal{A}$. By (2.2) and since \mathcal{X} is complete then by $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} \delta_n(x)$ exists for all $x \in \mathcal{A}$.

Let $m = 0$ and $n \rightarrow \infty$ in (2.7), we have

$$\|\delta(x) - f(x)\| \leq \frac{1}{32} \sum_{i=0}^{\infty} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) = \frac{1}{32} \tilde{\varphi}(x)$$

such that δ is defined $\delta : \mathcal{A} \rightarrow \mathcal{X}$ by $\delta(x) = \lim_{n \rightarrow \infty} \frac{1}{16^n} f(2^n x)$ for all $x \in \mathcal{A}$. Letting $z = t = 0$ and replacing x, y by $2^n x, 2^n y$, respectively, in the inequality (2.1), we get

$$\|D_f(2^n x, 2^n y)\| \leq \varphi(2^n x, 2^n y, 0, 0)$$

for all $x, y \in \mathcal{A}$, that is,

$$\left\| \frac{1}{16^n} D_f(2^n x, 2^n y) \right\| \leq \frac{1}{16^n} \varphi(2^n x, 2^n y, 0, 0)$$

for all $x, y \in \mathcal{A}$. Passing the limit $n \rightarrow \infty$, we have

$$D_\delta(x, y) = 0$$

for all $x, y \in \mathcal{A}$. Hence δ is a quartic functional equation. On the other hand, letting $x = y = 0$ and replacing z, t by $2^n z, 2^n t$, respectively, in (2.1), we obtain

$$\|f(2^{2n} zt) - 16^n z f(2^n t) - f(2^n z) 16^n t\| \leq \varphi(0, 0, 2^n z, 2^n t)$$

for all $z, t \in \mathcal{A}$. Hence

$$\left\| \frac{1}{16^{2n}} f(2^{2n} zt) - \frac{1}{16^n} z f(2^n t) - f(2^n z) \frac{1}{16^n} t \right\| \leq \frac{1}{16^{2n}} \varphi(0, 0, 2^n z, 2^n t)$$

for all $z, t \in \mathcal{A}$. Passing the limit $n \rightarrow \infty$, we obtain

$$\delta(zt) = z^4 \delta(t) + \delta(z) t^4$$

for all $z, t \in \mathcal{A}$.

Now, suppose there exists a function $\delta' : \mathcal{A} \rightarrow \mathcal{X}$ with

$$D_{\delta'}(x, y) = 0$$

for all $x, y \in \mathcal{A}$ and

$$\|\delta'(x) - f(x)\| \leq \frac{1}{32} \tilde{\varphi}(x)$$

for all $x \in \mathcal{A}$.

We have $\|\delta(x) - \delta'(x)\| = \frac{1}{16^n} \|\delta(2^n x) - \delta'(2^n x)\| = \frac{1}{16^n} (\|\delta(2^n x) - f(2^n x)\| + \|\delta'(2^n x) - f(2^n x)\|) \leq \frac{1}{16} \sum_{i=n}^{\infty} \frac{1}{16^i} \varphi(2^i x, 0)$ for all $x \in \mathcal{A}$. Passing the limit $n \rightarrow \infty$, we obtain $\delta(x) = \delta'(x)$ for all $x \in \mathcal{A}$. \square

Now, we establish the Ulam–Gavruta–Rassias stability of quadratic derivations as follows:

Corollary 2.4. *Let $p > 0, q_j > 0, (j = 1, 2, 3, 4)$ and θ be positive real numbers with*

$$\text{Max}\{p, \sum_{j=1}^4 q_j\} < 4.$$

If $f : \mathcal{A} \rightarrow \mathcal{X}$ with $f(0) = 0$ is a mapping such that

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\|$$

$$\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|t\|^p + \|x\|^{q_1} \|y\|^{q_2} \|z\|^{q_3} \|t\|^{q_4})$$

for all $x, y, z, t \in \mathcal{A}$, then there is a unique quartic derivation $\delta : \mathcal{A} \rightarrow \mathcal{X}$ such that

$$\|\delta(x) - f(x)\| \leq \frac{\theta}{32 - 2^{p+1}} \|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 2.1 taking

$$\varphi(x, y, z, t) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|t\|^p + \|x\|^{q_1} \|y\|^{q_2} \|z\|^{q_3} \|t\|^{q_4})$$

for all $x, y, z, t \in \mathcal{A}$. \square

Moreover, we investigate the superstability of quartic derivations as follows:

Corollary 2.5. *Let $q_j > 0, (j = 1, 2, 3, 4)$ with $\sum_{j=1}^4 q_j < 4$, and θ be positive real numbers. If $f : \mathcal{A} \rightarrow \mathcal{X}$ with $f(0) = 0$ is a mapping*

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\| \leq \theta(\|x\|^{q_1} \|y\|^{q_2} \|z\|^{q_3} \|t\|^{q_4})$$

for all $x, y, z, t \in \mathcal{A}$, then f is a quartic derivation.

Proof. It follows from Theorem 2.1 by putting

$$\varphi(x, y, z, t) := \theta(\|x\|^{q_1} \|y\|^{q_2} \|z\|^{q_3} \|t\|^{q_4})$$

for all $x, y, z, t \in \mathcal{A}$.

□

Theorem 2.6. Let $p_1 + p_2 < 4, q_1 + q_2 < 8$ and θ be positive real numbers. If $f : \mathcal{A} \rightarrow \mathcal{X}$ is a mapping

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\| \leq \theta(\|x\|^{p_1} \|y\|^{p_2} + \|z\|^{q_1} \|t\|^{q_2}) \quad (2.8)$$

for all $x, y, z, t \in \mathcal{A}$, then there is a unique quartic derivation $\delta : \mathcal{A} \rightarrow \mathcal{X}$ such that

$$\|\delta(x) - f(x)\| \leq \frac{\theta}{3^4 - 3^{p_1+p_2}} \|x\|^{p_1+p_2} \quad (2.9)$$

for all $x \in \mathcal{A}$.

Proof. In the inequality (2.8), let $y = x = z = t = 0$, then $23\|f(0)\| \leq 0$. Hence $f(0) = 0$. Letting $y = z = t = 0$ in (2.8), we see that $2f(x) = 2^4 f(x)$ for all $x \in \mathcal{A}$. In the inequality (2.8), put $z = t = 0$ and replace y with x . Then we obtain

$$\|f(3x) - 81f(x)\| \leq \theta\|x\|^{p_1+p_2} \quad (2.10)$$

for all $x \in \mathcal{A}$. Hence

$$\left\| \frac{f(3x)}{81} - f(x) \right\| \leq \frac{\theta}{81} \|x\|^{p_1+p_2} \quad (2.11)$$

for all $x \in \mathcal{A}$. By using the induction, we can get that

$$\left\| \frac{f(3^n x)}{81^n} - f(x) \right\| \leq \frac{\theta\|x\|^{p_1+p_2}}{81} \sum_{i=0}^{n-1} \frac{3^{i(p_1+p_2)}}{81^i} \quad (2.12)$$

for all $x \in \mathcal{A}$. It follows from $p_1 + p_2 < 4$ that the sequence $\{\frac{1}{81^n} f(3^n x)\}$ is Cauchy sequence and so it is convergent since \mathcal{X} is complete. Thus we can define a function $\delta : \mathcal{A} \rightarrow \mathcal{X}$ given by

$$\delta(x) := \lim_{n \rightarrow \infty} \frac{1}{81^n} f(3^n x) \quad (2.13)$$

for all $x \in \mathcal{A}$. In (2.12), passing the limit $n \rightarrow \infty$, we obtain the inequality (2.9). The proof of the uniqueness of δ , is similar to the proof of Theorem 2.1. □

Theorem 2.7. Let $p_1 + p_2 > 4$, $q_1 + q_2 > 8$ and θ be positive real numbers. If $f : \mathcal{A} \rightarrow \mathcal{X}$ is a mapping satisfying (2.8), then there is a unique quartic derivation $\delta : \mathcal{A} \rightarrow \mathcal{X}$ such that

$$\|f(x) - \delta(x)\| \leq \frac{\theta 3^{-(p_1+p_2)}}{1 - 3^{4-(p_1+p_2)}} \|x\|^{p_1+p_2} \quad (2.14)$$

for all $x \in \mathcal{A}$.

Proof. It follows from (2.10) that

$$\|f(x) - 81f\left(\frac{x}{3}\right)\| \leq \frac{\theta}{3^{p_1+p_2}} \|x\|^{p_1+p_2} \quad (2.15)$$

for all $x \in \mathcal{X}$. By using the induction, we can get that

$$\|f(x) - 81^n f\left(\frac{x}{3^n}\right)\| \leq \frac{\theta \|x\|^{p_1+p_2}}{81} \sum_{i=1}^n \frac{81^i}{3^{i(p_1+p_2)}} \quad (2.16)$$

for all $x \in \mathcal{A}$. It follows from $p_1 + p_2 > 4$ that the sequence $\{81^n f(\frac{x}{3^n})\}$ is Cauchy sequence and so it is convergent since \mathcal{X} is complete. Thus we can define a function $\delta : \mathcal{A} \rightarrow \mathcal{X}$ given by

$$\delta(x) := \lim_{n \rightarrow \infty} 81^n f\left(\frac{x}{3^n}\right)$$

for all $x \in \mathcal{A}$. The rest of the proof is similar to the proof of Theorem 2.3. \square

Theorem 2.8. Let $f : \mathcal{A} \rightarrow \mathcal{X}$ with $f(0) = 0$ be a mapping for which there exists function $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ such that

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\| \leq \varphi(x, y, z, t), \quad (2.17)$$

$$\tilde{\varphi}(x) := \sum_{i=1}^{\infty} 16^i \varphi(2^{-i}x, 0, 0, 0) < \infty, \quad (2.18)$$

$$\lim_{i \rightarrow \infty} 16^{2i} \varphi(2^{-i}x, 2^{-i}y, 2^{-i}z, 2^{-i}t) = 0 \quad (2.19)$$

for all $x, y, z, t \in \mathcal{A}$. Then there exists a unique quartic derivation $\delta : \mathcal{A} \rightarrow \mathcal{X}$ such that

$$\|f(x) - \delta(x)\| \leq \frac{1}{32} \tilde{\varphi}(x) \quad (2.20)$$

for all $x \in \mathcal{A}$.

Proof. It follows from (2.5) that

$$\|f(x) - 16f(2^{-1}x)\| \leq 2^{-1}\varphi(2^{-1}x, 0, 0, 0) \tag{2.21}$$

for all $x \in \mathcal{A}$. In (2.21), multiply the both sides by 16 and replace x with $2^{-1}x$, we have

$$\|16f(2^{-1}x) - 16^2f(2^{-2}x)\| \leq 2^{-1}16\varphi(2^{-2}x, 0, 0, 0) \tag{2.22}$$

for all $x \in \mathcal{A}$. From two inequalities (2.21) and (2.22), we get

$$\|f(x) - 16^2f(2^{-2}x)\| \leq 2^{-1}\varphi(2^{-1}x, 0, 0, 0) + 2^{-1}16\varphi(2^{-2}x, 0, 0, 0) \tag{2.23}$$

for all $x \in \mathcal{A}$. Continuing this way, we get

$$\|f(x) - 16^n f(2^{-n}x)\| \leq \frac{1}{32} \sum_{i=1}^n 16^i \varphi(2^{-i}x, 0, 0, 0) \tag{2.24}$$

for all $x \in \mathcal{A}$. For any positive integer m , multiply the both sides by 16^m and replace x by $2^{-m}x$ in (2.24), then we have

$$\|16^m f(2^{-m}x) - 16^{n+m} f(2^{-(n+m)}x)\| \leq \frac{1}{32} \sum_{i=1}^n 16^{i+m} \varphi(2^{-(i+m)}x, 0, 0, 0) \tag{2.25}$$

for all $x \in \mathcal{A}$. Passing the limit $m \rightarrow \infty$, the sequence $\{16^n f(2^{-n}x)\}$ is a Cauchy sequence in \mathcal{X} . By the completeness of \mathcal{X} , the sequence $\{16^n f(2^{-n}x)\}$ converges and so we can define a function $\delta : \mathcal{A} \rightarrow \mathcal{X}$ given by

$$\delta(x) = \lim_{n \rightarrow \infty} 16^n f(2^{-n}x)$$

for all $x \in \mathcal{A}$. The rest of the proof is similar to the proof of Theorem 2.1. \square

Corollary 2.9. Let $p > 0, q_j > 0, (j = 1, 2, 3, 4)$ and θ be positive real numbers with

$$\text{Min}\{p, \sum_{j=1}^4 q_j\} > 4.$$

If $f : \mathcal{A} \rightarrow \mathcal{X}$ with $f(0) = 0$ is a mapping such that

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\|$$

$$\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|t\|^p + \|x\|^{q_1} \|y\|^{q_2} \|z\|^{q_3} \|t\|^{q_4})$$

for all $x, y, z, t \in \mathcal{A}$, then there is a unique quartic derivation $\delta : \mathcal{A} \rightarrow \mathcal{X}$ such that

$$\|\delta(x) - f(x)\| \leq \frac{\theta}{32 - 2^{p+1}} \|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 2.5 taking

$$\varphi(x, y, z, t) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|t\|^p + \|x\|^{q_1} \|y\|^{q_2} \|z\|^{q_3} \|t\|^{q_4})$$

for all $x, y, z, t \in \mathcal{A}$. \square

Also, we obtain a superstability result for quartic derivations as follows:

Corollary 2.10. Let $q_j > 0$, ($j = 1, 2, 3, 4$) with $\sum_{j=1}^4 q_j > 4$, and θ be positive real numbers. If $f : \mathcal{A} \rightarrow \mathcal{X}$ with $f(0) = 0$ is a mapping

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\| \leq \theta(\|x\|^{q_1} \|y\|^{q_2} \|z\|^{q_3} \|t\|^{q_4})$$

for all $x, y, z, t \in \mathcal{A}$, then f is a quartic derivation.

In the following example, we show that the superstability of quartic derivations does not hold in general case.

Example 2.11. Let $x, y, z, t \in \mathcal{X}$ be fixed. We define $f : \mathcal{A} \rightarrow \mathcal{X}$ by $f(a) := a^4 x - xa^4 + y$ for all $a \in \mathcal{A}$,

$$\varphi(a, b, c, d) := \|D_f(x, y) - z^4 f(t) - f(z)t^4\| = \|y\| \|24 + z^4 + t^4\|.$$

Then we have

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i a, 0, 0, 0)}{16^i} = \sum_{i=0}^{\infty} \frac{\|y\| \|24 + z^4 + t^4\|}{16^i} = \frac{16}{15} \|y\| \|24 + z^4 + t^4\|,$$

$$\lim_{n \rightarrow \infty} \frac{1}{16^n} \varphi(2^n a, 2^n b, 2^n c, 2^n d) = 0$$

for all $a, b, c, d \in \mathcal{A}$. Hence $\delta(a) = \lim_{n \rightarrow \infty} \frac{f(2^n a)}{16^n} = a^4 x - xa^4$ for all $a \in \mathcal{A}$. On the other hand we have

$$\delta(ab) = (ab)^4 x - x(ab)^4 = a^4 b^4 x - xa^4 b^4,$$

$$a^4\delta(b) + \delta(a)b^4 = a^4(b^4x - xb^4) + (a^4x - xa^4)b^4 = a^4b^4x - xa^4b^4.$$

Thus

$$\delta(ab) = a^4\delta(b) + \delta(a)b^4$$

for all $a, b \in \mathcal{A}$. Furthermore,

$$\delta(2a + b) + \delta(2a - b) = [(2a + b)^4x - x(2a + b)^4] + [(2a - b)^4x - x(2a - b)^4].$$

On the other hand we have

$$\begin{aligned} & 4[\delta(a + b) + \delta(a - b)] + 24\delta(a) - 6\delta(b) \\ = & 4[((a + b)^4x - x(a + b)^4) + ((a - b)^4x - x(a - b)^4)] \\ & + 24[a^4x - xa^4] - 6[b^4x - xb^4]. \end{aligned}$$

Then δ is quartic, that is, $D_\delta(a, b) = 0$ for all $a, b \in \mathcal{A}$.

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