

GRAPHS r -POLAR SPHERICAL REALIZATION*

EDUARDO MONTENEGRO

EDUARDO CABRERA

JOSÉ GONZÁLEZ

ALEJANDRO NETTLE

and

RAMÓN ROBRES

UNIVERSIDAD DE PLAYA ANCHA, CHILE

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Abstract

The graph to considered will be in general simple and finite, graphs with a nonempty set of edges. For a graph G , $V(G)$ denote the set of vertices and $E(G)$ denote the set of edges. Now, let $P_r = (0, 0, 0, r) \in \mathbf{R}^4$, $r \in \mathbf{R}^+$. The r -polar sphere, denoted by S_{P_r} , is defined by $\{x \in \mathbf{R}^4 / \|x\| = 1 \wedge x \neq P_r\}$: The primary target of this work is to present the concept of r -Polar Spherical Realization of a graph. That idea is the following one: If G is a graph and $h : V(G) \rightarrow S_{P_r}$ is a injective function, then the r -Polar Spherical Realization of G , denoted by G^* , it is a pair $(V(G^*), E(G^*))$ so that $V(G^*) = \{h(v) / v \in V(G)\}$ and $E(G^*) = \{\text{arc}(h(u)h(v)) / uv \in E(G)\}$, in where $\text{arc}(h(u)h(v))$ it is the arc of curve contained in the intersection of the plane defined by the points $h(u)$, $h(v)$, P_r and the r -polar sphere.

Key words : Graph, Sphere.

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1. Introduction. The graph to be considered will be in general simple and finite, graphs with a nonempty set of edges. For a graph G , $V(G)$ denote the set of vertices and $E(G)$ denote the set of edges. The cardinality of $V(G)$ is called *order* of G and habitually it is written down by $|G|$, and the cardinality of $E(G)$ is called *size* of G and habitually it is written down by $|E(G)|$. A (p, q) graph has order p and size q . Two vertices u and v are called *neighbors* if $\{u, v\}$ is an edge of G . For any vertex v of G , denote by N_v the set neighbors of v and by $\deg v$ the degree of v . To simplify the notation, and edge $\{x, y\}$ is written as xy (or yx). Other concepts used in this work and not defined explicitly can be found in the references [1], [2], [3],[4],[5], [7], [9],[10], [11],[12],[13].

1.1.- **r - sphere:**The r - sphere, $\{x \in \mathbf{R}^4 / \|x\| = r\}$, $r \in \mathbf{R}^+$, where $\| \cdot \|$ is the usual norm of \mathbf{R}^4 , it will be denoted by $S(r)$. That is, $S(r) = \{x \in \mathbf{R}^4 / \|x\| = r\}$.

1.2.- **r - Polar sphere:**The r -polar sphere, denoted by S_{P_r} , $r \in \mathbf{R}^+$, where $P_r = (0, 0, 0, r) \in \mathbf{R}^4$, is defined by $S(r) - \{P_r\}$. That is $S_{P_r} = S(r) - \{P_r\} = \{x \in S(r) / x \neq P_r\}$.

2. Preliminaries.

Theorem 1: The r - polar sphere admits a real tridimensional vectorial space structure, isomorphic with \mathbf{R}^3 as a vectorial space.

Proof: Let $v = (x, y, z) \in \mathbf{R}^3$. We define a function f_r of \mathbf{R}^3 in S_{P_r} by $f_r(x, y, z) = \left(\frac{2rx}{\|v\|^2+1}, \frac{2ry}{\|v\|^2+1}, \frac{2rz}{\|v\|^2+1}, r \frac{\|v\|^2-1}{\|v\|^2+1} \right)$.

Now, we will try to prove that indeed f_r is function.

$$\begin{aligned} & \text{Since } \left(\frac{2rx}{\|v\|^2+1} \right)^2 + \left(\frac{2ry}{\|v\|^2+1} \right)^2 + \left(\frac{2rz}{\|v\|^2+1} \right)^2 + r^2 \left(\frac{\|v\|^2-1}{\|v\|^2+1} \right)^2 = \\ & = r^2 \frac{4x^2+4y^2+4z^2+\|v\|^4-2\|v\|^2+1}{(\|v\|^2+1)^2} = r^2 \frac{\|v\|^4+2\|v\|^2+1}{(\|v\|^2+1)^2} = r^2 \frac{(\|v\|^2+1)^2}{(\|v\|^2+1)^2} = r^2, \\ & \text{then } f_r(x, y, z) \in S_{P_r}. \end{aligned}$$

Therefore, each $v \in \mathbf{R}^3$ has an image in S_{P_r} , we say $f_r(v)$. On the other hand, if $u = (a, b, c) = v = (x, y, z)$ in \mathbf{R}^3 , then $a = x, b = y, c = z$, $\|u\| = \|v\|$ and then $f_r(u) = f_r(v)$.

If $w = (a, b, c, t) \in S_{P_r}$, then $t \neq r$ (Note that $a^2 + b^2 + c^2 + t^2 = r^2$ and $t = r$, then $a^2 + b^2 + c^2 = 0$) and we could define the real numbers $\frac{a}{r-t}, \frac{b}{r-t}, \frac{c}{r-t}$. Therefore, the relation $g_r(a, b, c, t) = \left(\frac{a}{r-t}, \frac{b}{r-t}, \frac{c}{r-t} \right)$ is a function from S_{P_r} in \mathbf{R}^3 .

It's true that, for every $v = (x, y, z) \in \mathbf{R}^3$

$$\begin{aligned}
(g_r \circ f_r)(v) &= (g_r \circ f_r)(x, y, z) = g_r(f_r(x, y, z)) = \\
&= g_r\left(\frac{2rx}{\|v\|^2+1}, \frac{2ry}{\|v\|^2+1}, \frac{2rz}{\|v\|^2+1}, r\frac{\|v\|^2-1}{\|v\|^2+1}\right) = \\
&= \left(\frac{\frac{2rx}{\|v\|^2+1}}{r-r\frac{\|v\|^2-1}{\|v\|^2+1}}, \frac{\frac{2ry}{\|v\|^2+1}}{r-r\frac{\|v\|^2-1}{\|v\|^2+1}}, \frac{\frac{2rz}{\|v\|^2+1}}{r-r\frac{\|v\|^2-1}{\|v\|^2+1}}\right) = \\
&= \left(\frac{\frac{2x}{\|v\|^2+1}}{1-\frac{\|v\|^2-1}{\|v\|^2+1}}, \frac{\frac{2y}{\|v\|^2+1}}{1-\frac{\|v\|^2-1}{\|v\|^2+1}}, \frac{\frac{2z}{\|v\|^2+1}}{1-\frac{\|v\|^2-1}{\|v\|^2+1}}\right) = (x, y, z) = v.
\end{aligned}$$

and for every $w = (a, b, c, t) \in S_{P_r}$, we have that

$$\begin{aligned}
(f_r \circ g_r)(w) &= (f_r \circ g_r)(a, b, c, t) = f_r(g_r(a, b, c, t)) = f_r\left(\frac{a}{r-t}, \frac{b}{r-t}, \frac{c}{r-t}\right) = \\
&= \left(\frac{\frac{2ra}{r-t}}{\left(\frac{a}{r-t}\right)^2 + \left(\frac{b}{r-t}\right)^2 + \left(\frac{c}{r-t}\right)^2 + 1}, \frac{\frac{2rb}{r-t}}{\left(\frac{a}{r-t}\right)^2 + \left(\frac{b}{r-t}\right)^2 + \left(\frac{c}{r-t}\right)^2 + 1}, \frac{\frac{2rc}{r-t}}{\left(\frac{a}{r-t}\right)^2 + \left(\frac{b}{r-t}\right)^2 + \left(\frac{c}{r-t}\right)^2 + 1}, \right. \\
& r \frac{\left(\frac{a}{r-t}\right)^2 + \left(\frac{b}{r-t}\right)^2 + \left(\frac{c}{r-t}\right)^2 - 1}{\left(\frac{a}{r-t}\right)^2 + \left(\frac{b}{r-t}\right)^2 + \left(\frac{c}{r-t}\right)^2 + 1} \left. = \left(\frac{2ra}{r-t} \cdot \frac{(r-t)^2}{a^2+b^2+c^2+r^2-2rt+t^2}, \frac{2rb}{r-t} \cdot \frac{(r-t)^2}{a^2+b^2+c^2+r^2-2rt+t^2}, \right. \right. \\
& \left. \left. \frac{2rc}{r-t} \cdot \frac{(r-t)^2}{a^2+b^2+c^2+r^2-2rt+t^2}, r \frac{a^2+b^2+c^2-r^2+2rt-t^2}{a^2+b^2+c^2+r^2-2rt+t^2}\right) = \\
&= \left(a, b, c, r \frac{r^2-t^2-r^2+2rt-t^2}{r^2-t^2+r^2-2rt+t^2}\right) = \left(a, b, c, r \frac{2rt-2t^2}{2r^2-2rt}\right) = \left(a, b, c, t \frac{2r-2t}{2r-2t}\right) = w.
\end{aligned}$$

Therefore, f_r is invertible and bijective.

Now, we define S_{P_r} as a real vectorial space structure.

To get this, we define a binary operation in S_{P_r} , defined by \oplus , as

$$\forall u, v \in \mathbf{R}^3, u = (a, b, c), v = (x, y, z) : f_r(u) \oplus f_r(v) = f_r(a, b, c) \oplus f_r(x, y, z) = \left(\frac{2r(a+x)}{\|u+v\|^2+1}, \frac{2r(b+y)}{\|u+v\|^2+1}, \frac{2r(c+z)}{\|u+v\|^2+1}, r\frac{\|u+v\|^2-1}{\|u+v\|^2+1}\right).$$

It is evident that $f_r(u) \oplus f_r(v) = f_r(u+v)$.

Operation \oplus satisfies the following properties:

(i) Closure: $\forall \tilde{u}, \tilde{v} \in S_{P_r}, \exists \tilde{w} \in S_{P_r} : \tilde{u} \oplus \tilde{v} = \tilde{w}$

Proof: Let $u = (a, b, c), v = (x, y, z) \in \mathbf{R}^3$ be so that $\tilde{u} = f_r(u)$ and $\tilde{v} = f_r(v)$. Define $\tilde{w} = \left(\frac{2r(a+x)}{\|u+v\|^2+1}, \frac{2r(b+y)}{\|u+v\|^2+1}, \frac{2r(c+z)}{\|u+v\|^2+1}, r\frac{\|u+v\|^2-1}{\|u+v\|^2+1}\right)$, then

$$\tilde{u} \oplus \tilde{v} = f_r(u) \oplus f_r(v) = f_r(u+v) = \tilde{w}.$$

(ii) Associative property: $\forall \tilde{u}, \tilde{v}, \tilde{w} \in S_{P_r} : (\tilde{u} \oplus \tilde{v}) \oplus \tilde{w} = \tilde{u} \oplus (\tilde{v} \oplus \tilde{w})$

Proof: Let $u, v, w \in \mathbf{R}^3$ be so that $\tilde{u} = f(u), \tilde{v} = f(v)$ and $\tilde{w} = f(w)$. Then $(\tilde{u} \oplus \tilde{v}) \oplus \tilde{w} = f_r((u + v) + w) = f_r(u + (v + w)) = \tilde{u} \oplus (\tilde{v} \oplus \tilde{w})$.

(iii) Existence of the neutral element:

$$\exists \tilde{0} \in S_{P_r}, \forall \tilde{u} \in S_{P_r}, : \tilde{u} \oplus \tilde{0} = \tilde{0} \oplus \tilde{u} = \tilde{u}$$

Proof: Define $\tilde{0} = f_r((0, 0, 0)) = (0, 0, 0, -r)$ and $u = (x, y, z) \in \mathbf{R}^3$ so

that $\tilde{u} = f_r(x, y, z) \in S_{P_r}$. Then $\tilde{u} \oplus \tilde{0} = \left(\frac{2r(x+0)}{\|u\|^2+1}, \frac{2r(y+0)}{\|u\|^2+1}, \frac{2r(z+0)}{\|u\|^2+1}, r \frac{\|u\|^2-1}{\|u\|^2+1} \right) = f_r(x, y, z) = \tilde{u}$. Similarly $\tilde{0} \oplus \tilde{u} = \tilde{u}$.

(iv) Existence of the inverse element:

$$\forall \tilde{u} \in S_{P_r}, \exists \tilde{v} \in S_{P_r} : \tilde{u} \oplus \tilde{v} = \tilde{v} \oplus \tilde{u} = \tilde{0}$$

Proof: Let be so that $\tilde{u} = f_r(x, y, z) \in S_{P_r}$ and $\mathbf{0} = (0, 0, 0) \in \mathbf{R}^3$.

Define $v = (-x, -y, -z) \in \mathbf{R}^3$. Then $\tilde{u} \oplus \tilde{v} = f_r(x, y, z) \oplus f_r(-x, -y, -z) = \left(\frac{2r(x-x)}{\|0\|^2+1}, \frac{2r(y-y)}{\|0\|^2+1}, \frac{2r(z-z)}{\|0\|^2+1}, r \frac{\|0\|^2-1}{\|0\|^2+1} \right) = \tilde{0}$. Similarly $\tilde{v} \oplus \tilde{u} = \tilde{0}$.

(v) Commutative property: $\forall \tilde{u}, \tilde{v} \in S_{P_r} : \tilde{u} \oplus \tilde{v} = \tilde{v} \oplus \tilde{u}$.

Proof: Let $u, v, w \in \mathbf{R}^3$ be so that $\tilde{u} = f_r(u)$ and $\tilde{v} = f_r(v)$. Then $\tilde{u} \oplus \tilde{v} = f_r(u + v) = f_r(v + u) = \tilde{v} \oplus \tilde{u}$.

Consequently $\langle S_{P_r}, \oplus \rangle$ is an abelian group.

Here, we would define an external operation $\odot : \mathbf{R} \times S_{P_r} \rightarrow S_{P_r}$ as

$$\forall \alpha \in \mathbf{R}, \forall v = (x, y, z) \in \mathbf{R}^3 : \alpha \odot f_r(v) = \left(\frac{2\alpha r x}{\alpha^2 \|v\|^2 + 1}, \frac{2\alpha r y}{\alpha^2 \|v\|^2 + 1}, \frac{2\alpha r z}{\alpha^2 \|v\|^2 + 1}, r \frac{\alpha^2 \|v\|^2 - 1}{\alpha^2 \|v\|^2 + 1} \right)$$

It is evident that $\alpha \odot f_r(v) = f_r(\alpha v)$ and likewise $\alpha \odot f_r(v) \in S_{P_r}$.

We should prove the following properties:

$$(i) \quad \forall \alpha, \beta \in \mathbf{R}, \forall v \in \mathbf{R}^3 : (\alpha + \beta) \odot f_r(v) = [\alpha \odot f_r(v)] \oplus [\beta \odot f_r(v)],$$

Proof: $(\alpha + \beta) \odot f_r(v) = f_r((\alpha + \beta)v) = f_r(\alpha v + \beta v) = f_r(\alpha v) \oplus f_r(\beta v) = [\alpha \odot f_r(v)] \oplus [\beta \odot f_r(v)]$

$$(ii) \quad \forall \alpha \in \mathbf{R}, \forall u, v \in \mathbf{R}^3 : \alpha \odot [f_r(u) \oplus f_r(v)] = [\alpha \odot f_r(u)] \oplus [\alpha \odot f_r(v)].$$

Proof: $\alpha \odot [f_r(u) \oplus f_r(v)] = \alpha \odot f_r(u + v) = f_r(\alpha(u + v)) = f_r(\alpha u + \alpha v) = f_r(\alpha u) \oplus f_r(\alpha v) = [\alpha \odot f_r(u)] \oplus [\alpha \odot f_r(v)]$.

$$(iii) \quad \forall \alpha, \beta \in \mathbf{R}, : \alpha \odot (\beta \odot f_r(v)) = (\alpha\beta) \odot f_r(v).$$

$$\text{Proof: } \alpha \odot (\beta \odot f_r(v)) = \alpha \odot f_r(\beta v) = f_r(\alpha\beta v) = (\alpha\beta) \odot f_r(v)$$

$$(iv) \quad \forall v \in \mathbf{R}^3 : 1 \odot f_r(v) = f_r(v) \quad (\text{Here } 1 \text{ is the neutral multiplicative from } \mathbf{R})$$

$$\text{Proof: } 1 \odot f_r(v) = f_r(1v) = f_r(v)$$

From the above points, we have that S_{P_r} is a \mathbf{R} -vectorial space.

Now, it will be proven that S_{P_r} has dimension three as real vectorial space and that f_r defines a linear transformation F of \mathbf{R}^3 in S_{P_r} .

First, we will prove that $\{w_1, w_2, w_3\}$ is a basis for S_{P_r} , where

$$w_1 = f_r(e_1) = (r, 0, 0, 0), w_2 = f_r(e_2) = (0, r, 0, 0), w_3 = f_r(e_3) = (0, 0, r, 0)$$

If $v = (x, y, z, t) \in S_{P_r}, t \neq r$, then there exist some $\alpha, \beta, \gamma \in \mathbf{R}$ so that $v = \alpha \odot f_r(e_1) \oplus \beta \odot f_r(e_2) \oplus \gamma \odot f_r(e_3)$.

Consequently $\alpha \odot f_r(e_1) = f_r(\alpha, 0, 0), \beta \odot f_r(e_2) = f_r(0, \beta, 0), \gamma \odot f_r(e_3) = f_r(0, 0, \gamma)$.

$$\begin{aligned} \text{Therefore } (x, y, z, t) &= f_r(\alpha, 0, 0) \oplus f_r(0, \beta, 0) \oplus f_r(0, 0, \gamma) = \\ &= f_r(\alpha, \beta, 0) \oplus f_r(0, 0, \gamma) = \left(\frac{2r\alpha}{\alpha^2 + \beta^2 + \gamma^2 + 1}, \frac{2r\beta}{\alpha^2 + \beta^2 + \gamma^2 + 1}, \frac{2r\gamma}{\alpha^2 + \beta^2 + \gamma^2 + 1}, r \frac{\alpha^2 + \beta^2 + \gamma^2 - 1}{\alpha^2 + \beta^2 + \gamma^2 + 1} \right). \end{aligned}$$

Therefore

$$x = \frac{2r\alpha}{\alpha^2 + \beta^2 + \gamma^2 + 1}, y = \frac{2r\beta}{\alpha^2 + \beta^2 + \gamma^2 + 1}, z = \frac{2r\gamma}{\alpha^2 + \beta^2 + \gamma^2 + 1}, t = r \frac{\alpha^2 + \beta^2 + \gamma^2 - 1}{\alpha^2 + \beta^2 + \gamma^2 + 1} \implies \alpha^2 + \beta^2 + \gamma^2 = \frac{r+t}{r-t} \implies$$

$$\implies x = \frac{2r\alpha}{\frac{r+t}{r-t} + 1}, y = \frac{2r\beta}{\frac{r+t}{r-t} + 1}, z = \frac{2r\gamma}{\frac{r+t}{r-t} + 1} \implies \alpha = \frac{x}{r-t}, \beta = \frac{y}{r-t}, \gamma = \frac{z}{r-t}$$

As a neutral element for the \oplus operation we have $-P_r = (0, 0, 0, -r)$, then $\alpha \odot f_r(e_1) \oplus \beta \odot f_r(e_2) \oplus \gamma \odot f_r(e_3) = -P_r \implies \alpha = \beta = \gamma = 0$.

Finally, the function F_r form \mathbf{R}^3 in S_{P_r} defined by $F_r(x, y, z) = f_r(x, y, z)$ is linear.

Corolary 1.1 : Any simple and finite graph G is realizable over S_{P_r} .

PROOF : Using a graph theorem that ensures all simple and finite graph G is realizable over \mathbf{R}^3 , its proof could be verified in [3], then we obtain the requested in the prior Theorem 1.

With the following definition we claim to obtain an always limited distance measure between two graphs, limited by a fixed number μ . In our research, $\mu = 3\pi^2 r^3$ [6],[8].

3 Graphs r -Polar Spheric Realization

If G is a simple and finite graph, and $h : V(G) \rightarrow S_{P_r}$ is an injective function, then the r -Polar Spheric Realization from G , denoted by G^* is a couple $(V(G^*), E(G^*))$ such that $V(G^*) = \{h(v)/v \in V(G)\} \subset S_{P_r}$ y $E(G^*) = \{arc(h(u)h(v))/uv \in V(G)\}$, where $arc(h(u)h(v))$ is the arc of the curve contained in the intersection of the plane, defined by the points $h(u), h(v), P$ and the r -polar sphere.

In the context of the polar spheric realization we consider the following restrictions:

- (1) As Ω we record the class of all finite and simple graphs class.
- (2) As Ω^* we record the families of all the polar spheric realizations form Ω .
- (3) If $G, H \in \Omega$ y $G \neq H$, then $V(G^*) \cap V(H^*) = \phi$ and $E(G^*) \cap E(H^*) = \phi$.
- (4) If $G \in \Omega$, $G^* \in \Omega^*$. (So that all the Ω and Ω^* elements are defined as univocal.).

Some r -polar spheric realization examples are:

1.- G is the graph defined by $V(G) = \{1, 2, 3\}$ and $E(G) = \{12, 23, 13\}$, then the 1-polar spheric realization from G is

$$V(G^*) = \{i, j, k\} \text{ and } E(G^*) = \{arc(ij), arc(ik), arc(jk)\},$$

where $h(1) = i = (1, 0, 0, 0)$, $h(2) = j = (0, 1, 0, 0)$, $h(3) = k = (0, 0, 1, 0)$

2.- G is the graph defined by $V(G) = \{a, b\}$ and $E(G) = \{ab\}$, then the 1-polar spheric realization from G is $V(G^*) = \left\{ \left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, 0, 0 \right), \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \right) \right\}$ and $E(G^*) = \{arc(h(a)h(b))\}$., where $h(a) = \left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, 0, 0 \right)$, $h(b) = \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \right)$.

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Eduardo Montenegro

Departamento de Matemáticas y Física
Facultad de Ciencias Naturales y Exactas
Universidad de Playa Ancha
Casilla 34 V
Valparaíso
Fax: (56)(32)286713
e-mail : emontene@upla.cl

Eduardo Cabrera

Departamento de Matemáticas y Física
Facultad de Ciencias Naturales y Exactas
Universidad de Playa Ancha
Casilla 34 V
Valparaíso
e-mail : ecabrera@upla.cl

José González

Departamento de Matemáticas y Física
Facultad de Ciencias Naturales y Exactas
Universidad de Playa Ancha
Casilla 34 V
Valparaíso
e-mail : egonzalez@upla.cl

Alejandro Nettle

Departamento de Matemáticas y Física
Facultad de Ciencias Naturales y Exactas
Universidad de Playa Ancha
Casilla 34 V
Valparaíso
e-mail : anettle@upla.cl

and

Ramón Robres

Departamento de Matemáticas y Física
Facultad de Ciencias Naturales y Exactas
Universidad de Playa Ancha
Casilla 34 V
Valparaíso
e-mail : rrobres@upla.cl