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θ -GENERALIZED SEMI-OPEN AND θ -GENERALIZED SEMI-CLOSED FUNCTIONS

GOVINDAPPA NAVALAGI
G. H COLLEGE, INDIA

and

Md. HANIF PAGE
B. V. B. COLLEGE OF ENG. AND TECH., INDIA
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Abstract

In this paper, we introduce and study the notions of θ -generalized-semi-open function, θ -generalized- semi-closed function, pre- θ -generalized-semi-open function, pre- θ -generalized-semi-closed function, contra pre- θ -generalized-semi-open, contra pre- θ -generalized-semi-closed function and θ -generalized-sem-homeomorphism in topological spaces and study their properties.

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1. INTRODUCTION

In 1970, Levine [10] first considered the concept of generalized closed (briefly, g-closed) sets were defined and investigated Arya and Nour [1] defined generalized semi-open (briefly, gs-open) sets using semi openness and obtained some characterization of s-normal space. The generalizations of generalized closed and generalized continuity were intensively studied in recent years by Balachandran, Devi ,Maki and Sundaram[2].Recently in [12] the notion of θ -generalized semi closed (briefly, θ gs-closed) set was introduced. The aim of this paper is to introduce the notions of θ -generalized semi open (briefly, θ gs-open) function, θ -generalized semi closed (briefly, θ gs-closed) function, θ -generalized-semi-homeomorphisms and study their simple properties.

2. PRELIMINARIES

Through out this paper (X, τ) and (Y, σ) (or simply X and Y)denote the spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of X , then $Cl(A)$ and $Int(A)$ denote the closure of A and the interior of A in X respectively.

Definition 2.1 : A subset A of a topological space X is called

- (i) a semi-open set[9] if $A \subset Cl(Int(A))$,
- (ii) a semi-closed set[5] if $A \subset Int(Cl(A))$.

Definition 2.2 : The semi-closure [5] of a subset of X is the intersection of all semiclosed sets that contain A and is denoted by $sCl(A)$.

Definition 2.3 : The θ -closure of a set A is denoted by $Cl_{\theta}(A)$ [16] and is defined by $Cl_{\theta}(A) = \{x \in X : Cl(U) \cap A \neq \phi, U \in \tau, x \in U\}$ and a set A is θ -closed if and only if $A = Cl_{\theta}(A)$.

Definition 2.4 : A point $x \in X$ is called a semi- θ -cluster point of A if $sCl(U) \cap A \neq \phi$, for each semi-open set U containing x .

Definition 2.5 : The set of all semi- θ -cluster point of A is called semi- θ -closure of A and is denoted by $sCl_{\theta}(A)$. A subset A is called semi- θ -closed set if $sCl_{\theta}(A) = A$. The complement of semi- θ -closed set is semi- θ -open set.

Definition 2.6 : A subset A of a space (X, τ) is called a generalized closed set (briefly g -closed) [10] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Definition 2.7 : A map $f : X \rightarrow Y$ is called

(i) a generalized continuous (briefly g -continuous)[2] if $f^{-1}(F)$ is g -closed in X for every closed set F of Y .

(ii) a gc -irresolute [2] if $f^{-1}(V)$ is g -closed set of X for every g - closed set V of Y .

Definition 2.8 : A function $f : X \rightarrow Y$ is called

(i) g -homeomorphism [11] if f is g -continuous and g -open.

(ii) gc -homeomorphism [11] if f and f^{-1} are gc -irresolute.

Definition 2.9 : A subset A of a topological space X is called θ -generalized-semi closed (briefly, θ gs-closed) [12] if $sCl_{\theta}(A) \subset U$ whenever $A \subset U$ and U is open. The complement of θ gs-closed set is θ generalized-semi open (briefly, θ gs-open).

We denote the family of θ gs-closed sets of X by $\theta GSC(X, \tau)$ and θ gs-open sets by $\theta GSO(X, \tau)$.

Definition 2.10 : For every set $A \subseteq X$, we define θ gs-closure of A [12]to be the intersection of all θ gs-closed sets containing A and is denoted by $\theta gsCl(A)$.

In symbol, $\theta gsCl(A) = \{\cap F : A \subseteq F \text{ where } F \text{ is } \theta \text{gs-closed in } X\}$.

Since every θ - g -closed set as well as semi- θ -closed set is θ gs-closed set and hence we have, $A \subset \theta gsCl(A) \subset Cl_{\theta}(A)$ and $A \subset \theta gsCl(A) \subset sCl_{\theta}(A) \subset Cl_{\theta}(A)$. A is θ gs-closed set, if $A = \theta gsCl(A)$.

Definition 2.11 : A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called:

(i) θ -semi-generalized-irresolute (briefly, θ -sg-irresolute)[14] if $f^{-1}(F)$ is θ -sg-closed in (X, τ) for every θ -sg-closed set F of (Y, σ) ,

(ii) θ -semi-generalized-continuous (briefly, θ -sg-continuous) if $f^{-1}(F)$ is θ -sg-closed in (X, τ) for every closed set F of (Y, σ) ,

Definition 2.12 : A space X is called $T_{\theta gs}$ -space [13] if every θ gs-closed set in it is closed set.

Theorem 2.13[13]: Intersection of arbitrary collection of θ gs-closed sets is θ gs-closed set.

Remark 2.14 :

- (i) Any intersection of θ gs-closed sets is θ gs-closed set. Hence, by complement, any union of θ gs-open sets is θ gs-open.
(ii) Union of θ gs-closed sets may fail to be θ gs-closed set.

Definition 2.15[15]: A topological space (X, τ) is called

- (i) θ gs- T_0 if for any pair of distinct points x and y of X , there exists a θ gs-open set containing x but not y or a θ gs-open set containing y but not x .
(ii) θ gs- T_1 if for every pair of distinct points x and y of X , there exists a θ gs-open set containing x but not y and a θ gs-open set containing y but not x .
(iii) θ gs- T_2 if for each pair of distinct points x and y of X , there exist disjoint θ gs-open sets, one containing x and the other containing y .

3. θ gs-OPEN and θ gs-CLOSED FUNCTIONS

Definition 3.1 : A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be θ gs-open (resp., θ gs-closed) if $f(V)$ is θ gs-open (resp., θ gs-closed) in Y for every open set (resp., closed) V in X .

Theorem 3.2 : A function $f: X \rightarrow Y$ is θ gs-closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$ there is a θ gs-open set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof : Assume that f is θ gs-closed. Let S be a subset of Y and U be an open set of X such that $S \subseteq f(U)$, that is, $f^{-1}(S) \subseteq U$. Now, U^C is closed set in X . Then $f(U^C)$ is θ gs-closed in Y , since f is θ gs-closed. So, $Y \setminus f(U^C)$ is θ gs-open in Y . Thus $V = Y \setminus f(U^C)$ is a θ gs-open set containing S such that $f^{-1}(V) \subseteq U$.

Conversely, suppose that F is a closed set in Y . Then $f^{-1}(Y \setminus f(F)) \subseteq X \setminus F$ and $X \setminus F$ is open. By hypothesis, there is a θ gs-open set V of Y such that $Y \setminus f(F) \subseteq V$ and $f^{-1}(V) \subseteq X \setminus F$ and so $F \subseteq X \setminus f^{-1}(V)$. Hence $Y \setminus V \subseteq f(F) \subseteq f(X \setminus f^{-1}(V)) \subseteq Y \setminus V$ which implies $f(F) = Y \setminus V$. Since $Y \setminus V$ is θ gs-closed, $f(F)$ is θ gs-closed and thus f is θ gs-closed.

Theorem 3.3 : If $f: X \rightarrow Y$ is bijective, θ gs-open function. If X is θ gs- T_1 space and $T_{\theta gs}$ - space, then Y is θ gs- T_1 space.

Proof : Let y_1 and y_2 be two distinct points of Y . Since f is bijective, there exists distinct points x_1 and x_2 of X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is a $\theta_{gs} - T_1$ space, there exists θ_{gs} -open sets G and H such that $x_1 \in G$ and $x_2 \notin G$ and $x_2 \in H$ and $x_1 \notin H$. Again since X is $T_{\theta_{gs}}$ -space, G and H are open sets in X . As f is θ_{gs} -open function, $f(G)$ and $f(H)$ are θ_{gs} -open sets such that $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \notin f(G)$ and $y_1 = f(x_1) \notin f(H)$, $y_2 = f(x_2) \in f(H)$. Hence Y is $\theta_{gs} - T_1$ space.

Definition 3.4 : A function $f: X \rightarrow Y$ is called pre- θ_{gs} -closed (resp., pre- θ_{gs} -open) if $f(V)$ is θ_{gs} -closed (resp., θ_{gs} -open) in Y for every θ_{gs} -closed (resp., θ_{gs} -open) set V of X .

Theorem 3.5 : If $f: X \rightarrow Y$ is continuous, pre- θ_{gs} -closed and A is θ_{gs} -closed subset of X , then $f(A)$ is θ_{gs} -closed

Proof : Let U be an open set such that $f(A) \subseteq U$. Since f is continuous $f^{-1}(U)$ is an open set containing A , that is $A \subseteq f^{-1}(U)$. Hence $sCl_{\theta}(A) \subseteq f^{-1}(U)$ as A is θ_{gs} -closed set in X . Since f is pre- θ_{gs} -closed, $f(sCl_{\theta}(A))$ is θ_{gs} -closed set. So, $f(sCl_{\theta}(A))$ is θ_{gs} -closed set contained in an open set U , that is, $f(sCl_{\theta}(A)) \subseteq U$. Now $sCl_{\theta}(f(A)) \subseteq sCl_{\theta}(f(\theta_{gs}Cl(A))) \subseteq sCl_{\theta}(f(sCl_{\theta}(A))) = f(sCl_{\theta}(A)) \subseteq U$. Hence $sCl_{\theta}(f(A)) \subseteq U$. Therefore $f(A)$ is a θ_{gs} -closed set in Y .

Theorem 3.6 : The property of a space being $\theta_{gs} - T_0$ space is preserved under one-one, onto, pre- θ_{gs} -open function and hence is a topological property.

Proof : Let X be a $\theta_{gs} - T_0$ space and Y be any other topological space. Let $f: X \rightarrow Y$ be one-one, onto, pre- θ_{gs} -open function from X to Y . Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$ and since f is one-one, onto, there exists distinct points $x_1, x_2 \in X$ such that $f(x_1) = y_1$, $f(x_2) = y_2$. Since X is $\theta_{gs} - T_0$ space, there exists a θ_{gs} -open set G in X such that $x_1 \in G$ but $x_2 \notin G$. Since f is pre- θ_{gs} -open, $f(G)$ is θ_{gs} -open set containing $f(x_1)$ but not containing $f(x_2)$. Thus there exists a θ_{gs} -open set $f(G)$ in Y such that $(y_1) \in f(G)$ but $(y_2) \notin f(G)$ and hence Y is $\theta_{gs} - T_0$ space. Again as the property of being $\theta_{gs} - T_0$ is preserved under one-one, onto mapping, it is also preserved under homeomorphism and hence is a topological property.

Theorem 3.7 : Let $f: X \rightarrow Y$ be a θ_{gs} -closed and $g: Y \rightarrow Z$ pre- θ_{gs} -closed and continuous, then their composition $g \circ f: X \rightarrow Z$ is θ_{gs} -closed.

Proof : Let A be a closed set of X . Then by hypothesis $f(A)$ is θ gs-closed set in Y . Since g is pre- θ gs-closed and continuous, by Theorem 3.5 $g(f(A) = (gof)(A)$ is θ gs-closed in Z . Hence (gof) is θ gs-closed.

4. CONTRA PRE- θ GS-CLOSED and CONTRA PRE- θ GS-OPEN FUNCTIONS

Definition 4.1 : A function $f: X \rightarrow Y$ is called contra pre- θ gs-open if for every θ gs-open set F of X , $f(F)$ is θ gs-closed in Y .

Definition 4.2 : A function $f: X \rightarrow Y$ is called contra pre- θ gs-closed if for every θ gs-closed set F of X , $f(F)$ is θ gs-open in Y .

The following example shows that contra pre- θ gs-closedness and contra pre- θ gs-openness are independent.

Example 4.3 : Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $\sigma = \{Y, \{a\}, \{b\}, \{a, b\}\}$. We have θ gs-open sets in X are $\{X, \{a\}, \{b, c\}\}$ and θ gs-open sets in Y are $\{Y, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Let $f: X \rightarrow Y$ is defined by $f(a) = f(c) = c$, $f(b) = b$ and $g: X \rightarrow Y$ is by $g(a) = g(b) = a$, $g(c) = b$. Then f is contra pre- θ gs-open but not contra pre- θ gs-closed and g is contra pre- θ gs-closed but not contra pre- θ gs-open.

Remark 4.4 : Contra pre- θ gs-closedness and contra pre- θ gs-openness are equivalent if the function is bijective.

Theorem 4.5 : For a function $f: X \rightarrow Y$ the following properties are equivalent.

- (i) f is contra pre- θ gs-open.
- (ii) For every subset B of Y and every θ gs-closed subset F of X with $f^{-1}(B) \subseteq F$, there exists a θ gs-open subset O of Y with $B \subseteq O$ and $f^{-1}(O) \subseteq F$.
- (iii) For every point $y \in Y$ and every θ gs-closed subset F of X with $f^{-1}(y) \subseteq F$, there exists a θ gs-open subset O of Y with $y \in O$ and $f^{-1}(O) \subseteq F$.

Proof : (i) \rightarrow (ii). Let B be a subset of Y and F be a θ gs-closed subset of X with $f^{-1}(B) \subseteq F$. For the case $B \neq \phi$, put $O = [f(F^C)]^C$. Then

$f^{-1}(O) = [f^{-1}(F^C)]^C \subseteq F$ and O is a θ gs-open subset of Y . We claim that $B \subseteq O$. There are two cases to be considered:

Case 1: $f^{-1}(B) \neq \phi$. Since $f^{-1}(B) \subseteq F$, we have $f(F^C) \subseteq B^C$ and $B \subseteq O$.

Case 2: $f^{-1}(B) = \phi$. Since $f^{-1}(B) = \phi \subseteq F$, we have $f(F^C) \subseteq f(X)$. We have $B \cap f(X) = \phi$, because $f^{-1}(B) = \phi$ and $B \neq \phi$. Thus $B \subseteq [f(X)]^C \subseteq [f(F^C)]^C = O$. For the case $B = \phi$, put $O = \phi$. Then the set O is required θ gs-open set of Y .

(ii) \rightarrow (iii). It suffices to put $B = y$ in (ii)

(iii) \rightarrow (i). Let A be a θ gs-open subset of X . Then $y \in [f(A)]^C$ and $F = A^C$. First we claim that $f^{-1}(y) \subset F$. For non empty set $f^{-1}(y)$, let $z \in f^{-1}(y)$; $f(z) = y \notin f(A)$. Suppose that $z \notin A$ and so $y = f(z) \in f(A)$. By contradiction, $z \in F$ for any $z \in f^{-1}(y)$, that is $f^{-1}(y) = \phi \subseteq F$. For the case where $f^{-1}(y) = \phi$, we have $f^{-1}(y) = \phi \subseteq F$. For both cases, we can use (iii) and get the following: there exists a θ gs-open set $O_y \subseteq Y$ such that $y \in O_y$ and $f^{-1}(O_y) \subseteq F = A^C$. Namely, (*) $f^{-1}(O_y) \cap A = \phi$ holds for each $y \in [f(A)]^C$. Finally we claim that $[f(A)]^C = \cup \{O_y : y \in [f(A)]^C\}$. Obviously, we have that $[f(A)]^C \subseteq \cup \{O_y : y \in [f(A)]^C\}$.

Conversely, let $z \in \cup \{O_y : y \in [f(A)]^C\}$. Then there exists a point $w \in [f(A)]^C$ such that $z \in O_w$. Suppose that $z \notin [f(A)]^C$. Then $z \in f(A)$ and there exists a point $b \in A$ such that $f(b) = z$. Thus we have that $f(b) \in O_w$ and so $b \in f^{-1}(O_w)$. We have a contradiction to (*) above, that is, $b \in f^{-1}(O_w) \cap A$. Hence, we show that $\cup \{O_y : y \in [f(A)]^C\} \subseteq [f(A)]^C$ and so $[f(A)]^C = \cup \{O_y : y \in [f(A)]^C\}$. Consequently, by Remark 2.12 (i), $f(A)$ is a θ gs-closed subset of Y .

Theorem 4.6 :

Let (1), (2) and (3) be properties of a function $f: X \rightarrow Y$ as follows.

1. f is contra pre- θ gs-closed.
2. For every set B of Y and every θ gs-open subset O of X with $f^{-1}(B) \subseteq O$, there exists a θ gs-closed subset F of Y with $B \subseteq F$ and $f^{-1}(F) \subseteq O$.
3. For every point $y \in Y$ and every θ gs-open subset O of X with $f^{-1}(y) \subseteq O$, there exists a θ gs-closed subset F of Y with $y \in F$ and $f^{-1}(F) \subseteq O$. Then

(i) The implications (1) \rightarrow (2) \rightarrow (3) hold.

(ii) Suppose $\theta GSC(Y, \sigma)$ is closed under arbitrary unions (i.e., union of any collection of θ gs-closed sets is θ gs-closed set). Then the implication (3) \rightarrow (1) holds.

Proof : (i)(1) \rightarrow (2). Let B be the subset of Y and O be a θ gs-open subset of X with $f^{-1}(B) \subseteq O$. Put $F = [f(O^C)]^C$. Since f is contra pre- θ gs-closed, then F is θ gs-closed set of Y . By $f^{-1}(B) \subseteq O$, we have $f(O^C) \subseteq B^C$. Moreover, $f^{-1}(F) \subseteq O$.

(2) \rightarrow (1). Let E be a θ gs-closed subset of X . Put $B = [f(E^C)]^C \subseteq E^C = O$. Hence $f^{-1}(B) = f^{-1}(f(E))^C = [f^{-1}(f(E))]^C \subseteq E^C = O$. By assumption there exists a θ gs-closed set $F \subseteq Y$ for which $B \subseteq F$ and $f^{-1}(F) \subseteq O$. It follows that $B = F$. If $y \in F$ and $y \notin B$, $y \in f(E)$. Therefore $y = f(x)$ for some $x \in E$ and we have that $x \in f^{-1}(F) \subseteq O = E^C$, which is a contradiction. Since $B = F$ (i.e., $[f(E)^C] = F$), $f(E)$ is a θ gs-open and hence f is contra pre- θ gs-closed.

(2) \rightarrow (3). It suffices to put $B = y$ for some $y \in Y$.

(ii)(3) \rightarrow (1). Let A be a θ gs-closed subset of X . Let $y \in [f(A)]^C$. Then we have $f^{-1}(y) \subseteq A^C$. By (3) there exists a θ gs-closed set $F_y \subseteq Y$ such that $y \in F_y$ and $f^{-1}(F_y) \subseteq A^C$. Namely, $f^{-1}(y) \cap A = \phi$ holds for each $y \in [f(A)]^C$. By an arrangement similar to that $[f(A)]^C = \cup \{F_y : y \in [f(A)]^C\}$ holds. Consequently, using assumption in (iii), $f(A)$ is θ gs-open in Y .

Theorem 4.7 : Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are θ gs-closed functions and Y be $T_{\theta gs}$ - space. Then their composition gof is θ gs-closed.

Proof : Let A be a closed set of X . Then by hypothesis $f(A)$ is a θ gs-closed set in Y . Since Y is $T_{\theta gs}$ - space, $f(A)$ is closed in Y . Since g is θ gs-closed, $g(f(A))$ is θ gs-closed in Z . But $g(f(A)) = (gof)(A)$. Hence gof is θ gs-closed.

Theorem 4.8 : The composition of a closed functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is θ gs-closed function from X to Z .

Proof : Trivial.

Theorem 4.9 : Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions such that their composition $gof : X \rightarrow Z$ is θ gs-closed function. Then the following statements holds;

- i) If f is continuous and surjective, then g is θ gs-closed.

ii) If g is θ gs-irresolute and injective, then f is θ gs-closed.

Proof : (i) Let A be a closed set in Y . Then $f^{-1}(A)$ is closed in X as f is continuous. Since gof is θ gs-closed and if f is surjective, $(gof)(f^{-1}(A)) = g(f(f^{-1}(A))) = g(A)$ is θ gs-closed in Z . Therefore g is a θ gs-closed function in X .

(ii) Let H be a closed set of X . Since gof is θ gs-closed, $(gof)(H)$ is θ gs-closed set in Z . Since g is θ gs-irresolute, $g^{-1}(gof)(H) = g^{-1}(g(f(H))) = f(H)$ is θ gs-closed in Y , since g is injective. Thus f is θ gs-closed function in X .

Theorem 4.10: For any bijection $f : X \rightarrow Y$, the following statements are equivalent (i) Inverse of f is θ gs-continuous.

(ii) f is a θ gs-open function.

(iii) f is a θ gs-closed function.

Proof : (i) \rightarrow (ii). Let U be an open set of X . By assumption θ gs-continuous, $(f^{-1})^{-1}(U) = f(U)$ is θ gs-open in Y and so f is θ gs-open function.

(ii) \rightarrow (iii). Let F be a closed set of X . Then F^C is open in X . By assumption $f(F^C)$ is θ gs-open in Y , that is, $f(F^C) = (f(F))^C$ is θ gs-open in Y and therefore $f(F)$ is θ gs-closed in Y . Hence f is θ gs-closed.

(iii) \rightarrow (i). Let F be a closed set in X . By assumption $f(F)$ is θ gs-closed in Y . But $f(F) = (f^{-1})^{-1}(F) = f(F)$ is θ gs-closed and therefore inverse image of f is θ gs-continuous.

5. θ gs-HOMEOMORPHISMS

Definition 5.1 : A bijection $f : X \rightarrow Y$ is called θ -generalized-semi-homeomorphism (briefly, θ gs-homeomorphism) if f is both θ gs-continuous and θ gs-open.

Remark 5.2 : Every g -homeomorphism is θ gs-homeomorphism.

But the converse is not always true by the following example.

Example 5.3 : Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, c\}\}$ and θ gs-open sets in Y are $\{Y, \phi, \{b\}, \{c\}, \{b, c\}\}$, g - open sets are $\{Y, \phi, \{b, c\}, \{a, c\}, \{b\}\}$. Let $f : X \rightarrow Y$ be a function defined by $f(a) = b$, $f(b) = c$, $f(c) = a$. It is easy to verify that f is θ gs-homeomorphism but not a g -homeomorphism since b is open in X , while $f(b) = c$ is not g -open in Y .

Theorem 5.4 : Let $f : X \rightarrow Y$ be a bijective and θ gs-continuous function. Then the following statements are equivalent

- (i) f is θ gs-open
- (ii) f is θ gs-homeomorphism.
- (iii) f is θ gs-closed.

Theorem 5.5 : (i) \rightarrow (ii). By assumption, f is bijective, θ gs-continuous and θ gs-open. Then by definition f is a θ gs-homeomorphism.

(ii) \rightarrow (iii). By assumption f is θ gs-open and bijective. By Theorem 4.10, f is θ gs-closed.

(iii) \rightarrow (iv). By assumption f is bijective and θ gs-closed. By Theorem 4.10, f is θ gs-open.

Now we introduce the class of functions which included in the class of θ gs-homeomorphism.

Definition 5.6 : A bijective $f : X \rightarrow Y$ is called θ gsc-homeomorphism if f and f^{-1} are θ gs-irresolute.

We say that the topological spaces X and Y are θ gsc-homeomorphic, if there exists a θ gsc-homeomorphism from X onto Y .

Remark 5.7 : Every θ gsc-homeomorphism is θ gs-homeomorphism.

However the converse is not true as shown by the following example.

Example 5.8 : Let $X = Y = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, c\}\}$. We have θ gs-closed sets in X are $\{X, \phi, \{a\}, \{b, c\}\}$ and θ gs-closed sets in Y are $\{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Let $f : X \rightarrow Y$ be a function defined by $f(a) = b$, $f(b) = c$, $f(c) = a$. It is easy to verify that f is θ gs-homeomorphism but not a θ gsc-homeomorphism since $\{a, b\}$ is θ gs-closed set in Y , while $f^{-1}(\{a, b\}) = \{a, c\}$ is not θ gs-closed set in X .

Theorem 5.9 : Every homeomorphism is a θ gsc-homeomorphism.

Proof : Let $f : X \rightarrow Y$ be a homeomorphism. To prove that f is θ gs-irresolute, let A be a θ gs-closed set of Y . Let U be an open set of X such that $f^{-1}(A) \subseteq U$. Then, $A = f \circ f^{-1}(A) \subseteq f(U)$ and $f(U) \in \sigma$. Since A is θ gs-closed in Y , we have $sCl_{\theta}(A) \subseteq f(U)$. Then $f^{-1}(sCl_{\theta}(A)) \subseteq f^{-1} \circ f(U) = U$. So, we are done if we show that $sCl_{\theta}(f^{-1}(A)) \subseteq f^{-1}(sCl_{\theta}(A))$. Let $x \notin f^{-1}(sCl_{\theta}(A))$. Since f is bijective, then $f(x) \notin sCl_{\theta}(A)$. Hence there exists an open set containing $f(x)$ such that $sCl(W) \cap A = \phi$. Then $f^{-1}(sCl(W) \cap A) = f^{-1}(sCl(W)) \cap f^{-1}(A) = \phi$, since f is homeomorphism. Since $f^{-1}(W)$ is an open set containing x , we obtain that $x \notin sCl_{\theta}(f^{-1}(A))$. Thus $sCl_{\theta}(f^{-1}(A)) \subseteq f^{-1}(sCl_{\theta}(A)) \subseteq U$. Hence $f^{-1}(A)$ is θ gs-closed in X , that is, f is θ gs-irresolute. Since f^{-1} is a homeomorphism, the θ gs-irresoluteness of f^{-1} is similarly proved as above. Thus f is θ gsc-homeomorphism.

However, the converse of the above Theorem need not be true as seen from the following example.

Example 5.10 : Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. We have θ gs-closed sets in X are $\{X, \phi, \{a\}, \{b, c\}\}$ and θ gs-closed sets in Y are $\{Y, \phi, \{a\}, \{b, c\}\}$. Let $f : X \rightarrow Y$ be an identity function. It is easy to verify that f is θ gsc-homeomorphism but not a homeomorphism.

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Govindappa Navalagi

Department of Mathematics

KLE Society's

G. H. College,

Haveri-581 110

Karnataka,

INDIA

e-mail : gnavalagi@hotmail.com

and

Md. Hanif Page

Department of Mathematics

B. V. B. College of Eng. and Tech.

Hubli-580031, Karnataka,

INDIA

e-mail : hanif01@yahoo.com