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SCHUR RING AND QUASI–SIMPLE MODULES

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Abstract

Let R be a ring of algebraic integers of an algebraic number field F and let $G \leq GL_n(R)$ be a finite group. In this paper we show that the R-span of G is just the matrix ring $M_n(R)$ of the $n \times n$-matrices over R if and only if $G/O_{p_i}(G)$ is absolutely simple for all $p_i \in \pi$, where π is the set of the positive prime divisors of $|G|$ and $O_{p_i}(G)$ is the largest normal p_i-subgroup.

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1. Introduction.

Let F be an algebraic number field with ring of algebraic integers R and let $\pi = \{p_1, ..., p_t\}$ be a set of positive prime numbers. Assume that the I_i are maximal ideals of R such that $p_i \in I_i (i = 1, \dots, t)$. Set $L_\pi = \{\frac{a}{b} \mid a, b \in R, b \notin I_i, i = 1, \cdots, t\}$. Then R_π denotes a localization of R at L_π. Thus R_π is a principal ideal having quotient field of characteristic zero and containing a unique prime ideal I_i such that $p_i \in I_i, i = 1, \dots, n$. We denote the Jacobson radical of R_π be $J(R_\pi)$. Therefore the residue ring $K_m = R_\pi / J(R_\pi)$ is a semi-simple ring of characteristic m. We may write

$$K_m = \bigoplus_{i=1}^{t} k_i \tag{1.1}$$

where the k_i are fields of characteristic $p_i (i = 1, \dots, t)$.

If G is a finite group then we obtain

$$K_m G = \bigoplus_{i=1}^{t} k_i G. \tag{1.2}$$

by (1.1).

From (1.2) it follows that

$$1 = f_1 + \cdots + f_t \tag{1.3}$$

where the f_i are orthogonal central idempotents in $K_m G$.

Therefore

K$_m G = \bigoplus_{i=1}^{t} k_i G$

$= \bigoplus_{i=1}^{t} K_m G f_i$ with $k_i G = K_m G f_i$.

Now, R_π is a Hausdorff space in its $J(R_\pi)$-topology, i. e., that

$$\bigcap_{j=1}^{\infty} J(R_\pi)^j = (0).$$

Therefore the $J(R_\pi)$-adic completion $\hat{R}_\pi$ of R_π is a complete semi-local ring such that

$$K_m = R_\pi / J(R_\pi) = \hat{R}_\pi / J(\hat{R}_\pi)$$

and

$$\hat{R}_\pi = R_1 \oplus \cdots \oplus R_t \tag{1.4}$$

where the R_i are complete local rings such that $R_i/J(R_i) \cong k_i$.

Observe that R_i is isomorphic to $\hat{R}_{v_{p_i}}$, where $\hat{R}_{v_{p_i}}$ is a complete valuation ring corresponding to the discrete valuation v_{p_i} associated to the maximal ideal I_i of R. Thus, we may write

$$\hat{R}_\pi \cong \hat{R}_{v_{p_1}} \oplus \cdots \oplus \hat{R}_{v_{p_t}} \tag{1.5}$$

by (1.4).

Therefore

$$\hat{R}_\pi G \cong \hat{R}_{v_{p_1}} G \oplus \cdots \oplus \hat{R}_{v_{p_t}} G. \tag{1.6}$$

From (1.6) we obtain

$$1 = \hat{f}_1 + \cdots + \hat{f}_n \tag{1.7}$$

where the $\hat{f}_i$ are orthogonal central idempotents in $\hat{R}_\pi G$.

Let π_l be any set of positive prime numbers. Assume that R_{π_l} is the localization of R at L_{π_l}. Then we have

$$\bigcap_{l=1}^{\infty} R_{\pi_l} = R \tag{1.8}$$

In the study of the Schur ring $M_n(R)$ of a commutative ring R the main problem is to find a finite group $G \le GL_n(R)$ such that the R-span of G coincides with the matrix ring $M_n(R)$ of the $n \times n$-matrices on K. The more precise question, in general sense, which Azumaya algebras over R are obtainable as an epimorphic image of the group-ring RG for some finite group G.

1.1. Notations and Definitions.

Throughout the paper F denote an algebraic number field and R denote the ring of algebraic integers of F. Moreover, K_m is semi-simple ring of characteristic m with maximal ideals K_i and residue fields $k_i = K_m/K_i$ of characteristic p_i. For an maximal ideal I of R we denote by ϕ the I-adic valuation on F. Here R_{v_p} denote the valuation ring of v_p and $\hat{R}_{v_p}$ denote the complete valuation ring corresponding to the discrete valuation v_p. Let $M_n(R)$ stand for the ring of $(n \times n)$-matrices over R. We write $GL_n(R)$ for the multiplicative group of the invertible elements of $M_n(R)$. For a finite subgroup G of $GL_n(R)$ we let $\langle G \rangle_R$ be the R-span of G in $M_n(R)$. Let π be a set of natural primes. We denote the fields of rational and complex numbers by $\mathbf{Q}$ and $\mathbf{C}$, respectively.

2. Preliminary Results.

Let R be a commutative ring and let $G \leq GL_n(R)$ be a finite group. Then the matrix ring $M_n(R)$ is called Schur ring if $\langle G\rangle_R = M_n(R)$.

Lemma 2.0.1. *Let k be a field of characteristic p and let $G \leq GL_n(k)$ be a finite group. Assume that V is the kG-module corresponding to G. Then G is absolutely simple if and only if $\langle G\rangle_k = M_n(k)$.*

Proof. If G is absolutely simple then by Burnside's theorem the assertion follows. Conversely, let $\dot{G}$ be a finite group with a representation $\varphi : \dot{G} \longrightarrow GL_n(k)$ such that $\varphi(\dot{G}) = G$. Consider the subjection of k-algebras $\psi : k\dot{G} \longrightarrow \langle G\rangle_k$, where $\psi(\dot{G}) = \varphi(\dot{G})$. Therefore $k\dot{G}/\ker\psi \cong M_n(k)$. Since $J(k\dot{G}) \subseteq \ker\psi$ if follows that the matrix algebra summand of $k\dot{G}/J(k\dot{G})$ corresponding to V is $M_n(k)$, so the result follows. □

Let K_m be a semi-simple ring of characteristic m and let $G \leq GL_n(K_m)$ be a finite group. Assume that V is the K_mG-module corresponding to G. Then G is called π-quasi-simple if each direct summand Vf_i is an absolutely simple k_iG-module.

Lemma 2.0.2. *Let K_m be a semi-simple ring of characteristic m and let $G \leq GL_n(K_m)$ be a finite group. Assume that V is the K_mG-module corresponding to G. Then $\langle G\rangle_{K_m} = M_n(K_m)$ if and only if G is π-quasi-simple.*

Proof. We have
$\langle G\rangle_{K_m} = \langle G\rangle_{k_1} \oplus \cdots \oplus \langle G\rangle_{k_t}$
$= M_n(K_m)$
$= M_n(k_1) \oplus \cdots \oplus M_n(k_t)$

Therefore $\langle G\rangle_{k_i} = M_n(k_i)$. Thus applying the last lemma the result follows. Conversely, applying again the lemma (2.0.1) we deduce that

$$\langle G\rangle_{k_i} = M_n(k_i)$$

for all i. Thus we may write

$$\begin{aligned}\langle G\rangle_{K_m} &= \langle G\rangle_{k_1} \oplus \cdots \oplus \langle G\rangle_{k_t}\\ &= M_n(k_1) \oplus \cdots \oplus M_n(k_t)\\ &= M_n(K_m)\end{aligned}$$

So we are done.
□

Proposition 2.0.3. *Let $G \leq GL_n(\mathbf{C})$ be a finite group. Assume that F is a finite extension of $\mathbf{Q}$ such that $G \leq GL_n(F)$. If R is the ring of integers of F then $G \leq GL_n(R_\pi)$, being R_π a localization of R at L_π.*

Proof. Let $\hat{R}_\pi$ be the completion of R_π. It is well known that $G \leq GL_n(\hat{R}_{v_{p_l}})(l = 1, \dots, t)$, where $\hat{R}_{v_{p_l}}$ is a complete valuation ring. Let $\hat{U}_i$ be the $\hat{R}_{v_{p_l}}G$-module corresponding to G. From (1.5) we deduce that there is an $\hat{R}_\pi G$-module minimal $\hat{R}_\pi$-free $\hat{U} = \hat{U}_1 \oplus \cdots \oplus \hat{U}_t$, so $G \leq GL_n(\hat{R}_\pi)$. Since $\hat{U}$ is a complete minimal $\hat{R}_\pi$- free module it follows that $\hat{U} = \hat{R}_\pi U$, being U an $R_\pi G$-module, which is a Hausdorff space for its $J(R_\pi)$-topology. So we are done. □

From (1.8) and the last proposition one can deduce an "Hasse Principle " for $\langle G \rangle_R$ to coincide with $M_n(R)$: this is the case if and only if $\langle G \rangle_{R_\pi} = M_n(R_\pi)$ for every set of positive prime numbers π.

Let $G \leq GL_n(\mathbf{C})$ be a finite group and let $U_{\mathcal{C}}$ be the $\mathbf{C}G$-module corresponding to G. Then there exists a finite extension F of $\mathbf{Q}$ with FG-module U_F conjugate to $U_{\mathcal{C}}$. According to the proposition (2.0.3) there is an $R_\pi G$-module U which is also conjugate to $U_{\mathcal{C}}$. Let $\hat{R}_\pi$ be the $J(R_\pi)$-adic completion of R_π. We know that there exists an $\hat{R}_\pi G$-module $\hat{U}$ which is the completion of U. From (1.7) it follows that

$$\hat{U} = \hat{U}\hat{f}_1 \oplus \cdots \oplus \hat{U}\hat{f}_t$$

where the $\hat{U}\hat{f}_i$ are $R_{v_{p_i}}G$-modules. The K_mG-module $\overline{U}_{K_m} = \hat{U}/J(R_\pi)\hat{U}$ $= \hat{U}\hat{f}_1/J(R_{v_{p_1}})\hat{U}\hat{f}_1 \oplus \cdots \oplus \hat{U}\hat{f}_t/J(R_{v_{p_t}})\hat{U}\hat{f}_t$

is called reduction of U modulo π.

The natural projection $GL_n(\hat{R}_\pi) \to GL_n(K_m)$ induces the homomorphism $\tau : G \to GL_n(K_m)$, where $\tau(G) = \overline{G}$. Furthermore the homomorphism $GL_n(\hat{R}_{v_{p_i}}) \to GL_n(k_i)$ induces the homomorphism $\tau_i : G \to GL_n(k_i)$ with $\tau_i(G) = \overline{G}_i$

Observe that if $\langle G \rangle_{R_{\pi_l}} = M_n(R_{\pi_l})$, where π_l is the set of the positive prime divisors of $|G|$, then $\langle G \rangle_{R_\pi} = M_n(R_\pi)$ for every set π of positive prime numbers. We use Nakayama's lemma to show that that $\langle G \rangle_{R_{\pi_l}} = M_n(R_{\pi_l})$ is equivalent to $\langle \overline{G} \rangle_{K_m} = M_n(K_m)$.

3. Main Results.

Let R be a ring of algebraic integers and let $G \leq GL_n(R)$ be a finite group. Assume that π is the set of the positive prime divisors of $|G|$ and that U is the $R_\pi G$-module corresponding to G. If the reduction of U module π is a π-quasi-simple $K_m G$-module, then we say that G is a π-globally simple.

Lemma 3.0.4. *Let R be a ring of algebraic integers with localization R_π and let $G \leq GL_n(R_\pi)$ be a finite group. Assume that π is a set of the positive prime divisors of $|G|$. Then $\langle G\rangle_{R_\pi} = M_n(R_\pi)$ if and only if G is π-globally simple.*

Proof. From $\langle G\rangle_{R_\pi} = M_n(R_\pi)$ we deduce that $\langle\overline{G}\rangle_{K_m} = M_n(K_m)$, being $K_m = R_\pi/J(R_\pi)$ a semi-simple ring. The result follows by lemma (2.0.2). Conversely, applying again the lemma (2.0.2) we obtain $\langle\overline{G}\rangle_{K_m} = M_n(K_m)$, so the assertion follows. □

Theorem 3.0.5. *Let R be a ring of algebraic integers and let $G \leq GL_n(R)$ be a finite group. Then $\langle G\rangle_R = M_n(R)$ if and only if G is π-globally simple.*

Proof. From $\langle G\rangle_R = M_n(R)$ we deduce that $\langle G\rangle_{R_\pi} = M_n(R_\pi)$. Hence we obtain $\langle\overline{G}\rangle_{K_m} = M_n(K_m)$. The result follows by lemma (2.0.2). Conversely, since

$$\langle G\rangle_{R_\pi} = M_n(R_\pi)$$

by lemma (3.0.4) we deduce that for any set of prime numbers π we obtain

$$\langle G\rangle_{R_\pi} = M_n(R_\pi).$$

So we are done. □

Theorem 3.0.6. *Let $G \leq GL_n(\mathbf{C})$ be a finite group. Then G is π-globally simple if and only if $G/O_{p_i}(G)$ is absolutely simple.*

Proof. Let F be a finite extension of $\mathbf{Q}$ such that $G \leq GL_n(F)$, and let R be the ring of integers of F. Consider the homomorphism $h_i : G \to \overline{G}_i$. Since $\overline{G}_i$ is absolutely simple it follows that $\ker h_i = O_{p_i}(G)$. Conversely, since $G/O_{p_i}(G)$ is absolutely simple for all i, we deduce that G is π-quasi-simple, being π the set of positive prime divisors of the order of G. So we are done. □

References

[1] P. Huu Tiep. Globally irreducible representations of the finite symplectic group Sp4(q), Comm. in Algebra 22, pp. 6439 - 6457, (1994).

[2] P. Huu Tiep. Basic spin representations of 2Sn and 2An as globally irreducible representations, Archiv Math. 64, pp. 103 - 112, (1995).

[3] P. Huu Tiep. Weil representations as globally irreducible representations, Math. Nachr. 184, pp. 313 - 327, (1997).

[4] P. Huu Tiep. Globally irreducible representations of finite groups and integral lattices, Geomet. Dedicata 64, pp. 85 - 123, (1997).

[5] P. Spiga and Q. Wang An answer to Hirasaka and Muzychuk : Every p-Schurring over C_p^3 p is Schurian, Discrete Mathematics 308, pp. 1760-1763, (2008).

[6] A. E. Zalesskii and F.Van Oystaeyen Finite Groups over Arithmetical Rings and Globally Irreducible Representations, J. Algebra 215, pp. 418-436, (1999).

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