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ARENS REGULARITY OF SOME BILINEAR MAPS

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Abstract

Let H be a Hilbert space. we show that the following statements are equivalent: (a) $B(H)$ is finite dimension, (b) every left Banach module action $l : B(H) \times H \longrightarrow H$, is Arens regular (c) every bilinear map $f : B(H)^* \longrightarrow B(H)$ is Arens regular. Indeed we show that a Banach space X is reflexive if and only if every bilinear map $f : X^* \times X \longrightarrow X^*$ is Arens regular.

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1. Introduction

The regularity of bilinear maps on norm spaces, was introduced by Arens in 1951. Let X , Y and Z be normed spaces and let $f : X \times Y \longrightarrow Z$ be a continuous bilinear map, then $f^* : Z^* \times X \longrightarrow Y^*$ (the transpose of f) is defined by

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x, y)\rangle \qquad (z^* \in Z^*\ ,\ x \in X\ ,\ y \in Y).$$

f^* is a continuous bilinear map.

Set $f^{**} = (f^*)^*$ and $f^{***} = (f^{**})^*, \ldots$. Then $f^{***} : X^{**} \times Y^{**} \longrightarrow Z^{**}$ is the unique extension of f such that $f^{***}(\cdot, y'') : X^{**} \longrightarrow Z^{**}$ is $weak^* - weak^*$ continuous for every $y'' \in Y^{**}$. Let $f^r : Y \times X \longrightarrow Z$ defined by $f^r(y, x) = f(x, y),\ \ (x \in X,\ y \in Y)$, then f^r is a continuous bilinear map. f is called Arens regular whenever $f^{***} = f^{r***r}$. It is easy to show that f is Arens regular if and only if $f^{***}(x'', \cdot) : Y^{**} \longrightarrow Z^{**}$ is $weak^* - weak^*$ continuous for every $x'' \in X^{**}$. For further details we refer the reader to [A], [E-F], [D-R-V], and [M-Y]. Let $\mathcal{A}$ be a Banach algebra and let $\pi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ be the product of $\mathcal{A}$. Then the second dual space $\mathcal{A}^{**}$ of $\mathcal{A}$ is Banach algebra with both products π^{***} and π^{r***r}. $\mathcal{A}$ is called Arens regular if π is Arens regular. See [A], [D-H] and [F-S]. An element $f \in \mathcal{A}^*$ is called weakly almost periodic if the maps

$$a \mapsto a.f, \quad a \mapsto f.a, \ \ \mathcal{A} \longrightarrow \mathcal{A}^*$$

are weakly compact. The collection of weakly almost periodic elements in $\mathcal{A}^*$ is denoted by $Wap(\mathcal{A}^*)$.

1.Theorem. Let X be a Banach space and let $l : B(X) \times X \longrightarrow X$ defined by $l(T, x) = T(x), \quad (T \in B(X), x \in X)$. If l is Arens regular, then $B(X)^{**}$ is isomorphic to a subalgebra of $B(X^*)$.

Proof: Let $\pi : B(X) \times B(X) \longrightarrow B(X)$ be the product of $B(X)$. Then for every $e \in X$, $e' \in X^*$, and $T, S \in B(X)$, we have

$$\begin{aligned}\langle l^*(e', \pi(T, S)), e\rangle &= \langle e', l(T, S(e)\rangle \\ &= \langle l^*(e', T), l(S, e)\rangle \\ &= \langle l^*(l^*(e', T)S), e\rangle.\end{aligned}$$

This means that

$$l^*(e', \pi(T, S)) = l^*(L^*(e', T), S) \quad (1).$$

By applying (1), for every $e'' \in X^{**}$, we have

$$\begin{aligned} \langle \pi^*(l^{**}(e'', e'), T), S\rangle &= \langle l^{**}(e'', e'), \pi(T, S)\rangle \\ &= \langle e'', l^*(e', \pi(T, S)\rangle \\ &= \langle e'', l^*(L^*(e', T), S)\rangle \\ &= \langle l^{**}(e'', L^*(e', T), S\rangle. \end{aligned}$$

Thus we have

$$\pi^*(l^{**}(e'', e'), T) = l^{**}(e'', L^*(e', T) \qquad (2).$$

Let $B \in B(X)^{**}$, then by (2), we have

$$\begin{aligned} \langle \pi^{**}(B, l^{**}(e'', e'), T)\rangle &= \langle B, \pi^*(l^*(e'', e'), T)\rangle \\ &= \langle B, l^{**}(e'', L^*(e', T))\rangle \\ &= \langle l^{***}(B, e''), L^*(e', T)\rangle \\ &= \langle l^{**}(l^{***}(B, e''), e'), T\rangle. \end{aligned}$$

This means that

$$\pi^{**}(B, l^{**}(e'', e') = l^{**}(l^{***}(B, e''), e') \qquad (3).$$

Let now $A \in B(X)^{**}$. By (3), we have

$$\begin{aligned} \langle l^{***}\pi^{***}(A, B), e''), e'\rangle &= \langle \pi^{***}(A, B), l^{**}(e'', e')\rangle \\ &= \langle A, \pi^{**}(B, l^{**}(e'', e'))\rangle \\ &= \langle A, l^{**}(l^{***}(B, e''), e')\rangle. \end{aligned}$$

Therefore

$$l^{***}(\pi^{***}(A, B), e'') = l^{***}(A, l^{***}(B, e'')).$$

This means that the mapping $f : B(X)^{**} \longrightarrow B(X^{**})$, defined by $f(A) = l^{***}(A, .) \quad (A \in B(X)^{**})$, is a Banach algebras homomorphism. On the other hand l is Arens regular, then $f(A) : X^{**} \longrightarrow X^{**}$ is $weak^* - weak^*$ continuous for every $A \in B(X)^{**}$. Let now

$$B_{w^*}(X^{**}) := \{U \in B(X^{**}) \mid U : X^{**} \to X^{**} \text{ is } weak^*{-}weak^* \text{ continuous } \},$$

and let $\phi : B(X^*) \longrightarrow B_{w^*}(X^{**})$ defined by $\phi(T) = T^* \quad (T \in B(X^*))$. Then $\phi^{-1} o f : B(X)^{**} \longrightarrow B(X^*)$ is a injective Banach algebras anti homomorphism.

2.Theorem. Let $\mathcal{A}$ be a Banach algebra and let L be a reflexive left Banach $\mathcal{A}$ module with module action $g : \mathcal{A}\times\mathcal{L} \longrightarrow \mathcal{L}$. Then $g^{**}(L^{**}\times L^{*}) \subseteq Wap(\mathcal{A}^{*})$.

Proof: Let $x'' \in L^{**}$, $x' \in X^{*}$, and let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathcal{A}$, in which the limits:

$$\lim_m \lim_n \langle g^{**}(x'', x'), a_m a_n\rangle, \quad \text{and} \quad \lim_n \lim_m \langle g^{**}(x'', x'), a_m a_n\rangle,$$

are both exist. Then we have

$$\begin{aligned}
\lim_m \lim_n \langle g^{**}(x'', x'), a_m b_n\rangle &= \lim_m \lim_n \langle x'', g^{*}(x', a_m b_n)\rangle \\
&= \lim_m \lim_n \langle x'', g^{*}(g^{*}(x', a_m), b_n)\rangle \\
&= \lim_m \lim_n \langle g^{**}(x'', g^{*}(x', a_m)), b_n\rangle \\
&= \lim_m \lim_n \langle g^{***}(\widehat{b_n}, x''), g^{*}(x', a_m)\rangle \\
&= \lim_n \lim_m \langle g^{***}(\widehat{b_n}, x''), g^{*}(x', a_m)\rangle \quad \text{(L is reflexive)} \\
&= \lim_n \lim_m \langle g^{**}(x'', g^{*}(x', a_m)), b_n\rangle \\
&= \lim_n \lim_m \langle x'', g^{*}(g^{*}(x', a_m), b_n)\rangle \\
&= \lim_n \lim_m \langle x'', g^{*}(x', a_m b_n)\rangle \\
&= \lim_m \lim_n \langle g^{**}(x'', x'), a_m b_n\rangle.
\end{aligned}$$

This means that $g^{**}(x'', x') \in Wap(\mathcal{A}^{*})$ [Pa, Theorem 1.4.11].

3.Corollary. Let $\mathcal{A}, \mathcal{L}$ and g are as above. If $g^{**}(L^{**} \times L^{*}) = \mathcal{A}^{*}$, then $\mathcal{A}$ is Arens regular.

4.Corollary [D]. Let X be a Banach space and let L be a reflexive left Banach $B(X)$ module. We define the map $\phi : L\widehat{\otimes}L^{*} \longrightarrow B(X)^{*}$ by

$$\langle \phi(f \otimes \mu), T\rangle = \langle \mu, T.f\rangle \quad (f \otimes \mu \in L\widehat{\otimes}L^{*}, T \in B(X)).$$

If ϕ is surjective then $B(X)$ is Arens regular.

5.Theorem. Let X be a Banach space. Then X is reflexive if and only if every bilinear map $f : X^{*} \times X \longrightarrow X^{*}$ is Arens regular.

Proof: Obviously every bilinear map $f : X^{*} \times X \longrightarrow X^{*}$ is Arens regular if X is reflexive. For the converse, let $0 \neq x_0 \in X$. Then by Hahn-Banach Theorem, there exists $g \in X^{*}$ such that $\langle g, x_0\rangle = 1$. Let $f : X \times X \longrightarrow X$ defined by $f((x, y)) = \langle g, y\rangle x$. f is a bilinear map. Then $f^{*} : X^{*}\times X \longrightarrow X^{*}$ is Arens regular. Let now $x''' \in X^{***}$ and let $x''_\alpha \xrightarrow{weak^{*}} x''$ in X^{**}, then $f^{****}(x''', x''_\alpha) \xrightarrow{weak^{*}} f^{****}(x''', x'')$ in X^{***}. Thus for every $y'' \in X^{**}$ we have

$$\begin{aligned}\lim_{\alpha}\langle x''', f^{***}(x''_{\alpha}, y'')\rangle &= \lim_{\alpha}\langle f^{****}(x''', x''_{\alpha}, y''\rangle \\ &= \langle f^{****}(x''', x'', y''\rangle \\ &= \langle x''', f^{***}(x'', y'')\rangle \quad (4).\end{aligned}$$

On the other hand for each $y'' \in X^{**}$, there exists a net (y_β) in Y such that $\widehat{y_\beta} \xrightarrow{weak^*} y''$ in X^{**}. We know that $f^{***}(\cdot, \widehat{x_0}) : X^{**} \longrightarrow X^{**}$ is $weak^* - weak^*$ continuous, then

$$f^{***}(y'', \widehat{x_0}) = w^* - \lim_{\beta} f^{***}(\widehat{y_\beta}, \widehat{x_0}) = w^* - \lim \widehat{y_\beta} = y'' \quad (5).$$

By (4) and (5), we have

$$\begin{aligned}\lim_{\alpha}\langle x''', x''_{\alpha}\rangle &= \lim_{\alpha}\langle x''', f^{***}(x''_{\alpha}, \widehat{x_0})\rangle \\ &= \langle x''', f^{***}(x'', \widehat{x_0})\rangle \\ &= \langle x''', x''\rangle.\end{aligned}$$

This means that $x''' : X^{**} \longrightarrow C$ is $weak^* - weak^*$ continuous. Thus $x''' \in \widehat{X^*}$, and X^* is reflexive. So X is reflexive.

6.Corollary. Let H be Hilbert space. Then the following assertions are equivalent:

(a) $B(H)$ is finite dimension.

(b) The mapping $l : B(H) \times H \longrightarrow H$ defined by $l(T, x) = T(x)$, $(T \in B(H), x \in H)$, is Arens regular.

(c) Every bilinear map $f : B(H)^* \times B(H) \longrightarrow B(H)^*$ is Arens regular.

Proof. $(a) \Leftrightarrow (b)$: Let $B(H)$ be finite dimension, the $B(H)$ is reflexive. Then by Theorem 1, l is Arens regular. For the converse we know that $H \cong H^*$ as Banach spaces. Then by Theorem 1, we have $B(H)^{**} = B(H)$, so $B(H)$ is finite dimension.

$(a) \Leftrightarrow (c)$: It follows from the Theorem 5.

Example. Let $H = l^1(N)$. Then the bilinear map $l : B(H) \times H \longrightarrow H$ defined by $l(T, x) = T(x)$, $(T \in B(H), x \in H)$, is not Arens regular.

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