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SOME REMARKS ON GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract

The principal aim of the paper is to establish the function $E_t(c,\nu,\gamma,q)$ and its properties by using Fractional Calculus. We also obtained some integral representations of the function $E^{\gamma,q}_{\alpha,\beta}(z)$ which is recently introduced by Shukla and Prajapati [6].

Key Words : *Fractional integral operators; fractional differential operators; generalized Mittag-Leffler function; integral representation.*

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1. INTRODUCTION

In 2007, Shukla and Prajapati [6] introduced the function $E_{\alpha,\beta}^{\gamma,q}(z)$ which is defined for $\alpha,\beta,\gamma \in \mathrm{C}$; $Re\,(\alpha) > 0$, $Re\,(\beta) > 0$, $Re\,(\gamma) > 0$ and $q \in (0,1) \cup \mathbf{N}$ as:

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\ z^n}{\Gamma\,(\alpha\, n+\beta)\ n\,!}\,, \tag{1.1}$$

where $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol (Rainville[5]) which in particular reduces to $q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma+r-1}{q}\right)_n$ if $q \in \mathrm{N}$.

Kilbas et. al [1] studied the several properties of generalized fractional calculus operators and the Mittag-Leffler function [3], the Wiman function [9] and its extension was discussed by Prabhakar and Suman [4].

We can write ordinary binomial expression (Rainville[5]) as,

$$(1-z)^{-\,a} = \sum_{n=0}^{\infty} \frac{(a)_n\, z^n}{n\,!}. \tag{1.2}$$

Shukla and Prajapati [7] also studied several properties of $E_{\alpha,\beta}^{\gamma,q}(z)$ in the light of Fractional Integral and Differential operators.

The fractional integral operator of order ν defined as (Miller and Ross [2])
for $Re\,\nu > 0$,

$$I^{\nu}\, f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(\xi)\, d\xi \tag{1.3}$$

and the fractional differential operator of order μ defined as

$$D^{\mu}\, f(t) = D^{n}\{I^{k-\mu}\, f(t)\}, \tag{1.4}$$

where $Re\,\mu > 0$ and if k is the smallest integer with the property that $k \geq Re\,\mu$.

2. FRACTIONAL OPERATORS AND GENERALIZED MITTAG–LEFFLER FUNCTION

Consider the function $f(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\,(ct)^n}{(n\,!)^2}$, where $\gamma \in \mathrm{C}$ $(Re(\gamma) > 0)$, $q \in (0,1) \cup \mathrm{N}$

and c is an arbitrary constant then using (1.3) the fractional integral operator of order ν is given as

$$I^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\,(c\xi)^n}{(n\,!)^2}\, d\xi$$

$$= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\, c^n}{(n\,!)^2} \int_0^t (t-\xi)^{\nu-1}\, \xi^n\, d\xi.$$

Above equation reduces to,

$$= t^{\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\,(ct)^n}{\Gamma(\nu+n+1)\, n\,!}\,. \tag{2.1}$$

Use of (1.1), the above equation can be written as,

$$= t^{\nu}\, E_{1,\,\nu+1}^{\gamma,q}(ct). \tag{2.2}$$

We denote the function (2.2) as $E_t(c,\nu,\gamma,q)$, i.e.

$$E_t(c,\nu,\gamma,q) = t^{\nu}\, E_{1,\nu+1}^{\gamma,q}(ct). \tag{2.3}$$

Now, using (1.4) the fractional differential operator of order μ is given as

$$D^{\mu} f(t) = D^n \left[I^{k-\mu} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\,(ct)^n}{(n\,!)^2} \right].$$

Applying (2.1), we can write

$$= D^n \left[t^{k-\mu} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\,(ct)^n}{\Gamma(k-\mu+n+1)\ n\,!} \right].$$

The simplification of above equation gives

$$= t^{-\mu} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\,(ct)^n}{\Gamma(n+1-\mu)\ n\,!}\,.$$

Use of (1.1), the above equation can be written as,

$$= t^{-\mu}\, E_{1,\,1-\mu}^{\gamma,q}(ct). \tag{2.4}$$

We denote the function (2.7) as $E_t(c,-\mu,\gamma,q)$, i. e.

$$E_t(c,-\mu,\gamma,q) = t^{-\mu}\, E_{1,1-\mu}^{\gamma,q}(ct). \tag{2.5}$$

3. PROPERTIES OF THE FUNCTIONS $E_t(c,\nu,\gamma,q)$ AND $E_t(c,-\mu,\gamma,q)$

Theorem 1. $\gamma \in C \quad (Re(\gamma) > 0)$,$q \in (0,1) \cup N$, c is an arbitrary constant and fractional integral operator of order ν then

$$I^{\lambda} E_t(c,\nu,\gamma,q) = E_t(c,\lambda+\nu,\gamma,q). \tag{3.1}$$

$$D^{\lambda} E_t(c,\nu,\gamma,q) = E_t(c,\nu-\lambda,\gamma,q). \tag{3.2}$$

The Laplace transform of $E_t(c,\nu,\gamma,q)$ is given as

$$L\{E_t(c,\nu,\gamma,q)\} = \frac{1}{s^{\nu+1}} \left(1-\frac{c}{s}\right)^{-\gamma,q}, \tag{3.3}$$

Shukla and Prajapati [8] introduced a new notation for binomial expression as

$$(1-z)^{-\gamma,q} \quad = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\, z^n}{n\,!}. \tag{3.4}$$

If $q = 1$ then (3.4) becomes (1.2) as

$$(1-z)^{-\gamma,1} \quad = (1-z)^{-\gamma}. \tag{3.5}$$

Proof. From (1.3), we get

$$I^{\lambda} E_t(c,\nu,\gamma,q) = \frac{1}{\Gamma(\lambda)} \int_0^t (t-\xi)^{\lambda-1}\, E_{\xi}(c,\nu,\gamma,q)\; d\xi.$$

Using (2.3), above equation becomes

$$= \frac{1}{\Gamma(\lambda)} \int_0^t (t-\xi)^{\lambda-1}\, \xi^{\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\,(c\xi)^n}{\Gamma(\nu+n+1)\; n\,!}\; d\xi$$

and substituting $\xi = xt$, which yields

$$= \frac{1}{\Gamma(\lambda)}\; t^{\lambda+\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\, c^n\, t^n}{\Gamma(\nu+n+1)\; n\,!} \int_0^1 (1-x)^{\lambda-1}\, x^{k+\nu}\; dx.$$

The simplification of above equation gives

$$= \; t^{\lambda+\nu}\, E^{\gamma,q}_{1,\,\lambda+\nu+1}\,(ct).$$

Again from (1.3), we get

$$= E_t(c, \lambda + \nu, \gamma, q).$$

This is the proof of (3.1).
From (1.4), we get

$$D^{\lambda} E_t(c, \nu, \gamma, q) = D^k\{I^{k-\lambda} E_t(c, \nu, \gamma, q)\}.$$

Using (3.1), we can write

$$= D^k\{t^{k+\nu-\lambda} E^{\gamma,q}_{1,k+\nu-\lambda+1}(ct)\}.$$

Applying (2.3), above equation can be written as

$$=D^k \left[\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\, c^n}{\Gamma\,(n + k + \nu - \lambda + 1)}\ \frac{t^{n+k+\nu-\lambda}}{n\,!}\right].$$

The above equation reduces to,

$$=\ t^{\nu-\lambda}\, E^{\gamma,q}_{1,\,\nu-\lambda+1}\,(ct).$$

Again from (1.3), we obtain

$$= E_t(c, \nu - \lambda, \gamma, q).$$

This is the proof of (3.2).
From (2.3), consider

$$L\{E_t(c, \nu, \gamma, q)\} = L\{t^{\nu}\, E^{\gamma,q}_{1,\nu+1}(ct)\}.$$

Therefore,
$= \frac{1}{s^{\nu+1}} \sum\limits_{n=0}^{\infty} \frac{(\gamma)_{qn}}{n\,!}\ \frac{c^n}{s^n}.$
Use of (3.4), we arrived at

$$= \frac{1}{s^{\nu+1}} \left(1 - \frac{c}{s}\right)^{-\gamma,q}.$$

This is the proof of Theorem 1.
In the light of Theorem 1, we can prove following Theorem 2.

Theorem 2. $\mu \in$ C $\quad (Re(\mu) > 0)$,$q \in (0, 1) \cup$ N, c is an arbitrary constant and fractional integral operator of order μ then

(3.6) $$I^{\lambda} E_t(c, -\mu, \gamma, q) = E_t(c, \lambda - \mu, \gamma, q).$$

(3.7) $$D^{\lambda}E_t(c,-\mu,\gamma,q) = E_t(c,-\lambda-\mu,\gamma,q).$$

(3.8) $$L\{E_t(c,-\mu,\gamma,q)\} = \frac{1}{s^{1-\mu}}\left(1-\frac{c}{s}\right)^{-\gamma,q}.$$

4. SOME INTEGRAL REPRESENTAIONS OF $E_{\alpha,\beta}^{\gamma,q}(z)$

In this section, we obtained three interesting integral representations of the function$E_{\alpha,\beta}^{\gamma,q}(z)$.

Theorem 3. If $\alpha,\beta,\gamma \in$ C ; $Re\,(\alpha) > 0$, $Re\,(\beta) > 0$, $Re\,(\gamma) > 0$, $\beta > \alpha > 0$ and $q \in (0,1) \cup$ N then

$$E_{\alpha,\beta}^{\gamma,q}(z) = k\,z^{\alpha-\beta}\int_0^{\infty}\exp\left(-\frac{t^k}{z^k}\right)t^{\beta-\alpha-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,t^n}{\Gamma(\alpha\,n+\beta)\,n!\;\Gamma\left(\frac{\beta-\alpha+n}{k}\right)}\,dt.$$
(4.1)

Proof. Consider,

$$\int_0^{\infty}\exp\left(-\frac{t^k}{z^k}\right)t^{\beta-\alpha-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,t^n}{\Gamma(\alpha\,n+\beta)\,n!\;\Gamma\left(\frac{\beta-\alpha+n}{k}\right)}\,dt.$$

Substituting$\frac{t^k}{z^k}=u$, we get

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,z^{\beta-\alpha+n}}{\Gamma(\alpha\,n+\beta)\,n!\,\Gamma\left(\frac{\beta-\alpha+n}{k}\right)}\,\frac{1}{k}\int_0^{\infty}e^{-u}\,u^{\frac{\beta-\alpha+n}{k}-1}dt.$$

Using (1.1), above equation immediately leads to,

$$=\frac{z^{\beta-\alpha}}{k}\;E_{\alpha,\beta}^{\gamma,q}(z).$$

This is the proof of Theorem 3.

Theorem 4. If $\alpha,\beta,\gamma \in$ C ; $Re\,(\alpha) > 0$, $Re\,(\beta) > 0$, $Re\,(\gamma) > 0$, $\beta > \alpha > 0$ and $q \in (0,1) \cup$ N then

(4.2) $$E_{\alpha,\beta}^{\gamma,q}(z) = \frac{1}{\alpha\,\Gamma\,(\beta-\alpha)}\int_0^1 (1-t^{\frac{1}{\alpha}})^{\,\beta-\alpha-1}\;E_{\alpha,\alpha}^{\gamma,q}(t\,z)\,dt.$$

Proof. Now consider,

$$\int_0^1 (1 - t^{\frac{1}{\alpha}})^{\beta-\alpha-1} \, E^{\gamma,q}_{\alpha,\alpha}(t\,z)\,dt.$$

Applying (1.1) and substituting$t^{\frac{1}{\alpha}} = u$, we get

$$= \alpha \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\; z^n}{\Gamma\,(\alpha\,n + \alpha)\; n\,!} \int_0^1 u^{\alpha\,n+\alpha-1}(1-u)^{\beta-\alpha-1}\; du.$$

Therefore we arrived at

$$= \alpha\,\Gamma(\beta-\alpha) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\; z^n}{\Gamma\,(\alpha\,n + \alpha)\; n\,!}\,.$$

Again use of (1.1), we arrived at

$$= \alpha\,\Gamma\,(\beta-\alpha)\,E^{\gamma,q}_{\alpha,\beta}(z).$$

This is the proof of Theorem 4.

Applying (1.1) in RHS of (4.3), It is easy to prove following Theorem.

Theorem 5. If $\alpha, \beta, \gamma \in \mathrm{C}$; $Re\,(\alpha) > 0$, $Re\,(\beta) > 0$, $Re\,(\gamma) > 0$, $\beta > \alpha > 0$ and $q \in (0,1) \cup \mathrm{N}$ then

$$E^{\gamma,q}_{\alpha,\beta}(z) = \frac{1}{\Gamma\,(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\; E^{\gamma,q}_{\alpha,\beta-\alpha}\,(z(1-t)^{\alpha})\;\, dt. \tag{4.3}$$

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