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FUNCTIONS OF BOUNDED (φ, p) MEAN OSCILLATION

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Abstract

In this paper we extend a result of Garnett and Jones to the case of spaces of homogeneous type.

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1. Introduction

The space of functions of bounded mean oscillation, or BMO, naturally arises as the class of functions whose deviation from their means over cubes is bounded. L_∞ functions have this property, but there exist unbounded functions with bounded mean oscillation, for instance the function $\log|x|$ is in BMO but it is not bounded. The space BMO shares similar properties with the space L_∞ and it often serve as a substitute for it. The space of the functions with bounded mean oscillation BMO, is well known for its several applications in real analysis, harmonic analysis and partial differential equations.

The definition of BMO is that $f \in BMO$ if $\sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx = \|f\|_{BMO} < \infty$where $\mathrm{f}_Q = \frac{1}{|Q|}\int_Q f(y)dy,$ $|Q|$ is the Lebesgue measure of Q and Q is a cube in R^n, with sides parallel to the coordinate axes.

In [1] Garnet and Jones gave comparable upper and lower bounds for the distance

$$dist\,(f, L_\infty) = \inf_{g\in L_\infty} \|f-g\|_{BMO}\,. \tag{1.1}$$

The bounds were expressed in terms of one constant in Jhon-Nirenberg inequality. Jhon and Nirenberg proved in [2] that $f \in BMO$ if and only if there is $\epsilon > 0$ and $\lambda_0 = \lambda_0(\epsilon)$ such that

$$\sup_Q \frac{1}{|Q|}\,|\{x \in Q : |f(x) - f_Q| > \lambda\}| \le e^{-\lambda/\epsilon}, \tag{1.2}$$

whenever $\lambda > \lambda_0 = \lambda_0(f,\epsilon)$. Indeed, when $f \in BMO$, (1.2) holds with $\epsilon = C\|f\|_{BMO}$, where the constant c depends only on the dimension.

Specifically, setting

$$\epsilon(f) = \inf\{\epsilon > 0 : f \text{ satisfies } (1.2)\}\,,$$

Garnett and Jones proved that

$$A_1 \epsilon(f) \leq dist\,(f, L_\infty) \leq A_2 \epsilon(f),$$

where A_1 and A_2 are constants depending only on the dimension. Also, they observed that $dist\,(f, L_\infty)$ can be related to the growth of

$$\sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{\frac{1}{p}}$$

as $p \to \infty$. This is because

$$\frac{\epsilon(f)}{e} = \lim_{p \to \infty} \frac{1}{p} \left(\sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{\frac{1}{p}} \tag{1.3}$$

Our latter end is to extend (1.3) to BMO^p_φ (see Preliminaries and Theorem 6.1) on spaces of homogeneous type. Also, we like to point out that (1.3) was announced in [1] without proof. Under the light of Remark 1 (see Preliminaries) we should note that if $|B| = \mu(B)$, then our main result coincide with the result of Garnett and Jones [1].

2. Spaces of homogeneous type

Let us begin by recalling the notion of space of homogeneous type.

Definition 2.1. A quasimetric d on a set X is a function $d : X \times X \to [0, \infty)$ with the following properties:

1. $d(x, y) = 0$ if and only if $x = y$.

2. $d(x, y) = d(y, x)$ for all $x, y \in X$.

3. There exists a constant K such that

$$d(x, y) \leq K \left[d(x, z) + d(z, y) \right],$$

   for all $x, y, z \in X$.

A quasimetric defines a topology in which the balls
$B(x,r)=\{y\in X : d(x,y)<r\}$ form a base. These balls may be not open in general; anyway, given a quasimetric d, is easy to construct an equivalent quasimetric d' such that the d'-quasimetric balls are open (the existence of d' has been proved by using topological arguments in [3]). So we can assume that the quasimetric balls are open. A general method of constructing families $\{B(x,\delta)\}$ is in terms of a quasimetric.

Definition 2.2. A space of homogeneous type (X,d,μ) is a set X with a quasimetric d and a Borel measure μ finite on bounded sets such that, for some absolute positive constant A the following doubling property holds

$$\mu\left(B(x,2r)\right)\leq A\mu\left(B(x,r)\right)$$

for all $x\in X$ and $r>0$.

Next, we are ready to give some example of a space of homogeneous type.

Example 1. Let $X\subset R^n$, $X=\{0\}\cup\{x:|x|=1\}$, put in X the euclidean distance and the following measure μ: μ is the usual surface measure on $\{x:|x|=1\}$ and $\mu\left(\{0\}\right)=1$. Then μ is doubling so that (X,d,μ) is a homogeneous space.

Example 2. In R^n, let C_k $(k=1,2,\cdots)$ be the point $(k^k+1/2,0,\cdots,0)$, for $k\geq 2$, let B_k be the ball $B(C_k,1/2)$ and $B_1=B(0,1/2)$. Let
$X=\cup_{k=1}^{\infty}B_k$ with the euclidean distance and the measure μ such that $\mu\left(B_k\right)=2^k$ and on each ball B_k, μ is uniformly distributed.
Claim 1. μ satisfies the doubling condition. Let $B_r=B(P,r)$ with

$P = (P_1, \dots, P_n)$ *and* $r > 0$.

Case 1. Assume for some k, $B_k \subset B_r$ and let $k_0 = \max\{k : B_k \subset B_r\}$. Then certainly $P_1 + r \le b_{k_0+1} = (k_0+1)^{k_0+1} + 1$ and $\mu(B_r) \ge 2^{k_0}$. But, then

$$\begin{aligned} P_1 + 2r &\le 2\left((k_0+1)^{k_0+1} + 1\right) \\ &\le (k_0+2)^{k_0+2} = a_{k_0+2}. \end{aligned}$$

Therefore $B_{2r} \subset B_{a_{k_0+2}}(0) \equiv B_0$. But

$$\mu(B_0) = \sum_{k=0}^{k_0+1} 2^k \le 2^{k_0+2} \le 4\mu(B_r).$$

Hence the doubling condition holds with $A = 4$.

Case 2. If for all k, $B_k \not\subseteq B_r$, then $r < 1$ so that B_r and B_{2r} intersect only one ball B_k. Then the doubling condition holds.

3. Preliminaries

In this section, we recall the definition of the space of functions of Bounded (φ, p) Mean Oscillation, $BMO_\varphi^{(p)}(X)$, where X is a space of homogeneous type (see [4]). Let φ be a nonnegative function on $[0, \infty)$. A locally μ-integrable function $f : X \to R$ is said to belong to the class $BMO_\varphi^{(p)}(X)$, $1 \le p < \infty$, if

$$\sup\left(\frac{1}{\mu(B)\,[\varphi(\mu(B))]^p}\int_B |f(x) - f_B|^p\, d\mu(x)\right)^{\frac{1}{p}} < \infty,$$

where the sup is taken over all balls $B \subset X$, and

$$f_B = \frac{1}{\mu(B)}\int_B f(y) d\mu.$$

Remark 1. *It is not hard to check that the expression*

$$(3.1)\|f\|_{BMO^p_\varphi} = \sup_B \left(\frac{1}{\mu(B)\left[\varphi\left(\mu(B)\right)\right]^p} \int_B |f(x) - f_B|^p \, d\mu(x) \right)^{\frac{1}{p}} < \infty,$$

define a norm on $BMO^{(p)}_\varphi(X)$*. For* $\varphi \equiv 1$ *and* $p = 1$*,* $\|\cdot\|_{BMO^p_\varphi}$ *coincide with* $\|\cdot\|_{BMO}$*.*

4. John-Nirenberg inequality on homogeneous type space

The proof of this theorem follows along the same lines as the proof of [4].

Theorem 4.1. There exist two positive constants β and b such that for any $f \in BMO_\varphi(X)$ and any ball $B \subset X$, one has

$$(4.1) \qquad \mu\left(\{x \in S : |f - f_B| > \lambda\}\right) \leq \beta \exp\left\{-b\lambda/\|f\|_{BMO_\varphi}\right\} \mu(B).$$

Proof. We follows the standard stopping time argument; that is, we assume that λ is large enough and fix some λ_1. Then we study the sets $\{x \in S : |f(x) - f_S| \leq \lambda_1\}$, $\{x \in S : |f(x) - f_S| \leq 2\lambda_1\}$ up to

$$\{x \in S : |f(x) - f_S| \leq m\lambda_1 \sim \lambda\}$$

in showing (4.1), we assume $\|f\|_\varphi = 1$ and fix $S = B(a, R)$. We define a maximal operator associated to S (if we replace S by another ball, then the maximal operator changes)

$$M_S f(x) = \sup_{B \text{ ball } , x \in B, B \subset B(a, \alpha R)} \left\{ \frac{1}{\varphi\left(\mu(B)\right)\mu(B)} \int_B |f(y) - f_S| \, d\mu(y) \right\}.$$

Using a Vitali-type covering lemma, one can prove that

$$\mu\left(\{x : M_S f(x) > t\}\right) \leq \frac{A}{t}\mu(S),$$

where A is a constant that depends only on K and k_2 but not on S. Take $\lambda_0 > A$ and consider the open set $U = \{x : M_S f(x) > \lambda_0\}$. We have

$$\mu(U \cap S) \leq \frac{A}{\lambda_0}\mu(S) < \mu(S),$$

and therefore $S \cap U^c \neq \emptyset$. Define

$$r(x) = \frac{1}{5K} dist\,(x, U^c)\,.$$

If $x, y \in S$, then $d(x,y) \leq 2KR$. Since $S \cap U^c \neq \emptyset$, if $x \in S$, we have $r(x) \leq 2KR/(5K) = 2R/5$.

Clearly,

$$U \cap S \subset \bigcup_{x \in U \cap S} B(x, r(x)) \subset U.$$

Again by a Vitali-type covering lemma (e. g, see [1, Theorem 3.1]), we can select a finite or countable sequence of disjoint balls $\{B(x_j, r_j)\}$ such that $r_j = r_j(x)$ and

$$U \cap S \subset \bigcup_j B(x_j, 4Kr_j) \subset U.$$

On the other hand, $B(x_j, 6Kr_j) \cap U^c \neq \emptyset$ and $B(x_j, 6Kr_j) \subset B(a, \alpha R)$ because $6kr_j \leq 12KR/5$. Thus, we get

$$\frac{1}{\varphi(\mu(B(x_j, 6Kr_j)))\,\mu(B(x_j, 6Kr_j))} \int_{B(x_j, 6Kr_j)} |f - f_S|\, d\mu \leq \lambda_0,$$

and consequently, if we write $S_j = B(x_j, 4Kr_j)$, we obtain

$$\begin{aligned} \left|f_S - f_{S_j}\right| &\leq \frac{1}{\mu(S_j)} \int_{S_j} |f - f_S|\, d\mu \\ &\leq \frac{\varphi(S_j)\, k_2^2}{\mu(B(x_j, Kr_j))} \lambda_0 := \lambda_1 \end{aligned}$$

because μ is a doubling measure.

By differentiation theorem, $|f(x) - f_S| \leq \lambda_0$ for μ-a.e. $x \in S \setminus \cup_j S_j$. Moreover,

$$\begin{aligned}
\sum \mu(S_j) &\leq k_2 \sum_j \mu(B(x_j, 2Kr_j)) \\
&\leq C \sum_j \mu(B(x_j, r_j)) \\
&\leq C\mu(U) \\
&\leq \frac{CA}{\lambda_0}\mu(S).
\end{aligned}$$

Now, we do the same construction for each S_j. Again $|f(x) - f_S| \leq \lambda_0$ for μ-a.e. $x \in S_j \setminus \cup_i S_i^{(2)}$ and therefore for these points

$$\begin{aligned}
|f(x) - f_S| &\leq \left|f(x) - f_{S_j}\right| + \left|f_{S_j} - f_S\right| \\
&\leq \lambda_0 + \frac{\varphi(S_j)\, k_2^2}{\mu(B(x_j, Kr_j))}\lambda_0 \\
&\leq \frac{2\varphi(S_j)\, k_2^2}{\mu(B(x_j, Kr_j))}\lambda_0,
\end{aligned}$$

taking $\lambda_0 = 2CA$, it is clear that

$$\begin{aligned}
\mu\left(\bigcup_k S_k^{(2)}\right) &\leq \sum_j \frac{CA}{\lambda_0}\mu(S_j) \\
&\leq \left(\frac{CA}{\lambda_0}\right)^2 \mu(S) = 2^{-2}\mu(S).
\end{aligned}$$

Continuing in this maner we get $N = 1, 2, \cdots$ a family of ball $\left\{S_j^N\right\}$ such that

$$\mu\left(\bigcup S_j^N\right) \leq 2^{-N}\mu(S),$$

finally

$$\begin{aligned}
\mu(\{x \in S : |f(x) - f_S| > \lambda\}) &\leq \mu(\{x \in S : |f(x) - f_S| > N\lambda_1\}) \\
&\leq \mu\left(\bigcup S_j^N\right) \leq 2^{-N}\mu(S) = e^{-b\lambda}\mu(S).
\end{aligned}$$

This complete the proof. □

5. Completeness

In this section we state some simple lemmas. The first one is showed by elementary calculations.

Lemma 5.1. *Let B_0 and B_1 be two balls such that $B_0 \subset B_1$ and $f \in BMO_\varphi$. Then there exists a constant C depending on B_0 and B_1 such that*

$$|f_{B_0} - f_{B_1}| \leq C\|f\|_{BMO_\varphi}.$$

Proof. Indeed,

$$\begin{aligned} |f_{B_0} - f_{B_1}| &= \left|\frac{1}{\mu(B_0)} \int_{B_0} (f(y) - f_{B_1})\, d\mu(y)\right| \\ &\leq \frac{1}{\mu(B_0)} \int_{B_1} |f(y) - f_{B_1}|\, d\mu(y) \\ &= \frac{\mu(B_1)}{\mu(B_0)} \frac{\varphi(\mu(B))}{\varphi(\mu(B))\,\mu(B_1)} \int_{B_1} |f(y) - f_{B_1}|\, d\mu(y) \\ &\leq \frac{\mu(B_1)\,\varphi(\mu(B))}{\mu(B_0)} \|f\|_{BMO_\varphi}. \end{aligned}$$

This complete the proof of Lemma 5.1. □

Lemma 5.2 (John-Nirenberg type). *Let $f \in BMO_\varphi^{(p)}(X)$, $1 \leq p < \infty$, then there exists a constant C_p such that*

$$\|f\|_{BMO_\varphi} \leq \|f\|_{BMO_\varphi^{(p)}} \leq C_p\|f\|_{BMO_\varphi}.$$

Proof. By Hölder's inequality we have

$$\frac{1}{\varphi(\mu(B))\,\mu(B)} \int_B |f(y) - f_B|\, d\mu(y) \leq \sup_B \left(\tfrac{1}{[\varphi(\mu(B))]^p \mu(B)} \int_B |f(y) - f_B|^p\, d\mu(y)\right)^{\frac{1}{p}}$$

for any ball, thus

$$\|f\|_{BMO_\varphi} \leq \|f\|_{BMO_\varphi^{(p)}}.$$

On the other hand

$$\int_B |f(y) - f_B|^p \, d\mu(y) \leq \int_0^\infty p\lambda^{p-1} \mu\left(\{x \in B : |f(x) - f_B| > \lambda\}\right) d\lambda.$$

By Theorem 4.1, we obtain

$$\int_B |f(y) - f_B|^p \, d\mu(y) \leq \int_0^\infty p\lambda^{p-1} \exp\left(-b\lambda/\|f\|_{BMO_\varphi}\right) \mu(B) d\lambda.$$

Therefore

$$\frac{1}{\left[\varphi\left(\mu(B)\right)\right]^p \mu(B)} \int_B |f(y) - f_B|^p \, d\mu(y) \leq p\Gamma(p) C \|f\|_{BMO_\varphi}$$

and thus

$$\|f\|_{BMO_\varphi^{(p)}} \leq C_p \|f\|_{BMO_\varphi}.$$

The Lemma is proved. □

Theorem 5.1. $BMO_\varphi^{(p)}$ equipped with the norm (3.1) is a Banach space.

Proof. We just need to prove that $BMO_\varphi^{(p)}$ is complete. To this end, let us take B_1 to be the unit ball centered at the origin. Let $f_k \in BMO_\varphi^{(p)}$, for each $k = 1, 2, 3, \cdots$, such that

$$\sum_{k=1}^{\infty} \|f_k\|_{BMO_\varphi^{(p)}} < \infty,$$

and assume that

$$\int_{B_1} f_k(y) d\mu(y) = 0. \tag{5.1}$$

Let B be any ball in X and let B_2 be a ball that contains both B_1 and B, then

$$\sum_{k=1}^{\infty} \left(\frac{1}{\mu(B)} \int_B |f_k(y)|^p \, d\mu(y) \right)^{\frac{1}{p}} = \left(\tfrac{\mu(B_2)}{\mu(B)} \right)^{\frac{1}{p}} \textstyle\sum_{k=1}^{\infty} \left(\tfrac{1}{\mu(B_2)} \int_{B_2} |f_k(y)|^p \, d\mu(y) \right)^{\frac{1}{p}}.$$

By Minkoswki's inequality and by (5.1), we have

$$\sum_{k=1}^{\infty}\left(\frac{1}{\mu(B)}\int_B |f_k(y)|^p\, d\mu(y)\right)^{\frac{1}{p}}\left(\frac{[\varphi(\mu(B_2))]^p\,\mu(B_2)}{\mu(B)}\right)^{\frac{1}{p}}$$

$$\leq \sum_{k=1}^{\infty}\left(\frac{1}{[\varphi(\mu(B_2))]^p\,\mu(B_2)}\int_{B_2} |f_k(y)-f_{B_2}|^p\, d\mu(y)\right)^{\frac{1}{p}} +$$

$$+\left(\frac{\mu(B_2)}{\mu(B)}\right)^{\frac{1}{p}}\sum_{k=1}^{\infty}\left(\frac{1}{\mu(B_2)}\int_{B_2}\left|(f_k)_{B_2}-(f_k)_{B_1}\right|^p d\mu(y)\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{[\varphi(\mu(B_2))]^p\,\mu(B_2)}{\mu(B)}\right)^{\frac{1}{p}}\sum_{k=1}^{\infty}\left[\|f_k\|_{BMO_\varphi^{(p)}}+\left(\frac{\mu(B_2)}{\mu(B)}\right)^{\frac{1}{p}}\left|(f_k)_{B_2}-(f_k)_{B_1}\right|\right].$$

By Lemma 5.1, we have

$$\sum_{k=1}^{\infty}\left(\frac{1}{\mu(B)}\int_B |f_k(y)|^p\, d\mu(y)\right)^{\frac{1}{p}}$$

$$\leq \left(\tfrac{\mu(B_2)}{\mu(B)}\right)^{\frac{1}{p}}\textstyle\sum_{k=1}^{\infty}\left[\|f_k\|_{BMO_\varphi}+[\varphi(\mu(B_2))]^p\,\|f_k\|_{BMO_\varphi^{(p)}}\right].$$

By Lemma 5.2 is easy to see that

$$\sum_{k=1}^{\infty}\left(\frac{1}{\mu(B)}\int_B |f_k(y)|^p\, d\mu(y)\right)^{\frac{1}{p}}$$

$$\leq \left(\tfrac{\mu(B_2)}{\mu(B)}\right)^{\frac{1}{p}}\textstyle\sum_{k=1}^{\infty}\left(1+[\varphi(\mu(B_2))]^p\right)\|f_k\|_{BMO_\varphi^{(p)}}.$$

Therefore $\sum_{k=1}^{\infty}\left(\frac{1}{\mu(B)}\int_B |f_k(y)|^p\, d\mu(y)\right)^{\frac{1}{p}} \leq \infty$. This means

$$\left(\frac{1}{\mu(B)}\right)^{\frac{1}{p}}\sum_{k=1}^{\infty}\|f_k\|_{L_p}<\infty, \tag{5.2}$$

and from (5.2), we obtain

$$f=\lim_{m\to\infty}\sum_{k=1}^{m} f_k, \quad \text{a. e.}$$

For $f\in L_p(B)$, clearly $f_B=\sum_{k=1}^{\infty}(f_k)_B$.

Finally, we want to show that:

(a) $f \in BMO_{\varphi}^{(p)}(X)$,

(b) $\|\sum_{k=1}^{m} f_k - f\|_{BMO_{\varphi}^{(p)}} \to 0$ as $m \to 0$.

To this end, observe that

$$\begin{aligned}
&\left(\frac{1}{[\varphi(\mu(B_2))]^p \mu(B)} \int_B |f(y) - f_B|^p \, d\mu(y)\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{[\varphi(\mu(B_2))]^p \mu(B)} \int_B \left|\sum_{k=1}^{\infty} (f_k(y) - (f_k)_B)\right|^p d\mu(y)\right)^{\frac{1}{p}} \\
&\leq \sum_{k=1}^{\infty} \left(\frac{1}{[\varphi(\mu(B_2))]^p \mu(B)} \int_B |f_k(y) - (f_k)_B|^p \, d\mu(y)\right)^{\frac{1}{p}} \\
&\leq \sum_{k=1}^{\infty} \|f_k\|_{BMO_{\varphi}^{(p)}} < \infty,
\end{aligned}$$

thus $\|f\|_{BMO_{\varphi}^{(p)}} < \infty$, then $f \in BMO_{\varphi}^{(p)}(X)$. This proves part (a).

On the other hand,

$$\begin{aligned}
&\left(\frac{1}{[\varphi(\mu(B_2))]^p \mu(B)} \int_B \left|\left(\sum_{k=1}^{\infty} f_k - f\right)(y) - \left(\sum_{k=1}^{m} f_k - f\right)_B\right|^p d\mu(y)\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{[\varphi(\mu(B_2))]^p \mu(B)} \int_B \left|\sum_{k=m+1}^{\infty} (f_k(y) - (f_k)_B)\right|^p d\mu(y)\right)^{\frac{1}{p}} \\
&\leq \sum_{k=m+1}^{\infty} \left(\frac{1}{[\varphi(\mu(B_2))]^p \mu(B)} \int_B |f_k(y) - (f_k)_B|^p \, d\mu(y)\right)^{\frac{1}{p}} \\
&\leq \sum_{k=m+1}^{\infty} \|f_k\|_{BMO_{\varphi}^{(p)}} \to 0, \quad \text{as } m \to \infty.
\end{aligned}$$

Hence $\|\sum_{k=1}^{m} f_k - f\|_{BMO_{\varphi}^{(p)}} \to 0$ as $m \to 0$. This proves part (b). This completes the proof of the Theorem 5.1. □

6. Main Result

Theorem 6.1. Let $f \in BMO_\varphi^{(p)}$, then there is a constant $\epsilon > 0$, such that

$$\sup \mu\left(\{x \in B : |f(x) - f_B| > \lambda\}\right)/\mu(B) \leq e^{-\lambda/\epsilon}, \tag{6.1}$$

where $\lambda > \lambda(\epsilon, f)$. Indeed by Theorem 4.1, we have $\epsilon = C\|f\|_{BMO_\varphi^{(p)}}$ and $\lambda(\epsilon, f) = C\|f\|_{BMO_\varphi^{(p)}}$. Now let

$$\epsilon(f) = \inf\left\{\epsilon : (6.1) \text{ holds }\right\}.$$

Then

$$\frac{\epsilon(f)}{e\varphi\left(\mu(B)\right)} = \lim_{p\to\infty} \frac{1}{p}\|f\|_{BMO_\varphi^{(p)}}.$$

Proof. Since

$$\begin{aligned}
\int_{B(x,r)} |f(x) - f_B|^p \, d\mu(x) &= p\int_0^\infty \lambda^{p-1}\mu\left(x \in B : |f(x) - f_B| > \lambda\right) d\lambda \\
&\leq p\mu(B)\int_0^\infty \lambda^{p-1}e^{-\lambda/\epsilon}d\lambda \\
&= \mu(B)\epsilon^p\int_0^\infty u^{p-1}e^u du.
\end{aligned}$$

Thus

$$\frac{1}{\mu(B)}\int_{B(x,r)} |f(x) - f_B|^p \, d\mu(x) \leq \epsilon^p p\Gamma(p).$$

Next, we obtain

$$\frac{1}{p}\sup\left(\frac{1}{\left[\varphi\left(\mu\left(B\right)\right)\right]^p \mu\left(B\right)}\int_B |f(y) - f_B|^p \, d\mu(y)\right)^{\frac{1}{p}} \leq \frac{\epsilon\left[p\Gamma(p)\right]^{\frac{1}{p}}}{\varphi\left(\mu(B)\right)p}$$

and then,

$$\lim_{p\to\infty}\frac{1}{p}\sup\left(\frac{1}{\left[\varphi\left(\mu\left(B\right)\right)\right]^p \mu\left(B\right)}\int_B |f(y) - f_B|^p \, d\mu(y)\right)^{\frac{1}{p}} \leq \frac{\epsilon(f)}{e\varphi\left(\mu(B)\right)}. \tag{6.2}$$

On the other hand, if $\epsilon < \epsilon(f)$ then there exists $B_0 \subset X$, such that

$$e^{-\lambda/\epsilon} \leq \mu\left(\{x \in B_0 : |f(x) - f_B| > \lambda\}\right)/\mu\left(B_0\right).$$

Thus

$$p\mu\left(B_0\right)\int_0^\infty \lambda^{p-1}e^{\lambda/\epsilon}d\lambda < p\int_0^\infty \lambda^{p-1}\mu\left(x\in B : |f(x)-f_B|>\lambda\right)d\lambda$$

and

$$\frac{\epsilon\left[\Gamma(p)\right]^{\frac{1}{p}}}{\varphi\left(\mu(B)\right)p} < \frac{1}{p}\left(\frac{1}{\left[\varphi\left(\mu\left(B\right)\right)\right]^p\mu\left(B\right)}\int_B |f(y)-f_B|^p\,d\mu(y)\right)^{\frac{1}{p}}.$$

It is follows that

$$(6.3)\quad \frac{\epsilon(f)}{e\varphi\left(\mu(B)\right)} < \lim_{p\to\infty}\frac{1}{p}\sup\left(\frac{1}{\left[\varphi\left(\mu\left(B\right)\right)\right]^p\mu\left(B\right)}\int_B |f(y)-f_B|^p\,d\mu(y)\right)^{\frac{1}{p}}.$$

Combining (6.2) and (6.3), we obtain the desired result. □

Remark 2. Theorem 6.1 together with Lemma 5.2 allow us to estimate the distance from $BMO_\varphi^{(p)}$ to L_∞ in the other words we can estimate

$$\inf_{g\in L_\infty}\|f-g\|_{BMO_\varphi^{(p)}}$$

with $f\in BMO_\varphi^{(p)}$.

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