THE UNIFORM BOUNDEDNESS PRINCIPLE FOR ARBITRARY LOCALLY CONVEX SPACES

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Abstract

We establish uniform boundedness principle for pointwise bounded families of continuous linear operators between locally convex spaces which require no assumptions such as barrelledness on the domain space of the operators. We give applications of the result to separately continuous bilinear operators between locally convex spaces.

In [LC] Li and Cho established a version of the Banach-Steinhaus Theorem which is valid for arbitrary locally convex spaces. Li and Cho showed that if $\{T_k\}$ is a sequence of continuous linear operators between locally convex spaces such that $\lim T_k x = Tx$ exists for every x, then, although the limit operator T may fail to be continuous with respect to the original topologies of the spaces, T is continuous when the domain space is equipped with the strong topology. Their result implies the version of the Banach-Steihaus Theorem for barrelled spaces since a barrelled space always carries the strong topology ([K]27.1,[S]24.7,[W]9.3.10). In this note we show that a result similar to that of Li and Cho holds for the Uniform Boundedness Principle. Namely, if Γ is a family of continuous linear operators between locally convex spaces which is pointwise bounded, then, although Γ may fail to be equicontinuous with respect to the original topologies of the spaces, Γ is equicontinuous when the domain space is equipped with the strong topology. Again, this result gives the version of the Uniform Boundedness Principle for barrelled spaces since a barrelled space always carries the strong topology. We use this result to establish hypocontinuity results for a single and families of separately continuous bilinear operators which are valid for arbitrary locally convex spaces.

Let (E, τ) and (F, η) be Hausdorff locally convex spaces and let $\sigma(E', E)$ be the weak topology on E' from E. If $A \subset E'$, the polar of A in E is $A_0 = \{x \in E : |\langle x', x \rangle| \leq 1 \text{ for every } x' \in A\}$. The strong topology $\beta(E, E')$ of E is the locally convex topology which has a local base at 0 consisting of the polars A_0 where A runs through the family of all $\sigma(E', E)$ bounded subsets of E' ([K]21.2,[S]17.5.19,[W]8.5). The topology η of F is generated by the polars A_0 where A runs through the family of all equicontinuous subsets of F' ([K]21.3,[S]17.7,[W]9.1). If T is a continuous linear operator from E into F, the adjoint (transpose) operator $T' : F' \rightarrow E'$ is defined by $\langle T'y', x \rangle = \langle y', Tx \rangle$ for $x \in E, y' \in F'$.

We first establish a lemma. Let Γ be a family of continuous linear operators from E into F.

Lemma 1. If Γ is pointwise bounded on E and $A \subset F'$ is an equicontinuous subset, then $\Gamma'A = \{T'y' : T \in \Gamma, y' \in A\}$ is $\sigma(E', E)$ bounded.

Proof : Let $x \in E$. Then $\{Tx : T \in \Gamma\}$ is η bounded in F so $\{|\langle y', Tx \rangle| = |\langle T'y', x \rangle| : T \in \Gamma, y' \in A\}$ is bounded or $\Gamma'A$ is $\sigma(E', E)$ bounded.

Theorem 2. If Γ is pointwise bounded on E, then Γ is $\beta(E, E') - \eta$ equicontinuous.

Proof : Let A be an equicontinuous subset of F' so that A_0 is a basic η neighborhood of 0. If $T \in \Gamma$, then $T^{-1}A_0 = (T'A)_0$ ([W]11.2.1) so

$$(1) \cap_{T \in \Gamma} T^{-1} A_0 = \cap_{T \in \Gamma}(T'A)_0 = (\cup_{T \in \Gamma} T'A)_0 = (\Gamma'A)_0.$$

But, $(\Gamma'A)_0$ is a basic $\beta(E, E')$ neighborhood of 0 by Lemma 1 so (1) gives the result.

Since a barrelled space always carries the strong topology ([K]27.1,[S]24.7,[W]9.3.10), we have the familiar version of the Uniform Boundedness Principle ([B]III.4.2.1,[S]24.11,[W]9.3.4).

Corollary 3.(Bourbaki) Suppose that E is barrelled. If Γ is pointwise bounded on E, then Γ is equicontinuous.

There exist locally convex spaces E such that $(E, \beta(E, E'))$ is not barrelled ([K]31.7,[W]15.4.6) so Theorem 2 gives a proper extension of the "usual" form of the Uniform Boundedness Principle. Theorem 2 immediately gives the result of Li and Cho.

Corollary 4. ([LC]) If $\{T_k\}$ is a sequence of continuous linear operators from E into F such that $\lim T_k x = Tx$ exists for every $x \in E$, then T is $\beta(E, E') - \eta$ continuous.

As noted earlier, Corollary 4 gives the "usual" form of the Banach-Steinhaus Theorem for barrelled spaces.

We now show that Theorem 2 can be used to derive a hypocontinuity result which is valid for arbitrary locally convex spaces.

Let (G, θ) be a Hausdorff locally convex space and let $b : E \times F \to G$ be a separately continuous bilinear operator. Let $\mathcal{B}_F$ be the family of all bounded subsets of F. Then b is $(\tau, \mathcal{B}_F)$ hypocontinuous if for every neighborhood of 0,W, in G and every bounded set $B \in \mathcal{B}_F$, there exists a τ neighborhood of 0,U, in E such that $b(U, B) \subset W$.

Theorem 5. The bilinear operator b is $(\beta(E, E'), \mathcal{B}_F)$ hypocontinuous.

Proof : Let W be a neighborhood of 0 in G and let $B \in \mathcal{B}_F$. Consider $\Gamma = \{b(\cdot, y) : y \in B\}$, a family of continuous linear operators from E into G. We claim that for every $x \in E$, Γx is bounded in G. For this, let $y_k \in B$ and $t_k \to 0$. Then $b(x, t_k y_k) = t_k b(x, y_k) \to 0$ since b is separately continuous and B is bounded so $t_k y_k \to 0$. Thus, Γx is bounded.

From Theorem 2, Γ is $(\beta(E, E'), \theta)$ equicontinuous. Therefore, there exists a $\beta(E, E')$ neighborhood of $0, U,$ in E such that $b(x, y) \in W$ for every $x \in U, y \in B$, i.e., b is $(\beta(E, E'), \mathcal{B}_F)$ hypocontinuous.

In general, a separately continuous bilinear operator may fail to be hypocontinuous with respect to the original topology of the space.

Example 6. Let $E = F = c_{oo}$ be the space of real sequences which are eventually equal to 0 equipped with the sup-norm topology $\|\|_{\infty}$. Define $b : E \times F \to \mathbf{R}$ by $b(\{x_j\}, \{y_j\}) = \sum_{j=1}^{\infty} x_j y_j$. Then b is separately continuous. Let e^j be the sequence with 1 in the jth coordinate and 0 in the other coordinates. Then $x^k = \frac{1}{k} \sum_{j=1}^{k} e^j \to 0$ and $y^k = \sum_{j=1}^{k} e^j$ is such that $\{y^k\}$ is bounded but $b(x^k, y^k) = 1$ so b is not $(\|\|_{\infty}, \mathcal{B}_{c_{00}})$ hypocontinuous.

Theorem 5 gives the "usual" form of the hypocontinuity result for barrelled spaces ([B]III.5.3.6,[K2]40.2.(3).a),[S]24.14).

Corollary 7. Suppose that E is barrelled. Then b is $(\tau, \mathcal{B}_F)$ hypocontinuous.

We can also obtain a general continuity result for metrizable spaces from Theorem 5.

Corollary 8. If E and F are metrizable, then b is $\beta(E, E') \times \eta$ continuous.

Proof : Let $(x_k, y_k) \to 0$ in $\beta(E, E') \times \eta$. By Theorem 5 $\lim_k b(x_k, y_j) = 0$ uniformly for $j \in \mathbf{N}$. In particular, $\lim_k b(x_k, y_k) = 0$ so b is continuous.

Corollary 8 gives the result in [K2]40.2.(1).a) as a corollary, where it is assumed that both E and F are barrelled so the operator b is continuous on $E \times F$.

It is interesting to compare the proofs of Theorems 2 and 5 with standard proofs of the corresponding results for barrelled spaces ([S]24.11 and 24.14,[W]9.3.4). In these proofs the definition of a barrel and the definition of barrelled space are employed whereas the proofs above use the strong topology directly.

Using the methods above, we can also treat families of separately continuous bilinear operators from $E \times F$ into G. Let Λ be a family of separately continuous bilinear operators from $E \times F$ into G. The family Λ is right (left) equicontinuous if for every $x \in E$ $(y \in F)$, the family $\{b(x, \cdot) : b \in \Lambda\}$

($\{b(\cdot, y) : b \in \Lambda\}$) is equicontinuous; Λ is separately equicontinuous if Λ is both right and left equicontinuous. The family Λ is $(\tau, \mathcal{B}_F)$ equihypocontinuous if for every neighborhood of 0, W, in G and $B \in \mathcal{B}_F$, there is a τ neighborhood of 0,U, in E such that $\Lambda(U, B) \subset W$.

We have a result analogous to Theorem 5 for families of bilinear operators.

Theorem 9. If Λ is right equicontinuous, then Λ is $(\beta(E, E'), \mathcal{B}_F)$ equihypocontinuous.

The proof follows from the proof of Theorem 5 where right equicontinuity is used in place of separate continuity.

Theorem 9 gives the result in [K2]40.2.(3).b) where it is assumed that F is barrelled. We can also obtain a generalization of another result of Köthe from Theorem 9.

Corollary 10. Assume that E and F are metrizable. If Λ is pointwise bounded on $E \times F$ and Λ is right equicontinuous, then Λ is $\beta(E, E') \times \eta$ equicontinuous.

Proof : Let $(x_k, y_k) \to 0$ in $\beta(E, E') \times \eta$. Then $\{y_k\}$ is η bounded and $x_k \to 0$ in $\beta(E, E')$ so by Theorem 9 $\lim_k b(x_k, y_j) = 0$ uniformly for $b \in \Lambda, j \in \mathbf{N}$. In particular, $\lim_k b(x_k, y_k) = 0$ uniformly for $b \in \Lambda$. Now apply [K]15.14.(4).

Corollary 10 gives the result in [K2]40.2.(1).b) where it is assumed that both E and F are barrelled so the pointwise boundedness assumption yields the right (left) equicontinuity condition by the version of the Uniform Boundedness Principle for barrelled spaces.

Finally, we can obtain a uniform boundedness result for bilinear operators.

Corollary 11. If Λ is right equicontinuous and pointwise bounded on $E \times F$, then Λ is uniformly bounded on sets of the form $A \times B$, where A is $\beta(E, E')$ bounded and B is η bounded.

Proof : Let $b_k \in \Lambda, x_k \in A, y_k \in B$ and $t_k \to 0$. Then $t_k b(x_k, y_k) = b(t_k x_k, y_k) \to 0$ by Theorem 9 since $t_k x_k \to 0$ in $\beta(E, E')$ and $\{y_k\}$ is η bounded.

In particular, it follows from Corollary 11 that if both E and F are barrelled, then a pointwise bounded family Λ is uniformly bounded on products of bounded subsets in $E \times F$.

In general, the strong topology cannot be replaced by the original topology in either Theorem 9 or Corollary 11.

Example 12. Define $b_i : c_{00} \times l^\infty \to \mathbf{R}$ by $b_i(x,y) = \sum_{j=1}^i x_j y_j$, where both c_{00} and l^∞ have the sup-norm. Then each b_i is separately continuous. Morover, if $x \in c_{00}$ has 0 coordinates after the nth entry, then for
$i \geq n$ $|b_i(x,y)| \leq \|y\|_\infty \sum_{j=1}^n |x_j|$ so that $\{b_i\}$ is right equicontinuous and pointwise bounded. However, if $x^k = \sum_{j=1}^k e^j$, then $A = \{\ x^k : k\}$ is bounded, $x^k/k \to 0$ and $b_i(x^k/k, x^k) = 1$ and $b_i(x^k, x^k) = k$ for $i \geq k$ so the conclusions of Theorem 9 and Corollary 11 both fail for the sup-norm topology.

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