

Proyecciones Journal of Mathematics
Vol. 29, N° 1, pp. 1-8, May 2010.
Universidad Católica del Norte
Antofagasta - Chile

SOME DIFFERENCE SEQUENCES DEFINED BY A SEQUENCE OF MODULUS FUNCTIONS

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Received : November 2009. Accepted : January 2010

Abstract

The idea of difference sequence spaces was introduced by Kizmaz [6], and this concept was generalized by Bektas and Colak [1]. In this paper, we define the sequence spaces $c_0(F, \Delta_u^m x)$ and $l_\infty(F, \Delta_u^m x)$, where $F = (f_k)$ is a sequence of modulus functions, and examine some inclusion relations and properties of these spaces.

Subclass [2000] : 40A05, 40C05, 46A45.

Keywords : *Difference sequence spaces, A sequence of modulus functions.*

1. Definitions and notations

Let w denote the set of all complex sequences $x = (x_k)$, and l_∞, c , and c_0 be the linear spaces of bounded, convergent, and null sequences with complex terms, respectively, normed by $\|x\| = \sup_k |x_k|$, where $k \in \mathbf{N}$, the set of positive integers.

Kizmaz [6] defined the sequence spaces

$$l_\infty(\Delta) = \{x \in w : \Delta x \in l_\infty\},$$

$$c(\Delta) = \{x \in w : \Delta x \in c\}, \text{ and}$$

$c_0(\Delta) = \{x \in w : \Delta x \in c_0\}$, where for any sequence $x = (x_k)$, the difference sequence Δx is defined by

$$\Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty,$$

$$\Delta^0 x = (\Delta^0 x_k)_{k=1}^\infty = (x_k)_{k=1}^\infty.$$

Kizmaz [6] proved that these are Banach spaces with the norm $\|x\| = \|x_1\| + \|x\|_\infty$. Also, he showed that $E \subset E(\Delta)$, where $E = \{l_\infty, c, c_0\}$, since there exists a sequence $x = (x_k)$ such that $x_k = k$, for each k , that is, $x = (1, 2, 3, \dots)$ for which $\Delta x = (-1, -1, -1, \dots)$, so that although x is not convergent, but it is Δ -convergent.

If m is a nonnegative integer and $x = (x_k)$ is any sequence, then the difference sequences $\Delta x, \Delta^2 x, \dots, \Delta^m x$ are defined by

$$\Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty,$$

$$\Delta^2 x = \Delta(\Delta x) = \Delta x_k - \Delta x_{k+1},$$

\vdots

$$\Delta^m x = \Delta(\Delta^{m-1} x) = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1},$$

so that

$$\Delta^m x_k = \sum_{r=0}^m (-1)^r \binom{m}{r} x_{k+r}.$$

Let U be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for each k . Then Gnanaseelan and Srivastava [4] defined and studied the sequence spaces

$$E(u, \Delta) = \{x \in w : u \Delta x \in E\},$$

$$\text{where } E = \{l_\infty, c, c_0\} \text{ and } u \Delta x = u_k x_k - u_{k+1} x_{k+1}.$$

After then, Et. and Colak [2] defined the sequence spaces

$$E(\Delta^m) = \{x \in w : \Delta^m x \in E\},$$

$$\text{where } E = \{l_\infty, c, c_0\}.$$

They proved that these are Banach spaces with the norm $\|x\| = \sum_{i=1}^m \|x_i\| + \|\Delta^m x\|_\infty$.

We recall that a modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

- (i) $f(x) = 0$ if and only if $x = 0$,
- (ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$,
- (iii) f is increasing,
- (iv) f is continuous from the right at 0.

Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from conditions (ii) and (iv) that f is continuous everywhere on $[0, \infty)$.

A modulus function may be bounded or unbounded. For example, $f(t) = \frac{t}{t+1}$ is bounded but $f(t) = t^p$ ($0 < p \leq 1$) is unbounded. Furthermore, we have $f(nx) \leq nf(x)$, from condition (ii), and so $f(x) = f(nx \frac{1}{n}) \leq nf(\frac{x}{n})$, hence $\frac{1}{n}f(x) \leq f(\frac{x}{n})$, for all $n \in \mathbf{N}$.

Ruckle [13] used the concept of modulus function to define the sequence space

$$L(f) = \{x \in w : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

If we take $f(x) = x^p$, then $L(f)$ reduces to the familiar space l_p which is given by

$$l_p = \{x \in w : \sum_{k=1}^{\infty} |x_k|^p < \infty\}.$$

In particular, if $f(x) = x$, then $L(f) = l_1$ which is given by

$$l_1 = \{x \in w : \sum_{k=1}^{\infty} |x_k| < \infty\}.$$

Several authors including Maddox ([8]-[11]), Ozturk and Bilgin [12], and some others studied some sequence spaces defined by a modulus function. Let X be a sequence space. Then the sequence space $X(f)$ is defined by

$$X(f) = \{x \in w : f(|x_k|) \in X\}.$$

Kolk [7] gave an extension of $X(f)$ by considering a sequence of modulus functions $F = (f_k)$ and defined the space

$$X(F) = \{x \in w : f_k(|x_k|) \in X\}.$$

Gaur and Mursaleen [3] defined and studied the following sequence spaces

$$l_{\infty}(F, \Delta) = \{x \in w : \Delta x \in l_{\infty}(F)\},$$

and

$$c_0(F, \Delta) = \{x \in w : \Delta x \in c_0(F)\}.$$

For a nonnegative integer m , Bektas and Colak [1] extended the above mentioned spaces to

$$l_{\infty}(F, \Delta^m) = \{x \in w : \Delta^m x \in l_{\infty}(F)\},$$

and

$$c_0(F, \Delta^m) = \{x \in w : \Delta^m x \in c_0(F)\}.$$

We further give an extension of the spaces of Bektas and Colak [1] and define the sequence spaces

$$l_\infty(F, \Delta_u^m) = \{x \in w : \Delta_u^m x \in l_\infty(F)\},$$

and

$$c_0(F, \Delta_u^m) = \{x \in w : \Delta_u^m x \in c_0(F)\},$$

where $u = (u_k)$ is any sequence such that $u_k \neq 0$ for each k , and

$$\Delta_u^0 x = u_k x_k,$$

$$\Delta_u^1 x = u_k x_k - u_{k+1} x_{k+1},$$

$$\Delta_u^2 x = \Delta(\Delta_u^1 x),$$

\vdots

$$\Delta_u^m x = \Delta(\Delta_u^{m-1} x),$$

so that

$$\Delta_u^m x = \Delta_{u_k}^m x_k = \sum_{r=0}^m (-1)^r \binom{m}{r} u_{k+r} x_{k+r}.$$

If $u = e = (1, 1, 1, \dots)$, then these spaces will give the spaces of Bektaş and Colak [1] as special cases.

2. Main results

For a sequence $F = (f_k)$ of modulus functions, we will give the necessary and sufficient conditions for the inclusion between $X(\Delta_u^m)$ and $Y(F, \Delta_u^m)$, where $X, Y = l_\infty$ or c_0 .

We need the following Lemmas (see Kolk [7]) :

Lemma 2.1. *The condition $\sup_k f_k(t) < \infty, t > 0$ holds if and only if there exists a point $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty$.*

Lemma 2.2. *The condition $\inf_k f_k(t) > 0, t > 0$ holds if and only if there exists a point $t_0 > 0$ such that $\inf_k f_k(t_0) > 0$.*

Now, we prove the following theorems

Theorem 2.3. *For a sequence $F = (f_k)$ of modulus functions, the following statements are equivalent :*

- (i) $l_\infty(\Delta_u^m) \subseteq l_\infty(F, \Delta_u^m)$
- (ii) $c_0(\Delta_u^m) \subseteq l_\infty(F, \Delta_u^m)$
- (iii) $\sup_k f_k(t) < \infty, t > 0$

Proof. (i) implies (ii) is obvious

(ii) implies (iii) : Let $c_0(\Delta_u^m) \subseteq l_\infty(F, \Delta_u^m)$. Suppose that (iii) is not true. Then by Lemma 2.1 , we see that $\sup_k f_k(t) = \infty$, for all $t > 0$, and therefore there is a sequence (k_i) of positive integers such that

$$(2.1) \quad f_{k_i}\left(\frac{1}{i}\right) > i \text{ for } i = 1, 2, 3, \dots$$

Define $x = (x_k)$ as follows

$$x_k = \begin{cases} \frac{1}{i}, & k = k_i \\ 0, & \text{otherwise} \end{cases}$$

Then $x \in c_0(\Delta_u^m)$, but by (2.1), $x \notin l_\infty(F, \Delta_u^m)$ which contradicts (ii). Hence (iii) must hold.

(iii) implies (i) : Let (iii) be satisfied and $x \in l_\infty(\Delta_u^m)$. If we suppose that $x \notin l_\infty(F, \Delta_u^m)$, then $\sup_k f_k(|\Delta_u^m x_k|) = \infty$ for $\Delta_u^m x \in l_\infty$. Now take $t = |\Delta_u^m x|$, then $\sup_k f_k(t) = \infty$ which contradicts (iii). Hence $l_\infty(\Delta_u^m) \subseteq l_\infty(F, \Delta_u^m)$. This completes the proof of the theorem. \square

Theorem 2.4. For a sequence $F = (f_k)$ of modulus functions, the following statements are equivalent :

- (i) $c_0(F, \Delta_u^m) \subseteq c_0(\Delta_u^m)$
- (ii) $c_0(F, \Delta_u^m) \subseteq l_\infty(\Delta_u^m)$
- (iii) $\inf_k f_k(t) > 0, (t > 0)$

Proof. (i) implies (ii) is obvious

(ii) implies (iii) : Let $c_0(F, \Delta_u^m) \subseteq l_\infty(\Delta_u^m)$. Suppose that (iii) does not hold. Then by Lemma 2.2, we see that

$$(2.2) \quad \inf_k f_k(t) = \infty, (t > 0)$$

and therefore there is a sequence (k_i) of positive integers such that $f_{k_i}(i^2) < \frac{1}{i}$ for $i = 1, 2, 3, \dots$.

Define $x = (x_k)$ as follows

$$x_k = \begin{cases} i^2, & k = k_i \text{ for } i = 1, 2, 3, \dots \\ 0, & \text{otherwise} \end{cases}.$$

Then by (2.2), $x \in c_0(F, \Delta_u^m)$ but $x \notin l_\infty(\Delta_u^m)$ which contradicts (ii). Hence (iii) must hold.

(iii) implies (i) : Let (iii) be satisfied and $x \in c_0(F, \Delta_u^m)$ that is $\lim_k f_k(|\Delta_u^m x_k|) = 0$. Suppose that $x \notin c_0(\Delta_u^m)$, then for some number $\varepsilon_0 > 0$ and positive integer k_0 , we have $|\Delta_u^m x_k| \varepsilon_0 > 0$ for $k \geq k_0$. Therefore $f_k(\varepsilon_0) \leq$

$f_k(|\Delta_u^m x_k|)$ for $k \geq k_0$ and hence $\lim_k f_k(\varepsilon_0) = 0$ which contradicts (iii). Thus $c_0(F, \Delta_u^m) \subseteq c_0(\Delta_u^m)$ and this completes the proof of the theorem. \square

Theorem 2.5. *The inclusion $l_\infty(F, \Delta_u^m) \subseteq c_0(\Delta_u^m)$ holds if and only if*

$$(2.3) \quad \lim_k f_k(t) = \infty \text{ for } t > 0$$

Proof. Let $l_\infty(F, \Delta_u^m) \subseteq c_0(\Delta_u^m)$ and suppose that (2.3) does not hold. Then there is a number $t_0 > 0$ and a sequence (k_i) of positive integers such that

$$(2.4) \quad f_{k_i}(t_0) \leq L < \infty$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0, & k = k_i \text{ for } i = 1, 2, 3, \dots \\ 0, & \text{otherwise} \end{cases}$$

Then $x \in l_\infty(F, \Delta_u^m)$ by (2.4) but $x \notin c_0(\Delta_u^m)$ so that (2.4) must hold if $l_\infty(F, \Delta_u^m) \subseteq c_0(\Delta_u^m)$.

Conversely, Suppose that (2.3) is satisfied. If $x \in l_\infty(F, \Delta_u^m)$, then $f_k(|\Delta_u^m x_k|) \leq L < \infty$ for $k = 1, 2, 3, \dots$

Now suppose that $x \notin c_0(\Delta_u^m)$. Then for some $\varepsilon_0 > 0$ and positive integer k_0 , we have $|\Delta_u^m x_k| > \varepsilon_0$ for $k \geq k_0$. Therefore $f_k(\varepsilon_0) \leq f_k(|\Delta_u^m x_k|) \leq L$ for $k \geq k_0$ which contradicts (2.3). Hence $x \in c_0(\Delta_u^m)$. \square

Theorem 2.6. *The inclusion $l_\infty(\Delta_u^m) \subseteq c_0(F, \Delta_u^m)$ holds if and only if*

$$(2.5) \quad \lim_k f_k(t) = 0 \text{ for } t > 0$$

Proof. Let $l_\infty(\Delta_u^m) \subseteq c_0(F, \Delta_u^m)$ and suppose that (2.5) does not hold. Then

$$(2.6) \quad \lim_k f_k(t) = l \neq 0$$

for some $t_0 > 0$.

Define the sequence $x = (x_k)$ by

$$x_k = t_0 \sum_{r=0}^m (-1)^m \binom{m+k-u-1}{k-u} \text{ for } k = 1, 2, 3, \dots$$

Then $x \notin c_0(F, \Delta_u^m)$ by (2.6). Hence (2.5) must hold.

Conversely, Suppose that (2.5) is satisfied. If $x \in l_\infty(\Delta_u^m)$, then $|\Delta_u^m x_k| \leq L < \infty$ for $k = 1, 2, 3, \dots$

Therefore $f_k(|\Delta_u^m x_k|) \leq f_k(L)$ for $k = 1, 2, 3, \dots$ and $\lim_k f_k(|\Delta_u^m x_k|) \leq \lim_k f_k(L) = 0$, by (2.5). Hence $x \in c_0(F, \Delta_u^m)$. \square

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