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A NOTE ON THE UPPER RADICALS OF SEMINEARRINGS

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Abstract

In this paper we work in the class of seminearrings. Hereditary properties inherited by the lower radical generated by a class M have been considered in [2, 5, 6, 7, 9, 10, 12]. Here we consider the dual problem, namely strong properties which are inherited by the upper radical generated by a class M.

1. Introduction and Preliminaries

V. G. Van Hoorn and B. Van Rootselaar [11] discussed general theory of seminearrings. The theory was further enriched by many authors (see [1, 3, 4, 13, 14]). The upper radicals were investigated by (see [2, 9, 12]) for radical classes of rings. Here we are interested in generalizing several results from [2, 5, 6, 7, 9, 10, 12] in the framework of seminearring, which is quite different from the ring theoretical approach discussed in [2, 5, 6, 7, 9, 10, 12]. Throughout this paper N will denote an seminearrings and ω be the universal class of all seminearrings. An semi-ideal I of N is denoted by $I \leq N$. In the following we shall be working within the class of all seminearrings.

Consider non-associative seminearrings as general algebras (N, +, .), where (N, +) is a semigroup, (N, .) is a groupoid, and only the one-side distributive law holds.

Lower radical classes for seminearrings can be constructed similar to the construction of lower radicals for rings (see [2, 5, 6, 7, 8, 9, 10, 12, 15,16]). First we include necessary preliminary, let ω be the universal class of all seminearrings and M be a sub-class of ω and let M_0 be the homomorphic closure of M in ω. For each $N \varepsilon \omega$, let $D_1(N)$ be the set of all semi-ideals of N. Inductively we define

$D_{n+1}(N) = \{I : I$ is an semi-ideal of some seminearring in $D_n(N)\}$
Let $D(N) = \bigcup_{n \in N} D_n\ (N)$, n =1, 2, 3 . . . . By using rings theoretical approach discussed in [12], we have

$$\pounds M = \{N\varepsilon\omega : D(N/I) \cap M_0 \neq 0, for\, each\, proper\, semi-ideal\, I\, of\, N\},$$

is the Lee construction for lower radical determined by M, and $M \subseteq \pounds M$ (see also [8, 15, 16]).

First we give a construction for the upper radical, dual to the construction of [10] for the lower radical. From this the theorem on the inheritance of the left strong property is deduced.

We define the following classes from a given class M of seminearrings:
$IM = \{N : N$ is a subsemi-ideal of some seminearring of $M\}$;

$TM = \{N : N$ contains an semi-ideal B such that $B\varepsilon M$ and $N/B\varepsilon M\}$;

$SM = \{N : N$ contains a descending chain of semi-ideals B_i such that $B_i = 0$ and $N/B_i\varepsilon M\}$.
It is clear that M is contained in IM and SM and that M is contained in TM if 0 belongs to M. The class M is semi-ideally closed if and only if $M = IM$. If M is semi-ideally closed then it follows easily that TM and SM are also semi-ideally closed. For undefined terms of seminearrings we may refer (see [1, 3, 4, 9, 11, 12, 13, 14, 16]).

2. Upper Radicals

We extend the results of [2, 5, 6, 7, 9, 10, 12] by using the above construction of upper radical for seminearring which is indeed provides an excellent and different approach to handle the many results of [2, 5, 6, 7, 9, 10, 12] in the frame work of seminearring.

Definition 2.1. If ρ is a radical class of seminearring then it admits a semisimple class:

$$S\rho = \{N\varepsilon\omega : \rho(N) = 0\}.$$

The following theorems were proved by N. J. Divinsky [2] for rings. Here we generalize it for seminearring which can be obtained on the line of rings theoretical approach discussed in [2].

Theorem 2.2. For any radical property N, every semi-ideal of an N-semisimple is itself N-semisimple.

Proof. The proof of our Theorem 2.2 is very similar to the proof of [2].

Theorem 2.3. The class M is the class of all S-semisimple seminearrings with respect to some radical property S if and only if M satisfies the following conditions:

(1) Every non-zero semi-ideal of a seminearring of M can be mapped homomorphically on to some non-zero seminearring of M.
(2) If every non-zero semi-ideal of a seminearring N can be mapped homomorphically onto some non-zero seminearring of M, then the seminearring

N must be in M.

Proof. The proof of our Theorem 2.3 is very similar to the proof of [2].

Theorem 2.4. A non-empty class of seminearrings M is the semisimple class with respect to some radical if and only if $M = IM = TM = SM$.
Proof. By Theorem 2.2 a semisimple class is semi-ideally closed. Also if B and N/B are semisimple and $\rho(N)$ is the radical of N, $(\rho(N) + B)/B$ is semisimple, being an semi-ideal of N/B and is also radical being isomorphic to $\rho(N)/(\rho(N) \cap B^*)$, (where B^* is a k-semi-ideal generated by B (see [8, 15, 16]). Hence $\rho(N) \subseteq B$. As B is semisimple we have $\rho(N) = 0$. Therefore N is semisimple. If B_i is a family of semi-ideals of N such that N/B_i is semisimple and $\cap B_i = 0$, then as above, $\rho(N) \subseteq B_i$ and so $\rho(N) \subseteq \cap B_i$. Therefore N is semisimple. It follows that a semisimple class M satisfies $M = IM = TM = SM$.

Conversely let M be a class of seminearrings such that $M = IM = TM = SM$. We show that M is a semisimple class by verifying the conditions (1) and (2) of Theorem 2.3. Since $M = IM$ condition (1) of Theorem 2.3 is clear. Now let N be a seminearring such that every non-zero semi-ideal of N can be mapped onto some non-zero seminearring of M. To complete the proof we must show that N is in M. Consider the family of proper semi-ideals G of N such that $N/G \varepsilon M$. From $M = SM$ and using Zorn's Lemma we see that there is an semi-ideal B minimal in this family. If $B = 0$, we are finished. If not there is an semi-ideal B minimal in the family of proper semi-ideals of B whose quotients belong to M. J is not an semi-ideal of N. Since $M = TM$ would then imply $N/J \varepsilon M$, contradicting the minimality of B. Hence either NJJ or JNJ. We may assume without loss of generality that there exists $n \varepsilon N$ with nJJ. Consider $(nJ + J)/J$. We have $nJB \subseteq nJ$ and $BnJ \subseteq BJ \subseteq J$. Hence $(nJ + J)/J$ in an semi-ideal of B/J. Since $B/J \varepsilon M$ and $IM = M$ we have $(nJ + J)/J \varepsilon M$. Consider the mapping from J to $(nJ + J)/J$ given by $\eta(x) = nx + J$, $\eta(xy) = nxy + J \subseteq NJJ + J \subseteq BJ + J \subseteq J$; $\eta(x)\eta(y) = nxny + J \subseteq NJNJ + J \subseteq BJ + J \subseteq J$. Therefore η is an epimorphism. Let K be the kernel of η , i.e. $K = \{x \varepsilon J : nx \varepsilon J\}$. Let $x \varepsilon K$, $b \varepsilon B$; then $nbx \subseteq BJ \subseteq J$, $nxb \subseteq JB \subseteq J$. Hence K is an semi-ideal of B. However we have $B/J \varepsilon M$ and

$$J/K \cong (nJ + J)/J \varepsilon M$$

Since $TM = M$ it follows that $B/K \varepsilon M$. This contradicts the minimality of J. Therefore $N \varepsilon M$ and M is a semisimple class.

Now we can give the upper radical construction. Let M be a non-empty class of seminearrings. Let $M_1 = IM$. We define M_α inductively on ordinals $\alpha > 1$ as follows. If α is not a limit ordinal $M_\alpha = TM_{\alpha-1}$. If α is a limit ordinal $M_\alpha = \mathrm{S}(\bigcup_{\beta<\alpha} M_\beta)$. Finally we set $\bar{M} = \cup M_\alpha$.

Theorem 2.5. For any non-empty class M of seminearrings, $\bar{M}$ is the smallest semisimple class containing M.

Proof. It is clear that if M is contained in semisimple class so also are IM, TM, and SM. Therefore $\bar{M}$ is contained in every semisimple class containing M. It remains to show that $\bar{M}$ is a semisimple class. It is clear that $IM_\alpha = M_\alpha$ for all α and hence that $I\ \bar{M} = \bar{M}$. Let $N \varepsilon T\bar{M}$. Then N contains an semi-ideal B with B and N/B in $\bar{M}$. Therefore there exists an ordinal α with both B and $N/B \varepsilon M_\alpha$. Therefore $N \varepsilon TM_\alpha = M_{\alpha+1}$. Hence $N \varepsilon \bar{M}$. Therefore $\bar{M} = T\bar{M}$. Let $N \varepsilon S\bar{M}$. Let B_i be the descending chain of semi-ideals of N with $\cap B_i = 0$ and $N/B \varepsilon \bar{M}$. Then $N/B_i \varepsilon M_{\alpha_i}$, for some α_i. Since the indices I form a set there exists a limit ordinal α with α_i for all i. Then $N/B_i \varepsilon \bigcup_{\beta<\alpha} M_\beta$ and so $N \varepsilon\ S(\bigcup_{\beta<\alpha} M_\beta) = M_\alpha$. Therefore $A \varepsilon \bar{M}$ and $\bar{M} = S\bar{M}$. It follows from Theorem 2.4 that $\bar{M}$ is a semisimple class of seminearrings.

Definition 2.6. A radical ρ is said to be left strong, if every radical left semi-ideal of a seminearring N is contained in the radical of N. Equivalently semisimple seminearrings contain no non-zero radical left semi-ideal of every seminearring in M has a non-zero image in IM. Clearly a radical is strong if and only if its semisimple class is left strong.

Theorem 2.7. If a non-empty class M of seminearrings is left strong then the upper radical generated by $M$is left strong.

Proof. Let ρ denote the upper radical generated by M. By Theorem 2.5 the semisimple class of the radical ρ is $\bar{M}$. We need to show that if K is a non-zero left semi-ideal of a seminearring N in $\bar{M}$ then K has a non-zero image in $\bar{M}$. Let $N \varepsilon M_\alpha$. The proof is by transfinite induction on α. First suppose that $N \varepsilon M_1 = IM$. Then N is a subsemi-ideal of a seminearring in M. Let $N = N_1 \subseteq N_2 \subseteq ... \subseteq N_n$, where $N_n \varepsilon M$ and N_i is an semi-ideal of N_{i+1}.We prove this case by induction on n. If n =1 the required result holds. $K + N_2K$ is a left semi-ideal of N_2. By the inductive assumption $K + N_2K$ has a non-zero image $(K + N_2K)/J$ in $\bar{M}$. If $K \subseteq J$ then

$$K/(K \cap J^*) \cong (K + J)/J$$

(where J^* is a k-semi-ideal generated by J (see [8, 15, 16])) is a non-zero semi-ideal of $(K+N_2K)/J$ and so is in $\bar{M}$. If $K \subseteq J$ then there exist $b\varepsilon N_2$ with bKJ; then $(J+bK)/J$ is an semi-ideal of $(K+N_2K)/J$ and so is in $\bar{M}$. As before, the mapping η from K to $(J+bK)/J$given by $\eta(x) = bx+J$ is an epimorphism and so K has a non-zero image in M as required. Now let $N\varepsilon M_\alpha$ and assume that the result has been proved for ordinals less then α. If α in not a limit ordinal then N contains an semi-ideal B and N/B in $M_{\alpha-1}$. If $K \subseteq B$ the required result holds. Otherwise

$$K/(K \cap B^*) \cong (K + B)/B,$$

(where B^* is a k-semi-ideal generated by B (see [8, 15, 16])) which is a non-zero left semi-ideal of N/B. Again the required result follows. If α is a limit ordinal then N contains a descending chain of semi-ideals Bi with $\cap Bi = 0$ and $N/B_i \varepsilon M_{\alpha_i}$, $\alpha_i < \alpha$. For some i, KB_i. Then $(K + B_i)/B_i$ is a non-zero left semi-ideal of N/B_i. Thus in all cases K has a non-zero image in $\bar{M}$. Therefore the upper radical generated by M is left strong.

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