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A NEW APPROACH TO ALMOST FUZZY COMPACTNESS *

FU-GUI SHI
BEIJING INSTITUTE OF TECHNOLOGY, CHINA
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Abstract

A new definition of almost fuzzy compactness is introduced in L-topological spaces by means of open L-sets and their inequality when L is a complete DeMorgan algebra. It can also be characterized by closed L-sets, regularly closed L-sets, regularly open L-sets and their inequalities. When L is a completely distributive DeMorgan algebra, its many characterizations are presented.

Keywords : *L-topology, fuzzy compactness, almost fuzzy compactness, almost continuous, weakly continuous*

Subjclass [2000] : *03E72, 54A40, 54D35*

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1. Introduction

Almost compactness has also been generalized to L-topological spaces by many authors (see [2, 3, 4, 6, 9, 10, 14, 15, 16]). These notions of almost fuzzy compactness rely on the structure of the basis lattice L, where $L = [0, 1]$ or L is a completely distributive DeMorgan algebra. In [21], a new definition of fuzzy compactness was presented in L-fuzzy topological spaces by means of open L-sets and their inequality.

In this paper, based on [19, 21], we shall introduce a new definition of almost fuzzy compactness in L-topological spaces. When L is a completely distributive DeMorgan algebra, its many characterizations are presented. From these characterizations we know that it is a generalization of the notion of almost fuzzy compactness in [3, 9].

2. Preliminaries

Throughout this paper, $(L, \bigvee, \bigwedge, ')$ is a complete DeMorgan algebra and X is a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets for short) on X. The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$.

An element a in L is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in L is called a co-prime element if a' is a prime element [7]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $M(L)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [5]. In a completely distributive DeMorgan algebra L, each element b is a supremum of $\{a \in L \mid a \prec b\}$. In the sense of [11, 23], $\{a \in L \mid a \prec b\}$ is the greatest minimal family of b, denoted by $\beta(b)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations in [20].

$$A^{(a)} = \{x \in X \mid A(x) \not\leq a\}, \quad A_{(a)} = \{x \in X \mid a \in \beta(A(x))\},$$
$$A_{[a]} = \{x \in X \mid A(x) \geq a\}.$$

An L-topological space (or L-space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\underline{0}$, $\underline{1}$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Each member of $\mathcal{T}$ is called an open L-set and its quasi-complement is called a closed L-set.

Definition 2.1 ([11, 23]). *For a topological space (X, τ), let $\omega_L(\tau)$ denote the family of all the lower semi-continuous maps from (X, τ) to L, i.e.,*

$\omega_L(\tau) = \{A \in L^X \mid A^{(a)} \in \tau, a \in L\}$. *Then* $\omega_L(\tau)$ *is an* L*-topology on* X*, in this case,* $(X, \omega_L(\tau))$ *is called topologically generated by* (X, τ).

Definition 2.2 ([11, 23]). *An* L*-space* $(X, \mathcal{T})$ *is called weakly induced if* $\forall a \in L$, $\forall A \in \mathcal{T}$*, it follows that* $A^{(a)} \in [\mathcal{T}]$*, where* $[\mathcal{T}]$ *denotes the topology formed by all crisp sets in* $\mathcal{T}$.

It is obvious that $(X, \omega_L(\tau))$ is weakly induced.

Lemma 2.3 ([20]). *Let* $(X, \mathcal{T})$ *be a weakly induced* L*-space,* $a \in L, A \in \mathcal{T}$*. Then* $A_{(a)}$ *is an open* L*-set in* $[\mathcal{T}]$.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ.

Definition 2.4 ([19, 21]). *Let* $(X, \mathcal{T})$ *be an* L*-space.* $G \in L^X$ *is called fuzzy compact if for every family* $\mathcal{U} \subseteq \mathcal{T}$*, it follows that*

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Lemma 2.5 ([19, 21]). *Let* L *be a complete Heyting algebra,* $f : X \to Y$ *be a map,* $f_L^{\rightarrow} : L^X \to L^Y$ *is the extension of* f*, then for any family* $\mathcal{P} \subseteq L^Y$*, we have:*

$$\bigvee_{y \in Y} \left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^{\leftarrow}(B)(x) \right).$$

Definition 2.6 ([1]). *Let* $(X, \mathcal{T}_1)$ *and* $(Y, \mathcal{T}_2)$ *be two* L*-spaces. A map* $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ *is called*

(1) almost continuous if $f_L^{\leftarrow}(G) \in \mathcal{T}_1$ *for all regularly open* L*-set* G *in* $(Y, \mathcal{T}_2)$*;*

(2) weakly continuous if $f_L^{\leftarrow}(G) \leq \mathrm{int}(f_L^{\leftarrow}(\mathrm{cl}(G)))$ *for every open* L*-set* G *in* $(Y, \mathcal{T}_2)$.

Lemma 2.7 ([1]). *Let* $(X, \mathcal{T}_1)$ *and* $(Y, \mathcal{T}_2)$ *be two* L*-spaces. A map* $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ *is:*

(1) *almost continuous if and only if* $f_L^{\leftarrow}(G)$ *is closed in* $(X, \mathcal{T}_1)$ *for all regularly closed* L*-set* G *in* $(Y, \mathcal{T}_2)$.

(2) *weakly continuous if and only if* $f_L^{\leftarrow}(G) \geq \mathrm{cl}(f_L^{\leftarrow}(\mathrm{int}(G)))$ *for every closed* L*-set* G *in* $(Y, \mathcal{T}_2)$.

Lemma 2.8 ([1]). *The closure of an open L-set is regularly closed and the interior of a closed L-set is regularly open.*

Definition 2.9 ([8]). *An L-space $(X, \mathcal{T})$ is said to be regular if every open L-set G is a supremum of open L-sets whose closure is less that G.*

3. Definition and characterizations of almost fuzzy compactness

Definition 3.1. *Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is called almost fuzzy compact if for every family $\mathcal{U} \subseteq \mathcal{T}$, it follows that*

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} \mathrm{cl}(A)(x) \right).$$

Definition 3.2. *Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is called almost countably fuzzy compact if for every countable family $\mathcal{U} \subseteq \mathcal{T}$, it follows that*

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} \mathrm{cl}(A)(x) \right).$$

For an open L-set A, by $A \leq \mathrm{int}(\mathrm{cl}(A))$ we can obtain the following theorem.

Theorem 3.3. *Fuzzy compactness $\Rightarrow$ almost fuzzy compactness $\Rightarrow$ almost countable fuzzy compactness.*

From Definition 3.1 and Definition 3.2 we can obtain the following theorem by using quasi-complement.

Theorem 3.4. *Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is almost (countably) fuzzy compact if and only if for every (countable) family $\mathcal{P} \subseteq \mathcal{T}'$, it follows that*

$$\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{A \in \mathcal{P}} A(x) \right) \geq \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{A \in \mathcal{F}} \mathrm{int}(A)(x) \right).$$

Definition 3.5 ([21]). *Let $(X, \mathcal{T})$ be an L-space, $a \in L \backslash \{1\}$ and $G \in L^X$. A family $\mathcal{A} \subseteq L^X$ is said to be*

(1) an a-shading of G if for any $x \in X$*, it follows that* $\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \not\leq a$.

(2) a strong a-shading of G if $\bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \not\leq a$.

(3) an a-remote family of G if for any $x \in X$*, it follows that* $\left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} B(x)\right) \not\geq a$.

(4) a strong a-remote family of G if $\bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} B(x)\right) \not\geq a$.

From Definition 3.1, Definition 3.2, Theorem 3.4 and Theorem 3.5 we immediately obtain the following result.

Theorem 3.6. *Let* $(X, \mathcal{T})$ *be an L-space and* $G \in L^X$*. Then the following conditions are equivalent:*

(1) *G is almost (countably) fuzzy compact.*

(2) *For any* $a \in L\backslash\{1\}$*, each (countable) open strong a-shading* $\mathcal{U}$ *of G has a finite subfamily* $\mathcal{V}$ *such that* $\mathcal{V}^-$ *is a strong a-shading of G, where* $\mathcal{V}^- = \{\mathrm{cl}(A) \mid A \in \mathcal{V}\}$.

(3) *For any* $a \in L\backslash\{0\}$*, each (countable) closed strong a-remote family* $\mathcal{P}$ *of G has a finite subfamily* $\mathcal{F}$ *such that* $\mathcal{F}^\circ$ *is a strong a-remote family of G, where* $\mathcal{F}^\circ = \{\mathrm{int}(A) \mid A \in \mathcal{F}\}$.

Moreover by means of regularly open L-sets and regularly closed L-sets, we can give the following characterizations of almost (countable) fuzzy compactness.

Theorem 3.7. *Let* $(X, \mathcal{T})$ *be an L-space and* $G \in L^X$*. Then the following conditions are equivalent:*

(1) *G is almost (countably) fuzzy compact.*

(2) *For each (countable) family* $\mathcal{U}$ *of regularly open L-sets, it follows that*

$$\bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{A\in\mathcal{V}} \mathrm{cl}(A)(x)\right).$$

(3) *For each (countable) family* $\mathcal{U}$ *of regularly closed L-sets, it follows that*

$$\bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{A\in\mathcal{U}} A(x)\right) \geq \bigwedge_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{A\in\mathcal{V}} \mathrm{int}(A)(x)\right).$$

Proof. (2) ⇔ (3) is obvious. Because a regularly open L-set is open, we easily obtain (1) ⇒ (2). Now we prove (2) ⇒ (1). Suppose that $\mathcal{U}$ is a family of open L-sets. From Lemma 2.8 we know that $\mathrm{int}(\mathrm{cl}(A))$ is a regularly open L-set for each $A \in \mathcal{U}$. Hence by (2) we obtain

$$\begin{aligned}
& \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \\
= & \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} \mathrm{int}(A)(x)\right) \\
\leq & \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} \mathrm{int}(\mathrm{cl}(A))(x)\right) \\
\leq & \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))(x)\right) \\
\leq & \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} \mathrm{cl}(\mathrm{cl}(A))(x)\right) \\
= & \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} \mathrm{cl}(A)(x)\right).
\end{aligned}$$

This shows that (1) is true.

Analogous to Theorem 3.6 we have the following result.

Theorem 3.8. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G is almost (countably) fuzzy compact.*

(2) *For any $a \in L\backslash\{1\}$, each (countable) regularly open strong a-shading $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is a strong a-shading of G.*

(3) *For any $a \in L\backslash\{0\}$, each (countable) regularly closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ such that $\mathcal{F}^\circ$ is a strong a-remote family of G.*

Theorem 3.9. *Let $(X, \mathcal{T})$ be a regular L-space and $G \in L^X$. Then G is fuzzy compact if and only if it is almost fuzzy compact.*

Proof. The necessity is obvious. Now we prove the sufficiency. Let $\{A_i\}_{i\in\Omega}$ be a family of open L-sets. By regularity of $(X, \mathcal{T})$, we know that for each $i \in \Omega$, there exists a family $\{B_{ij} \mid j \in \Delta_i\}$ of open L-sets such that $A_i = \bigvee_{j\in\Delta_i} B_{ij}$ and $\mathrm{cl}(B_{ij}) \leq A_i$. By almost fuzzy compactness of G, we know

$$\begin{aligned}\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{i\in\Omega}A_i(x)\right) &= \bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{i\in\Omega}\bigvee_{j\in\Delta_i}B_{ij}(x)\right)\\ &\le \bigvee_{\Gamma\in 2^{(\Omega)}}\bigvee_{\Theta_i\in 2^{(\Delta_i)}}\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{i\in\Gamma}\bigvee_{j\in\Theta_i}\mathrm{cl}(B_{ij})(x)\right)\\ &\le \bigvee_{\Gamma\in 2^{(\Omega)}}\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{i\in\Gamma}A_i(x)\right).\end{aligned}$$

Therefore G is fuzzy compact.

4. Some properties of almost fuzzy compactness

Theorem 4.1. *Let L be a complete Heyting algebra. If both G and H are almost (countably) fuzzy compact, then $G\vee H$ is almost (countably) fuzzy compact.*

Proof. For any family $\mathcal{P}$ of closed L-sets, by Theorem 3.4 we have

$$\begin{aligned}&\bigvee_{x\in X}\left((G\vee H)(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\\ =\ &\left\{\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\right\}\vee\left\{\bigvee_{x\in X}\left(H(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\right\}\\ \ge\ &\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}\mathrm{int}(B)(x)\right)\right\}\vee\\ &\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(H(x)\wedge\bigwedge_{B\in\mathcal{F}}\mathrm{int}(B)(x)\right)\right\}\\ =\ &\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left((G\vee H)(x)\wedge\bigwedge_{B\in\mathcal{F}}\mathrm{int}(B)(x)\right).\end{aligned}$$

This shows that $G\vee H$ is almost fuzzy compact. □

Theorem 4.2. *If G is almost (countably) fuzzy compact, and H is clopen, then $G\wedge H$ is almost (countably) fuzzy compact.*

Proof. Since G is almost fuzzy compact, for any family $\mathcal{P}$ of closed

L-sets, by Theorem 3.4 we have

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\wedge H)(x)\wedge \bigwedge_{B\in\mathcal{P}} B(x)\right)\\
=\;&\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{P}\bigcup\{\mathcal{H}\}} B(x)\right)\\
\geq\;&\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P}\cup\{\mathcal{H}\})}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{F}} \mathrm{int}(B)(x)\right)\\
=\;&\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{F}} \mathrm{int}(B)(x)\right)\right\}\\
&\wedge\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge \mathrm{int}(H)(x)\wedge \bigwedge_{B\in\mathcal{F}} \mathrm{int}(B)(x)\right)\right\}\\
=\;&\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge \mathrm{int}(H)(x)\wedge \bigwedge_{B\in\mathcal{F}} \mathrm{int}(B)(x)\right)\right\}\\
=\;&\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left((G\wedge H)(x)\wedge \bigwedge_{B\in\mathcal{F}} \mathrm{int}(B)(x)\right).
\end{aligned}$$

This shows that $G\wedge H$ is almost fuzzy compact. □

Theorem 4.3. *Let L be a complete Heyting algebra, and let $f:(X,\mathcal{T}_1)\to(Y,\mathcal{T}_2)$ be almost continuous. If G is almost (countably) fuzzy compact in $(X,\mathcal{T}_1)$, then so is $f_L^{\rightarrow}(G)$ in $(Y,\mathcal{T}_2)$.*

Proof. Suppose that $\mathcal{P}$ be a family of regularly closed L-sets, by Lemma 2.5 and almost fuzzy compactness of G, we have

$$\begin{aligned}
&\bigvee_{y\in Y}\left(f_L^{\rightarrow}(G)(y)\wedge \bigwedge_{B\in\mathcal{P}} B(y)\right)\\
=\;&\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{P}} f_L^{\leftarrow}(B)(x)\right)\\
\geq\;&\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{F}} \mathrm{int}(f_L^{\leftarrow}(B))(x)\right)\\
\geq\;&\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{F}} f_L^{\leftarrow}(\mathrm{int}(B))(x)\right)\\
=\;&\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{y\in Y}\left(f_L^{\rightarrow}(G)(y)\wedge \bigwedge_{B\in\mathcal{F}} \mathrm{int}(B)(y)\right).
\end{aligned}$$

Therefore $f_L^{\rightarrow}(G)$ is almost fuzzy compact.

Theorem 4.4. *Let L be a complete Heyting algebra, and let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be weakly continuous. If G is (countably) fuzzy compact in $(X, \mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is almost (countably) fuzzy compact in $(Y, \mathcal{T}_2)$.*

Proof. Let $\mathcal{P}$ be a family of regularly closed L-sets, by Lemma 2.5 and fuzzy compactness of G, we have

$$
\begin{aligned}
& \bigvee_{y\in Y}\left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B\in\mathcal{P}} B(y)\right) \\
= & \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} f_L^{\leftarrow}(B)(x)\right) \\
\geq & \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} \mathrm{cl}\left(f_L^{\leftarrow}(\mathrm{int}(B))\right)(x)\right) \\
\geq & \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{F}} \mathrm{cl}\left(f_L^{\leftarrow}(\mathrm{int}(B))\right)(x)\right) \\
\geq & \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{F}} f_L^{\leftarrow}(\mathrm{int}(B))(x)\right) \\
= & \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{y\in Y}\left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B\in\mathcal{F}} \mathrm{int}(B)(y)\right).
\end{aligned}
$$

Therefore $f_L^{\rightarrow}(G)$ is almost fuzzy compact.

5. Further characterizations of almost fuzzy compactness

In this section, we assume that L is a completely distributive DeMorgan algebra.

Definition 5.1 ([21]). *Let $(X, \mathcal{T})$ be an L-space, $a \in L\backslash\{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a β_a-cover of G if for any $x \in X$, it follows that $a \in \beta\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)$. $\mathcal{U}$ is called a strong β_a-cover of G if $a \in \beta\left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)\right)$.*

Definition 5.2 ([21]). *Let $(X, \mathcal{T})$ be an L-space, $a \in L\backslash\{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a Q_a-cover of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) \geq a$.*

Analogous to [21] we can obtain the following theorem.

Theorem 5.3. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent.*

(1) *G is almost (countably) fuzzy compact.*

(2) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$), each (countable) closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily $\mathcal{F}$ such that $\mathcal{F}^\circ$ is an (a strong) a-remote family of G.*

(3) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$) and any (countable) closed strong a-remote family $\mathcal{P}$ of G, there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ (or $b \in \beta^*(a)$) such that $\mathcal{F}^\circ$ is a (strong) b-remote family of G.*

(4) *For any $a \in L\backslash\{1\}$ (or $a \in P(L)$), each (countable) open strong a-shading $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is an (a strong) a-shading of G.*

(5) *For any $a \in L\backslash\{1\}$ (or $a \in P(L)$) and any (countable) open strong a-shading $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha(a)$ (or $b \in \alpha^*(a)$) such that $\mathcal{V}^-$ is a (strong) b-shading of G.*

(6) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$), each (countable) open strong β_a-cover $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is a (strong) β_a-cover of G.*

(7) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$) and any (countable) open strong β_a-cover $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ (or $b \in M(L)$) with $a \in \beta(b)$ such that $\mathcal{V}^-$ is a (strong) β_b-cover of G.*

(8) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$) and any $b \in \beta(a)\backslash\{0\}$, each (countable) open Q_a-cover of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is a Q_b-cover of G.*

(9) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$) and any $b \in \beta(a)\backslash\{0\}$ (or $b \in \beta^*(a)$), each (countable) open Q_a-cover of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is a (strong) β_b-cover of G.*

Remark 5.4. *In Theorem 5.3, 'open' can be replaced by 'regularly open', and 'closed' can be replaced by 'regularly closed'.*

Remark 5.5. *From (2) of Theorem 5.3 we know that our notion of almost fuzzy compactness is a generalization of almost F-compactness in [3, 9].*

The following theorem shows that almost (countable) fuzzy compactness is a good extension.

Theorem 5.6. *Let (X, τ) be a topological space and $(X, \omega(\tau))$ be generated topologically by (X, τ). Then $(X, \omega(\tau))$ is almost (countably) fuzzy compact if and only if (X, τ) is almost (countably) compact.*

Proof. (Necessity) Let $\mathcal{A}$ be an open cover of (X,τ). Then $\{\chi_A \mid A \in \mathcal{A}\}$ is a family of open L-sets in $(X,\omega(\tau))$ with $\bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{U}} \chi_A(x)\right) = 1$. From almost fuzzy compactness of $(X,\omega(\tau))$ we know that

$$\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}} \chi_{\mathrm{cl}(A)}(x)\right) = \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}} \mathrm{cl}(\chi_A)(x)\right) = 1.$$

This implies that there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $\bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}} \chi_{\mathrm{cl}(A)}(x)\right) = 1$. Hence $\{\mathrm{cl}(A) \mid A \in \mathcal{V}\}$ is a cover of (X,τ). Therefore (X,τ) is almost compact.

(Sufficiency) Let $\mathcal{U}$ be a family of open L-sets in $(X,\omega(\tau))$ and let $\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}} B(x)\right) = a$. If $a = 0$, then obviously we have

$$\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}} B(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}} \mathrm{cl}(B)(x)\right).$$

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a)\backslash\{0\}$ we have

$$b \in \beta\left(\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}} B(x)\right)\right) \subseteq \bigcap_{x\in X} \beta\left(\bigvee_{B\in\mathcal{U}} B(x)\right) = \bigcap_{x\in X}\bigcup_{B\in\mathcal{U}} \beta\left(B(x)\right).$$

From Lemma 2.3 this implies that $\{B_{(b)} \mid B \in \mathcal{U}\}$ is an open cover of (X,τ). From almost fuzzy compactness of (X,τ) we know that there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $\{\mathrm{cl}(B_{(b)}) \mid B \in \mathcal{V}\}$ is a cover of (X,τ). From [17] we can obtain that $\mathrm{cl}(B_{(b)}) \subseteq \mathrm{cl}(B)_{[b]}$. This shows that $\{\mathrm{cl}(B)_{[b]} \mid B \in \mathcal{V}\}$ is a cover of (X,τ). Hence $b \leq \bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}} \mathrm{cl}(B)(x)\right)$. Further we have

$$b \leq \bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}} \mathrm{cl}(B)(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}} \mathrm{cl}(B)(x)\right).$$

This implies

$$\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}} B(x)\right) = a = \bigvee\{b \mid b \in \beta(a)\} \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}} \mathrm{cl}(B)(x)\right).$$

Therefore $(X,\omega(\tau))$ is almost fuzzy compact.

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Fu-Gui Shi
Department of Mathematics
Beijing Institute of Technology
Beijing 100081,
P. R. China
e-mail : fuguishi@bit.edu.cn or f.g.shi@263.net