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CHARACTERIZATION OF LALLEMENT ORDER ON A REGULAR SEMIGROUP

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Abstract

In this paper a study of properties of the Mitsch order relation 'μ' on a regular semigroup and Nambooripads order 'ν' on any arbitrary regular semigroup is made. Mainly a characterization of Lallement order on a regular semigroup is obtained. The necessary and sufficient condition for the restriction of Lallement order 'λ' to $B(S)$ to be usual order on an orthodox semigroup is also obtained.

Key Words : *Nambooripads Order ν, Mitsch Order 'μ', Lallement order λ.*

AMS Subject Classification 2000 : *20 M 18.*

1. Introduction

In this paper a study of properties of the Mitsch Order relation 'μ' on a regular semigroup S, Lallement order 'λ' on a regular semi group and Nambooripads order ν on a regular semigroup of 'S is made. The common property enjoyed by all the three partial orders λ, 'μ' and ν is that the set E (S) is an initial segment (See def. 1.5) of 'S' under each of these partial orders. It is interesting to note that λ is a compatible partial order on a regular semigroup such that $\lambda \cap [E(S)XE(S)]$ is contained in the usual order on E(S), where as Nambooripads partial order 'ν' is not in general compatible but $\nu \cap [E(S)xE(S)]$ is the usual order on E(S). It is obtained in theorem (2.15) that the restriction of Lallement order λ to $E(S)$ is the usual order on $E(S)$. Nambooripad himself proved that on a regular semi group 'ν' is compatible iff 'S' is a locally inverse semigroup (See def. 1.4). A necessary and sufficient condition for restriction of Lallement order λ to $E(S)$ is the usual order on $E(S)$ is also the same i.e., 'S' is a locally inverse semi group is also obtained and in this case both λ and ν are same. It is also observed that in order to show that ν is compatible on 'S' it is enough to show that $(xey, xfy) \in \nu$ whenever $\forall(e, f) \in \nu\forall$ and $\forall x, y \in S^1$. The necessary and sufficient condition for the restriction of lallement order 'λ' to $B(S)$ to be the usual order on $B(S)$ is that $B(S)$ is a normal band. It is also obtained as a corollary that a band B is narmal band if and only if 'λ' is equal to usual partial order on $B(S)$.

First we start with the following preliminaries

Definition 1.0 : Suppose 'S' is a semigroup. An element $a \in S$ is said to be regular if there exists $x \in S$ such that $axa = a$. If every element of 'S' is regular then 'S' is called a regular semigroup.

Definition 1.1 : On a semigroup 'S' define the relation '$\leq$' on 'S' by $a \leq b$ if there exists two idempotents e, f in S^1 (S^1 is the monoid obtained from 'S' by adjoining an identity 1) such that $a = eb = bf$.

Definition 1.2 : Suppose (S,.) is a regular semigroup. Then the Lallement order λ on 'S' is defined by the rule that $a\lambda b$ iff for all x, y in 'S' $x\mathbf{R}xa \Rightarrow xa = xb$ and $y\mathcal{L}ay \Rightarrow ay = by$.

Definition 1.3 : Suppose (S,.) is a regular semigroup and $E(S)$ is the set of all Idempotents of 'S'. For any two elements a, b of S, define a relation ν on 'S' by $a\nu b$ if $R_a \leq R_b$ and (there exist $e \in E \cap Ra)a = eb$. 'ν' is called Nambooripads order on 'S', Ra is the principal right ideal containing a.

Definition 1.4 : A regular semigroup S with set E of Idempotents will be called locally inverse if eSe is an inverse semigroup for every e in E.

Definition 1.5 : Suppose $(X, \leq)$ is a partially order set. Then a subset A of X is called an initial segment of $X if x \in A$ whenever $x \leq a \in A$.

Definition 1.6 : Suppose (S,.) is a semigroup. Then the Mistch order relation "μ" on 'S' is defined for any a, b $\in S, a'\mu$' b, iff there exists $s, t \in S^1$ such that $sa = sb = a = at = bt$. Now we start with the following lemma

Lemma 2.0 : Suppose "μ" is a Mitsch order relation on 'S'.Then the restriction of "μ" to the set of Idempotents of 'S' coincides with the usual order on E(S).

Proof : Let $(e, f) \in' \mu$', where $e, f \in E(S)$, so that $se = sf = e = et = ft$ for some $s, t \in S^1$ and hence $ef = sef = sf2 = sf = e$ so that fe = ef = e therefore $e \leq f$. Conversely if $e, f \in E(s)$ and $e.f = f.e = e$, then by choosing s = t = e, we have $se = sf = e = et = ft$ and hence $(e, f) \in' \mu$', so that the restriction of the order 'μ' to $E(S)$ coincides with the usual order on $E(S)$.

Lemma 2.1 : Suppose 'S' is a semigroup and a me where e is an Idempotent of 'S' then a is also an Idempotent of 'S'. In other words, the set of all Idempotents of 'S' is an Initial segment of 'S' under 'μ'.

Proof : Let $a\mu e$, where $e \in E(S)$, so that there exists $s, t \in S^1$ such that $sa = se = a = at = et$, and now $a^2 = seet = set = sa = a$ and hence a is an Idempotent of 'S'.

Theorem 2.2 : Suppose a is a regular element of a semigroup 'S' and if $(a, b) \in' \mu$' then $a \leq bi.e.a = eb = bf$ for $e, f \in E(S1)$.

Proof : Let $(a, b) \in' \mu$' so that there exists s, $t \in S^1$ such that $sa = sb = a = at = bt$. For any inverse a' of a, we have aa' and a'a are Idempotents of 'S'. Now, $a = aa'a = bta'a$ (as $a = bt$) $= b$ $(ta'a)$. Also $(ta'a)$ $(ta'a) = ta'ata'a = ta'aa'a = ta'a$ so that $ta'a \in E(S^1)$. Similarly $aa's \in E(S^1)$ for s $\in S^1$ and hence $a = aa'a = aa'sb = (aa's)b$. Thus $a = eb = bf$, Where $e = aa's$ and $f = ta'a$ and hence $a \leq b$.

Remark 2.3 : It can be easily observed that on any semigroup'S'; we have as a relation $\leq \subseteq' \mu$'. $\leq$ is not in general a transitive relation. However if a is a regular element of 'S' and $a \leq b$, $b \leq c$ then $a \leq c$.

Corollary 2.4 : Suppose 'S' is a semigroup and a is a regular element of 'S' such that $a = eb = bf$ and $b = gc = ch$ for $e, f, g, h \in E(S1)$ then $a \leq c$.

Proof : We have obviously $\leq \subseteq' \mu$'. Since $(a, b) \in \leq \subseteq' \mu$' and $(b, c) \in \leq \subseteq' \mu$' hence $(a, c) \in' \mu$' as 'μ' is transitive and hence $(a, c) \in \leq$.

Corollary 2.5 : The Mitsch order relation 'μ' on a semigroup 'S' is such that its restriction to the set $E(S)$ of Idempotents of 'S' is the usual order on $E(s)$.

Proof : Proof is obvious.

Remark 2.6 : If '$\leq$' is the binary relation defined on a semigroup 'S' by $a \leq b$ if and only if $a = b$ or $a = eb = bf$ for some Idempotents e, f of 'S' and if the Idempotents form a subsemigroup of 'S' then it can be easily verified that '$\leq$' is a partial order relation on 'S'.

Lemma 2.7 : Suppose 'S' is a semigroup and '$\leq$' on 'S' is defined by $a \leq b$ if either $a = b$ or $a = eb = bf$ for some Idempotents e, f of 'S'. If to each c$\in$ $S and e \in E(S) there exists$g,h $\in E(S)$ such that $ce = gc$ and $fc = ch$, then '$\leq$' is compatible with multiplication.

Proof : If e, f are Idempotents of 'S', then from the given condition there exists an Idempotent g of 'S' such that $ef = ge$ so that $efe = ge2 = ge = ef$ and hence $ef.\ ef = ef2 = ef$. Thus the set of Idempotents of 'S' is a subsemigroup of 'S'. Hence by remark 2.6, '$\leq$' is a partial order relation on 'S'. For $a, b \in S$ with $a \leq b$, then $a = eb = bf$. Since $a = eb = bf$ so that $ac = ebc = bfc$. We have from the given condition $fc = ch$ for some Idempotent h and therefore, $ac = e(bc) = (bc)h$ so that $ac \leq bc$. Similarly it can be shown that $ca \leq cb$.

Remark 2.8 : The above condition is only sufficient but not necessary because of the following example

Example 2.9 : Let $S = a, b, c, d, e, f$ Define . on 'S' as follows : .. a b c d e f a b b e e e b b b b e e e b c f f d d d f d f f d d d f e b b e e e b f f f d d d f In this example, the above condition is not satisfied one can easily verify that '$\leq$' is compatible with multiplication and $b < a, d < c$ where a and c are not regular elements of 'S'. In this example, idempotents form a subsemigroup of 'S'.

Theorem 2.10 : Suppose 'S' is a regular semigroup then Lallement order 'λ' on 'S' is a compatible partial order on 'S'.

Proof : We have $a\lambda b$ if and only if $x\mathbf{R}xa \Rightarrow xa = xb$ and $yLay \Rightarrow ay = by$ for $x, y \in S$; we have for any $a \in S$, x **R** x a $\Rightarrow xa = xa$ and $yLay \Rightarrow ay = ay$ so that $a\lambda a$ for $aya \in S$. Now, let $a\lambda b$ and $b\lambda a(a, b \in S)$, so that $x\mathbf{R}xa \Rightarrow xa = xb$ and $yLay \Rightarrow ay = by$ and $x\mathbf{R}xa \Rightarrow xb = xa$, $y\mathcal{L}byy \Rightarrow by = ay$, taking $x = a'$ (where a' is inverse of a), we have $a'a = a'b$, and by taking $y = b$, we have $ab = bb$ and therefore $a = aa'a = aa'b$ (as $a'a = a'b$) $= aa'bbb$ (where $b$is any inverse of b) $= aa'abb$ (as $a'b = a'a$) $=$ $abb = bbb$ (as $ab = bb$) $= b$. Now, let $a\lambda b$ and $b\lambda c$ for any $a, b, c \in S$. Let $4x\mathbf{R}xa$ and y Lay for $x, yx \in S$, then since $a\lambda b$, we have $xa = xb$ and $ay = by$. Since $b\lambda c$, we have $xb = xc$ (since xa = xb and x **R** xa) and there fore $xa = xc$. We also have $by = cy$ (since ay = by and $yLay$). Now, let $a\lambda b$ for any $c \in S$, We have to show that a $c\lambda bc$ and $ca\lambda cb$. Let $x\mathbf{R}xac$ and $y\mathcal{L}acys!$ o that $xac = xbc$. $y\mathcal{L}acy$ we have $Sy = Sacy$, so that y= zacy for $z \in S$, then $cy = czacy$ so that $Scy = Sczacy \subseteq Sacy \subseteq Scy$. Therefore $Scy = Sacy$ and hence $cy\mathcal{L}acy$ implies that $acy = bcy$ so that $ac\lambda bc$. Now let $x\mathbf{R}xca$ and $y\mathcal{L}cay$, so that $cay = cby$. Since $x\mathbf{R}xca$, we have $x = xcaz$ for $z \in S$, so that $xc = xcazc, impliesthatxcS = xcazcS \subseteq xcaS \subseteq xcS$ and hence $xcS = xcaS$ so that $x\mathbf{R}xca$. Therefore $xc\mathbf{R}xca$ implies $xca = xcb$ and hence $ca\lambda cb$, so that 'λ' is compatible on 'S' under multiplication.

Corollary 2.11 : Suppose 'S' is a regular semigroup and if $(e, f) \in \lambda$ for $e, f \in E(S)$, then $e \leq f$ i.e, $e.f = f.e = e$.

Proof : Let $e\lambda f$ for $e, f \in E(S)$, so that $x\mathbf{R}xe$ implies $xe = xf$ and $xLey$ implies $ey = fy$. Taking $x = y = e$, then $e = ef = fe$.

The following is an example to show that $(e, f) \notin \lambda$, even though $e.f = f.e = e$.

Example 2.12 : Let $S = a, b, c, d$, Define . On 'S' by the composition table as follows : $E(S) = a, b, c, d$, since $b.d = d.b = b$ so that $b \leq d$, but $a = bc \neq dc = d$, and hence $(b, d) \notin \lambda$. .. a b c d a a b a a b a b a b c a b c c d a b d d

Lemma 2.13 : Suppose ale for e $\in E(S)thena \in E(S)$

Proof : It is obvious The following theorem is due to [2]. For the sake of definiteness, we stated the following.

Theorem 2.14 : Suppose S is a regular semigroup with set E of Idempotents and let the relation λ' defined by the rule that $a\lambda' b$ iff
$\forall a' \in V(a)$, $\forall e \in E \cap aa'S$, $\forall f \in ESa'a$, $ea = eb$, $af = bf4$, then the following holds
a. $\lambda \subseteq \lambda'$
b. Suppose $(a, b) \in \lambda'$, then $x\mathbf{R}xa$ implies that $ga = gb$ where g is the sandwich set $S(x'x, aa')$ and $y\mathcal{L}ay$ implies that $ah = bh$ where $h \in S(a'a, yy')$
c. $\lambda = \lambda'$
d. If $\forall S\forall$ is orthodox then $\lambda = (a, b) \in SS : \forall a' \in V(a), \forall b' \in V(b)$,
$a'ea = a'eb$ and $aea' = bea'$.

Theorem 2.15 : Suppose S is an orthodox semigroup then the restriction of Lallement order 'λ' to $B(S)$ is the usual order on $B(S)$ iff $B(S)$ is a normal band.

Proof : Suppose S is an orthodox semigroup and the restriction to $B(S)$ is the usual order on $B(S)$. As 'λ' is compatible partial order and hence the restriction to $B(S)$ is a compatible partial order. So we have A band B(S) is a normal band iff it is compatible w.r.t multiplication, so that $B(S)$ is a normal band. Hence restriction of 'λ' to B(S) is the usual order on $B(S)$. Conversly suppose that the restriction 'λ' to B(S) is a normal band and if for any a, b $\in E(S) witha.b = b.a = a$.

Now we claim that $a\lambda b$ i.e. $aea' = bea'$ and $a'ea = a'eb \forall e \in E$, $a' \in V(a)$. Consider $bea' = bea'aa' = bea'(ab)a'(asab = a) = [b(ea')ab]a' = b(a)(ea')ba'(asabca = acba) = aea'ba'$ $(asb.a = a) = aea'ba'aa' = aea'ba'aa' = aba'ea'aa'$ (by using normality) $= aba'ea' = aa'ea'(asa.b = a) = aa'ea'aa' = aa'ea'aa' = aea'a'aa'$ (by using normality) $= aea'a' = aea'asaea' \in E$. Hence for all $e \in E$, $aea' = bea'$. Now we have to show that $a'ea = a'eb$.

Consider $a'eb = a'aa'eb = a'aba'eb(asab = a) = a'aba'eb = a'baa'eb$ (normality) $= a'ba'eab = a'ba'ea(asa.b = a) = a'ba'eaa'a = a'ba'eaa'a = a'aba'ea'a$ (normality) $= a'aa'ea'a(asa.b = a) = a'ea'e = a'aa'ea'a = a'aa'ea'a = a'eaa'a'a$ (by using normality) $= a'eaa'a = a'ea$. Hence $\lambda = (a, b) \in SS : \forall a' \in V(a), \forall b' \in V(b), a'ea = a'eb$ and $aea' = bea'$ Hence λ restricted to $B(S)$ is the usual order on $B(S)$ (by using theorem 2.14)

Corollary 2.16 : A band B is a normal band iff λ is equal to usual partial order on $B(S)$.

Proof : Suppose B is a normal band, then B is an orthodox and by theorem 2.15 λ restricted to $B(S)$ is the usual partial order on $B(S)$.

Conversely if λ restricted to $B(S)$ is the usual partial order on $B(S)$ and λ is compatible, hence the usual partial order on $B(S)$ is also compatible. Hence $B(S)$ is a normal band.

Theorem 2.17 : Suppose R is a partial order relation on a semigroup 'S'. Then Rb is defined by the rule that a R$^b b$ if $(xay, xby) \in R \forall x, y, \in S^1$, then R^b is the largest compatible relation contained in R.

Proof : We have a $R^b b$ for any $a, b \in S \Rightarrow (xay, xby) \in R \forall x, y \in S^1$.

Reflexive : We have $(a, a) \in R$ imply that $(a, a) \in R^b$ as $a = 1.a.1$ and $b = 1.b.1$. Antisymmetric : Let $aRbb$ and $bRba$, for any $a, b \in S$. Since $aR^b b \Rightarrow (xay, xby) \in R \forall x, y \in S1.Since$b R$^b a \Rightarrow (xby, xay) \in R$. In particular if $x = y = 1$, then $(a, b) \in R(b, a) \in R$ so that $a = b$ as R is antisymmetirc.

Transitive : Let $(a, b) \in Rb$ and $(b, c) \in R^b$ for any $a, b, c \in S$. Since $(a, b) \in R^b$ so that $(xay, xby) \in R$ and $(b, c) \in R^b$ so that
$(xay, xby) \in R \forall x, y \in S^1$. As R is transitive, $(xay, xcy) \in R$. Hence R is a partially order relation on 'S'. For $(a, b) \in R^b$ imply that $(xay, xby) \in R \forall x, y \in S^1$ so that $(a, b) \in R f or x = y = 1$, Hence $R^b \subseteq R$.

Compatibility: Let $(a, b) \in R^b$ so that $(xay, xby) \in R$ for all $x, y \in S^1$ imply that $(xcay, xcby) \in R$ and hence $(ca, cb) \in R^b$ so that left compatibility holds. Similarly for $(a, b) \in R$ imply that $(xay, xby) \in R \forall x, y \in S^1$ so that $(xacy, xbcy) \in R$. Hence $(ac, ca) \in R^b$. Here $R^b$is a compatible relation on 'S' contained in R. Let be any compatible partial order relation on 'S' which is contained in R. For $(a, b) \in r \Rightarrow (a, b) \in R$. Since r in compatible so that $(xay, xby) \in r$ and hence $(xay, xby) \in R$. Hence $(a, b) \in R^b$ so that $r \subseteq R^b$. Hence R^b is the largest compatible partial order relation contained in R.

Theorem 2.18 : Suppose 'S' is a regular semigroup, then $\lambda = \nu^b$ where 'ν' is Nambooripads order on 'S'.

Proof : Let $(a, b) \in nb$ then $(xay, xby) \in \nu$ for all $x, y \in S^1$. Now, for $x = u, y = 1$, $(ua, ub) \in \nu$ and so that $ue = eub$ and $e \in \mathbf{R}_u a$...(1). We have $R_u = R_u a \leq R_u$ so that $R_u = R_u a$ and hence $uS = uaS$. Since $e \in Rua$ so that Rua = Re as Ru = Rua = Re, we have uS = uaS = ubS $\subseteq uS.Sothat$uS = uaS = ubS.$Hence$Rub = Re$implythat$(e, ub) $\in \mathbf{R}$ so that $eub = eb$. But from (1), $eub = ua$ and hence $ua = ub$. Similarly $v\mathcal{L}av \Rightarrow av = bv$. Hence

$(a,b) \in \nu^b$, we have $(a,b) \in \lambda$ so that $nb \subseteq \lambda$...(2). On the other hand, let $(a,b) \in \lambda$, we have $x\mathbf{R}xa \Rightarrow xa = xb$ For $x = a'$, we have $(a', a'a) \in \mathbf{R}$. Now $xa = xb$ so that $a'a = a'b$. Since $y\mathcal{L}ay \Rightarrow ay = by$, Choose $y = a'$, so that $aa' = ba'$. Now, a =aa'a=ba'a$\in bS, so that a \in bS. But for$a $\in bS$ and $a' \in V(a)$. So that $(a,b) \in \nu$. Hence $\lambda \subseteq \nu$. Hence 'λ' is compatible partial order which is contained in ν. But nb is the largest compatible partial order relation which is contained in ν. Hence $\lambda \subseteq \nu^b$...(3). From (2) and (3) we have $\lambda = \nu^b$

Theorem 2.19 : Suppose 'S' is a regular semigroup, then the following conditions are equivalent. (1) $(xey, xfy) \in \nu$ where $(e,f) \in \nu \cup (E(S) \times E(S))$ for all $x, y \in S^1$. (2) 'ν' is compatible with multiplication. (3) 'S' is a locally inverse semigroup. (4) $\lambda = \nu$. (5)Restriction of Lallement order λ to $E(S)$ is the usual order on $E(S)$

Proof : (1) $\Rightarrow$ (2)$Assume (1) holds i.e.$(xey, xfy)$\in \nu$ whenever $(e,f) \in \nu E(S)E(S)$ for all $x, y \in S^1$. Let $(a,b) \in \nu$, by using $[N1]$, for every $f \in E(Rb)$, there exists $e \in Ra$ such that $e.f = f.e = e$ and $a = eb$. By the assumption $(xey, xfy) \in \nu$ for all $x, y \in S^1$ so that $(xeb, xfb) \in \nu$ for all $x, y \in S^1$ so that $(xeb, xfb) \in \nu$ (by taking y = b)$and hence$(xa, xb) $\in \nu$...$(*)$. And also we have $(a,b) \in \nu$, by using proposition [3] for each $f \in E(Lb)$ there exists $e' \in E(La)$ such that $e'.f' = e' = f'e'$ and $a = be'$ since $(e', f') \in \nu$ and $(xe'y, xf'y) \in \nu$ such that $(be'y, bf'y) \in \nu$ (by taking $x = b$) and hence $(ay, by) \in \nu$ (Since $a = be'$ and $f' \in E(Lb)$). Therefore $(ay, by) \in \nu$...$(**)$. From (*) and (**) we have $(xay, xby) \in \nu$ for any $(a,b) \in \nu$ and hence 'ν' is compatible with multiplication.

(2) $\Rightarrow$ (3) : (2) and (3) are equivalent from [3] (3) $\Rightarrow$ (4) Assume (3), since 'S' is a locally inverse semigroup by using (Exercise 6.4; 3 of [2]) $\lambda = nb$ where ν^b is the largest compatible relation on 'S' which is contained in ν and by using [2] $\lambda = \nu$. (4) $\Rightarrow$ (5)$Suppose \lambda = \nu$, since the restriction of $\nu to E(S)$ is the usual order on $E(S)$ and hence the restriction of $\lambda to E(S)$ is also usual order on $E(S)$. (5)$\Rightarrow$ (1). Assume (5), i.e the restriction of to $E(S)$ in the usual order on $E(S)$ and let (e, f) $\in \nu \cap E(S)xE(S)$ so that $e.f = f.e = e$ and hence $e \leq f in \lambda$. Since the restriction of λ to $E(S)$ is the usual order on $E(S)$, we have $(e,f) \in nb$. Hence $(xey, xfy) \in \nu$ for all $x, y \in S^1$. Hence the given conditions on any regular semigroup 'S' are equivalent.

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