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## ON SUMS OF BINOMIAL COEFFICIENTS

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### Abstract

*We investigate the integral representation of infinite sums involving the ratio of binomial coefficients. We also recover some well-known properties of  $\zeta(3)$  and extend the range of results given by other authors.*

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## 1. Introduction

In this paper we investigate the summation of the ratio of products of combinatorial coefficients. In particular, we develop integral representations for

$$\sum_{n=0}^{\infty} \frac{t^n n^p}{\binom{an+j}{j} \binom{bn+k}{k} \binom{cn+l}{l}}, \quad \sum_{n=0}^{\infty} \frac{t^n n^p \binom{n+m-1}{n}}{\binom{an+j}{j} \binom{bn+k}{k} \binom{cn+l}{l}}.$$

For the representation of sums of reciprocals of single and double binomial coefficients, in integral form, one may refer to some results in the papers [5], [4] and [7], see also the book [6].

For designated cases of the parameter values  $(a, b, c, j, k, l, m, p, t)$ , various particular sums may be expressed in terms of  $\zeta(2)$  and  $\zeta(3)$ . For many interesting properties of the Zeta function the interested reader is referred to the internet site [9].

The representation of sums in terms of integrals is extremely useful because it allows one to estimate bounds on the sums in cases they cannot be written in closed form. Convexity properties for sums may also be investigated, see [8].

Apéry's [1], see also Beukers [2], proof of the irrationality of  $\zeta(3)$  uses an elementary and quite complicated construction of the approximants  $\frac{\alpha_n}{\beta_n} \in \mathbb{Q}$  to this number based on a recurrence relation. The integral representation

$$\int_0^1 \int_0^1 \int_0^1 \frac{\{x(1-x) y(1-y) z(1-z)\}^n}{(1 - (1-x)y)z} dx dy dz = 2\beta_n \zeta(3) - 2\alpha_n$$

for the sequence  $\{\alpha_n, \beta_n\}$  was proposed.

More recently Rhin and Viola [3] introduced the integral

$$\int_0^1 \int_0^1 \int_0^1 \frac{u^h (1-u)^l v^k (1-v)^s w^j (1-w)^q}{(1 - (1-uv)w)^{q+h-r+1}} du dv dw \in \mathbb{Q}\zeta(3) + \mathbb{Q}$$

in their study of an irrationality measure for  $\zeta(3)$ .

## 2. The Main Results

In this section we develop integral identities for reciprocals of triple products of binomial coefficients.

The following lemma is given

**Lemma 1.** For  $a, b$  and  $c$  positive real numbers and  $t \in \mathbf{R}$  let

$$(2.1) \quad f = tx^a y^b z^c$$

and

$$\lambda(f) = \sum_{n=0}^{\infty} f^n = \frac{1}{1-f}$$

The consecutive partial derivative operator of the continuous function  $(1-f)^{-1}$  for  $(x, y, z) \in \{[0, 1] \times [0, 1] \times [0, 1]\}$  is defined as

$$[\lambda(f)]^{(0)} = \frac{1}{1-tx^a y^b z^c}$$

$$\vdots$$

$$[\lambda(f)]^{(p)} = \underbrace{x \frac{\partial}{\partial x} \left( x \frac{\partial}{\partial x} \left( \dots x \frac{\partial}{\partial x} \left( \frac{1}{1-f} \right) \right) \right)}_{p\text{-times}}$$

so that

$$(2.2) \quad [\lambda(f)]^{(p)} = a^p \sum_{n=0}^{\infty} n^p f^n = \frac{a^p}{(1-f)^{1+p}} \sum_{r=1}^p S(p, r) r! f^r (1-f)^{p-r},$$

where

$$S(p, r) = \left\{ \begin{matrix} p \\ r \end{matrix} \right\} = \frac{1}{r!} \sum_{\mu=0}^r (-1)^\mu \binom{r}{\mu} (r-\mu)^p$$

are Stirling numbers of the second kind.

**Proof.** The proof follows by noting that  $x \frac{\partial f}{\partial x} = af$  and

$$[\lambda(f)]^{(1)} = a \sum_{n=0}^{\infty} n f^n = \frac{af}{(1-f)^2}$$

$$[\lambda(f)]^{(2)} = a^2 \sum_{n=0}^{\infty} n^2 f^n = \frac{a^2}{(1-f)^3} \{f(1-f) + 1.2f^2\}$$

$$[\lambda(f)]^{(3)} = a^3 \sum_{n=0}^{\infty} n^3 f^n = \frac{a^3}{(1-f)^4} \{f(1-f)^2 + 3.2!f^2(1-f) + 3!f^3\}$$

.....

By induction we see that  
 $[\lambda(f)]^{(p+1)}$

$$\begin{aligned} &= x \frac{\partial}{\partial x} [\lambda(f)]^{(p)} = a^p x \frac{\partial}{\partial x} \left[ \sum_{r=1}^p S(p, r) r! f^r (1-f)^{-1-r} \right] \\ &= a^{p+1} \left[ \begin{aligned} &S(p, 1) 1! f (1-f)^{-2} + \dots + p! f^p (1-f)^{-1-p} \\ &\times \{S(p, p-1) + pS(p, p)\} + (p+1)! S(p, p) f^{p+1} (1-f)^{-2-p} \end{aligned} \right]. \end{aligned}$$

We may write

$$S(p, 1) = S(p+1, 1) = 1, \quad S(p, p) = S(p+1, p+1) = 1$$

and by the recurrence of Stirling numbers of the second kind,  $S(p+1, p) = S(p, p-1) + pS(p, p)$  we have that

$$\begin{aligned} &a^{p+1} \left[ \begin{aligned} &S(p, 1) 1! f (1-f)^{-2} + \dots + p! f^p (1-f)^{-1-p} \\ &\times \{S(p, p-1) + pS(p, p)\} + (p+1)! S(p, p) f^{p+1} (1-f)^{-2-p} \end{aligned} \right] \\ &= \frac{a^{p+1}}{(1-f)^{p+2}} \left[ \begin{aligned} &S(p+1, 1) 1! f (1-f)^p + \dots \\ &S(p+1, 1) p! f^p (1-f) + (p+1)! S(p+1, p+1) f^{p+1} \end{aligned} \right] \\ &= \frac{a^{p+1}}{(1-f)^{p+2}} \sum_{r=1}^{p+1} S(p+1, r) r! f^r (1-f)^{p+1-r}, \end{aligned}$$

so that (2.7) follows.  $\square$

Now we investigate the following theorem

**Theorem 1.** For  $a, b$  and  $c$  positive real numbers and  $j, k, l \geq 0, t \in \mathbf{R}, p \geq 0$  then

$$\begin{aligned} (2.3) \quad R &= \sum_{n=0}^{\infty} \frac{t^n n^p}{\binom{an+j}{j} \binom{bn+k}{k} \binom{cn+l}{l}} \\ &= jkl \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^{j-1} (1-y)^{k-1} (1-z)^{l-1}}{(1-f)^{1+p}} \\ &\quad \sum_{r=1}^p S(p, r) r! f^r (1-f)^{p-r} dx dz \end{aligned} \tag{2.4}$$

$$(2.5) \quad = T O_{j+k+l+p} F_{j+k+l+p-1} \left[ \begin{array}{c} \frac{a+j}{a}, \dots, \frac{a+1}{a}, \frac{b+k}{b}, \dots, \frac{b+1}{b}, \frac{c+l}{c}, \dots, \frac{c+1}{c}, 2, \dots, 2 \\ \frac{2a+j}{a}, \dots, \frac{2a+1}{a}, \frac{2b+k}{b}, \dots, \frac{2b+1}{b}, \frac{2c+l}{c}, \dots, \frac{2c+1}{c}, 1, \dots, 1 \end{array} \middle| t \right],$$

where

$$(2.6) \quad T_0 = \frac{t}{\binom{a+j}{j} \binom{b+k}{k} \binom{c+l}{l}}$$

**Proof.** Consider (2.3)

$$\begin{aligned} R &= \sum_{n=0}^{\infty} \frac{t^n}{\binom{an+j}{j} \binom{bn+k}{k} \binom{cn+l}{l}} \\ &= jkl \sum_{n=0}^{\infty} \frac{t^n \Gamma(an+1) \Gamma(j) \Gamma(bn+1) \Gamma(k) \Gamma(cn+1) \Gamma(l)}{\Gamma(an+j+1) \Gamma(bn+k+1) \Gamma(cn+l+1)} \\ &= jkl \sum_{n=0}^{\infty} t^n B(an, j+1) B(bn, k+1) B(cn, l+1) \end{aligned}$$

where  $\Gamma(\cdot)$  is the classical Gamma function and  $B(\cdot, \cdot)$  is the Beta function.

$$\begin{aligned} R &= jkl \sum_{n=0}^{\infty} t^n \int_{x=0}^1 (1-x)^{j-1} x^{an} dx \int_{y=0}^1 (1-y)^{k-1} y^{bn} dy \int_{z=0}^1 (1-z)^{l-1} z^{cn} dz \\ &= jkl \int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^1 (1-x)^{j-1} (1-y)^{k-1} (1-z)^{l-1} \sum_{n=0}^{\infty} (tx^a y^b z^c)^n dx dy dz \end{aligned}$$

by an allowable change of integral and sum.

By Lemma 1

$$[\lambda(f)]^{(p)} = a^p \sum_{n=0}^{\infty} n^p f^n = \frac{a^p}{(1-f)^{1+p}} \sum_{r=1}^p S(p, r) r! f^r (1-f)^{p-r},$$

so that

$$R = jkl \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^{j-1} (1-y)^{k-1} (1-z)^{l-1}}{(1-f)^{1+p}} \\ \sum_{r=1}^p S(p, r) r! f^r (1-f)^{p-r} dx dz.$$

which is the result (2.4).  $\square$

The hypergeometric representation (2.5) can be obtained by the consideration of the ratio of successive terms (2.3).

We may also note that from known properties of the hypergeometric function, we may write, from (2.5):

$${}_{j+k+l+p}F_{j+k+l+p-1} \left[ \begin{matrix} \frac{a+j}{a}, \dots, \frac{a+1}{a}, \frac{b+k}{b}, \dots, \frac{b+1}{b}, \frac{c+l}{c}, \dots, \frac{c+1}{c}, 2, \dots, 2 \\ \frac{2a+j}{a}, \dots, \frac{2a+1}{a}, \frac{2b+k}{b}, \dots, \frac{2b+1}{b}, \frac{2c+l}{c}, \dots, \frac{2c+1}{c}, 1, \dots, 1 \end{matrix} \middle| t \right] \\ = {}_{a+b+c+p}F_{a+b+c+p-1} \left[ \begin{matrix} \frac{2a}{a}, \dots, \frac{a+1}{a}, \frac{2b}{b}, \dots, \frac{b+1}{b}, \frac{2c}{c}, \dots, \frac{c+1}{c}, 2, \dots, 2 \\ \frac{j+2a}{a}, \dots, \frac{j+a+1}{a}, \frac{k+2b}{b}, \dots, \frac{k+b+1}{b}, \frac{l+2c}{c}, \dots, \frac{l+c+1}{c}, 1, \dots, 1 \end{matrix} \middle| t \right]$$

**Example:**

1. For  $a = 1/2, b = 1, c = 1/2, j = 1, k = 2, l = 3, p = 1, t = 1/2$

$$R = \sum_{n=0}^{\infty} \frac{\binom{n}{n/2+1} \binom{n}{n+2} \binom{n}{n/2+3}}{2^n} \\ = 192 \sum_{n=0}^{\infty} \frac{n}{2^n (n+1)(n+2)^3 (n+4)(n+6)} \\ = 168\zeta(3) - 24 \{1 + 4 \ln 2\} \zeta(2) - 4 \left\{ \frac{73}{5} \ln 2 - 6 (\ln 2)^2 - 8 (\ln 2)^3 \right\} - \frac{2599}{75} \\ = \frac{16}{315} {}_6F_5 \left[ \begin{matrix} 7, 5, 3, 3, 2, 2 \\ 8, 6, 4, 4, 4 \end{matrix} \middle| \frac{1}{2} \right] \\ = 3 \int_0^1 \int_0^1 \int_0^1 \frac{(1-y)(1-z)^2 x^{1/2} y z^{1/2}}{\left(1 - \frac{x^{1/2} y z^{1/2}}{2}\right)^2} dx dy dz.$$

2. For  $a = b = c = j = k = l = t = 1, p = 0$ , we obtain the classical result

$$R = \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} = \zeta(3)$$

Now consider the following lemma and theorem, which is a generalisation of Theorem 1.

**Lemma 2.** For  $a, b, c$  and  $m$  positive real numbers and  $t \in \mathbf{R}$  let

$$\lambda_m(f) = \sum_{n=0}^{\infty} \binom{n+m-1}{n} f^n = \frac{1}{(1-f)^m}$$

where  $f$  is given by (2.1).

The consecutive partial derivative operator of the continuous function  $(1-f)^{-m}$  for  $(x, y, z) \in \{[0, 1] \times [0, 1] \times [0, 1]\}$  is defined as

$$\begin{aligned} [\lambda_m(f)]^{(0)} &= \frac{1}{(1-tx^ay^bz^c)^m} \\ &\vdots \\ [\lambda_m(f)]^{(p)} &= \underbrace{x \frac{\partial}{\partial x} \left( x \frac{\partial}{\partial x} \left( \dots x \frac{\partial}{\partial x} \left( \frac{1}{1-f} \right) \right) \right)}_{p\text{-times}} \end{aligned}$$

so that

$$\begin{aligned} [\lambda_m(f)]^{(p)} &= a^p \sum_{n=0}^{\infty} n^p \binom{n+m-1}{n} f^n \\ (2.7) \quad &= \frac{a^p}{(1-f)^{m+p}} \sum_{r=1}^p S(p, r) (m)_r f^r (1-f)^{p-r}, \end{aligned}$$

where  $S(p, r)$  are Stirling numbers of the second kind.

**Proof.** We note that  $x \frac{\partial f}{\partial x} = af$  and

$$\begin{aligned} [\lambda_m(f)]^{(1)} &= a \sum_{n=0}^{\infty} n \binom{n+m-1}{n} f^n = \frac{maf}{(1-f)^{m+1}} \\ [\lambda_m(f)]^{(2)} &= a^2 \sum_{n=0}^{\infty} n^2 \binom{n+m-1}{n} f^n = \frac{a^2}{(1-f)^{m+2}} \{mf(1-f) + m(m+1)f^2\} \\ [\lambda_m(f)]^{(3)} &= a^3 \sum_{n=0}^{\infty} n^3 \binom{n+m-1}{n} f^n = \frac{a^3}{(1-f)^{m+3}} \{mf(1-f)^2 \\ &\quad + 3.m(m+1)f^2(1-f) + m(m+1)(m+2)f^3\} \\ &\dots\dots\dots \end{aligned}$$

By induction we see that

$$[\lambda_m(f)]^{(p+1)} = x \frac{\partial}{\partial x} [\lambda_m(f)]^{(p)} = a^p x \frac{\partial}{\partial x} \left[ \sum_{r=1}^p S(p, r) (m)_r f^r (1-f)^{-m-r} \right]$$

$$= a^{p+1} \left[ \begin{array}{l} S(p, 1) m f (1-f)^{-m-1} + \dots + f^p (1-f)^{-m-p} \\ \times \{(m+p-1)(m)_{p-1} S(p, p-1) + p(m)_p S(p, p)\} \\ + (m+p)(m)_p S(p, p) f^{p+1} (1-f)^{-m-p-1} \end{array} \right].$$

We may write

$$\begin{aligned} (m+p-1)(m)_{p-1} &= (m)_p, \quad (m+p)(m)_p = (m)_{p+1} \\ S(p, 1) &= S(p+1, 1) = 1, \quad S(p, p) = S(p+1, p+1) = 1 \end{aligned}$$

and by the recurrence of Stirling numbers of the second kind,  $S(p, p-1) + pS(p, p) = S(p+1, p)$  we have that

$$\begin{aligned} & a^{p+1} \left[ \begin{array}{l} S(p, 1) m f (1-f)^{-m-1} + \dots + f^p (1-f)^{-m-p} \\ \times \{(m+p-1)(m)_{p-1} S(p, p-1) + p(m)_p S(p, p)\} \\ + (m+p)(m)_p S(p, p) f^{p+1} (1-f)^{-m-p-1} \end{array} \right] \\ &= \frac{a^{p+1}}{(1-f)^{m+p+1}} \left[ \begin{array}{l} S(p+1, 1)(m)_1 f (1-f)^p + \dots \\ + S(p+1, p)(m)_p f^p (1-f) + (m)_{p+1} S(p+1, p+1) f^{p+1} \end{array} \right] \\ &= \frac{a^{p+1}}{(1-f)^{m+p+1}} \sum_{r=1}^{p+1} S(p+1, r) (m)_r f^r (1-f)^{p+1-r} \end{aligned}$$

so that (2.7) follows.  $\square$

Now we investigate the following theorem

**Theorem 2.** For  $a, b, c$  and  $m$  positive real numbers  $j, k, l \geq 0$  with  $j + k + l \geq m + p$ ,  $t \in \mathbf{R}$ , and  $p \geq 0$  then

$$(2.8) \quad Q = \sum_{n=0}^{\infty} \frac{t^n n^p \binom{n+m-1}{n}}{\binom{an+j}{j} \binom{bn+k}{k} \binom{cn+l}{l}}$$

$$(2.9) \quad = \sum_{n=0}^{\infty} \frac{t^n n^p j!k!l! (n+1)_{m-1}}{(m-1)! (an+1)_j (bn+1)_k (cn+1)_l}$$

$$(2.10) \quad = jkl \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^{j-1} (1-y)^{k-1} (1-z)^{l-1}}{(1-f)^{m+p}} \sum_{r=1}^p S(p, r) (m)_r f^r (1-f)^{p-r} dx dy dz$$



$$= T_{\nabla} \quad j+k+l+p F_{j+k+l+p-1}$$

$$(2.11) \left[ \begin{array}{c} m+1, \frac{a+j}{a}, \dots, \frac{a+1}{a}, \frac{b+k}{b}, \dots, \frac{b+1}{b}, \frac{c+l}{c}, \dots, \frac{c+1}{c}, \overbrace{2, \dots, 2}^{(p-1)\text{terms}} \\ \frac{2a+j}{a}, \dots, \frac{2a+1}{a}, \frac{2b+k}{b}, \dots, \frac{2b+1}{b}, \frac{2c+l}{c}, \dots, \frac{2c+1}{c}, 1, \dots, 1 \end{array} \middle| t \right],$$

where

$$(w)_{\alpha} = w(w+1) \cdots (w+\alpha-1) = \frac{\Gamma(w+\alpha)}{\Gamma(w)}$$

is Pochhammer's symbol, and  $T_{\nabla} = mT_0$  where  $T_0$  is given by (2.6).

**Proof.** Consider (2.8)

To arrive at the result (2.10) consider

$$\begin{aligned} Q &= jkl \sum_{n=0}^{\infty} t^n \binom{n+m-1}{n} \frac{\Gamma(an+1)\Gamma(j)\Gamma(bn+1)\Gamma(k)\Gamma(cn+1)\Gamma(l)}{\Gamma(an+j+1)\Gamma(bn+k+1)\Gamma(cn+l+1)} \\ &= jkl \sum_{n=0}^{\infty} t^n \binom{n+m-1}{n} B(an+1, j) B(bn+1, k) B(cn+1, l) \\ &= jkl \sum_{n=0}^{\infty} t^n \binom{n+m-1}{n} \int_0^1 (1-x)^{j-1} x^{an} dx \int_0^1 (1-y)^{k-1} y^{bn} dy \\ &\quad \times \int_0^1 (1-z)^{l-1} z^{cn} dz \\ &= jkl \int_0^1 \int_0^1 \int_0^1 (1-x)^{j-1} (1-y)^{k-1} (1-z)^{l-1} \sum_{n=0}^{\infty} \binom{n+m-1}{n} \\ &\quad \times (tx^a y^b z^c)^n dx dy dz \end{aligned}$$

by an allowable change of sum and integral.

By Lemma 2

$$\begin{aligned} [\lambda_m(f)]^{(p)} &= a^p \sum_{n=0}^{\infty} n^p \binom{n+m-1}{n} f^n \\ &= \frac{a^p}{(1-f)^{m+p}} \sum_{r=1}^p S(p, r) (m)_r f^r (1-f)^{p-r}, \end{aligned}$$

so that

$$\begin{aligned} S &= jkl \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^{j-1} (1-y)^{k-1} (1-z)^{l-1}}{(1-f)^{m+p}} \\ &\quad \sum_{r=1}^p S(p, r) (m)_r f^r (1-f)^{p-r} dx dy dz. \end{aligned}$$

which is the result (2.10).  $\square$

The hypergeometric representation (2.11) can be obtained by the consideration of the ratio of successive terms (2.8) respectively.

In the case when  $m = 1$ , Theorem 2 reduces to Theorem 1.

**Example:** For  $a = 2, b = 2, c = 4, j = 4, k = 3, l = 3, m = 5, p = 2, t = -1$

$$\begin{aligned} Q &= \sum_{n=0}^{\infty} \frac{(-1)^n n^2 \binom{n+4}{4}}{\binom{2n+4}{4} \binom{2n+3}{3} \binom{4n+3}{3}} \\ &= 180 \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^3 (1-y)^2 (1-z)^2 x^2 y^2 z^4 (1-5x^2 y^2 z^4)}{\left(1 + \frac{x^2 y^2 z^4}{2}\right)^7} dx dy dz \\ &= -\frac{1}{1050} {}_9F_8 \left[ \begin{matrix} 6, \frac{3}{3}, \frac{3}{2}, \frac{3}{2}, \frac{7}{4}, \frac{5}{4}, 2, 2, 2 \\ 4, \frac{7}{2}, \frac{7}{2}, \frac{5}{2}, \frac{11}{4}, \frac{7}{4}, 3, 1 \end{matrix} \middle| 1 \right] \\ &= \frac{3591}{2560} + \left( \frac{1341\sqrt{2}}{160} - \frac{34173}{5120} \right) \pi + \frac{2223}{256} G - \frac{9}{2} \ln(2) \\ &\quad - \frac{945}{2098} \pi \zeta(2) + \frac{2583\sqrt{2}}{320} \ln(3 - 2\sqrt{2}). \end{aligned}$$

where  $G$  is Catalans constant.

### 3. Conclusion

We have provided triple integral identities for sums of the reciprocal of triple binomial coefficients. In doing so we have recovered the standard representation for  $\zeta(2)$  and  $\zeta(3)$  and have generalised and extended some results published previously by other authors.

In another forum we shall extend our results to consider more general sums of binomial coefficients.

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