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ON THE ALGEBRAIC DIMENSION OF BANACH SPACES OVER NON-ARCHIMEDEAN VALUED FIELDS OF ARBITRARY RANK

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Abstract

Let K be a complete non-archimedean valued field of any rank, and let E be a K-Banach space with a countable topological base. We determine the algebraic dimension of E (2.3, 2.4, 3.1).

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Introduction

It is a well-known fact that the (algebraic) dimension of the Hilbert space l^2 is the power of the continuum. (See the Appendix for an elegant and general proof, kindly pointed out to us by A. van Rooij).
Now consider a Banach space E with a topological base $e_1, e_2, \ldots$ over a field K of any cardinality, with a non-archimedean valuation of arbitrary rank. To compute the dimension of E a new approach is needed. In fact we extend Köthe's method ([2] Ch.2, Sec.9.5) used to determine the dimension of the algebraic dual of a vector space.

The results are striking. They depend strongly on whether or not K is metrizable (see 2.4 and 3.1). It is also noteworthy that, if K is non-metrizable the dimension of E turns out to be so small and independent of the cardinality of K!

1. Preliminaries

We will use notations and terminology from [3], but for convenience we quote a few basics here.

Throughout K is a field. Let G be a totally ordered multiplicatively written abelian group with unit 1, augmented with an element 0 satisfying $0 < g,\ 0 \cdot g = g \cdot 0 = 0 \cdot 0 = 0$ for all $g \in G$. (We point out that G is not necessarily a subgroup of the positive real numbers).

A *valuation* on K with *value group* G is a surjective map $|\ | :\ K \to G \cup \{0\}$ satisfying

(i) $|\lambda| = 0$ if and only if $\lambda = 0$

(ii) $|\lambda\mu| = |\lambda|\,|\mu|$

(iii) $|\lambda + \mu| \leq \max(|\lambda|,\ |\mu|)$

for all $\lambda,\ \mu \in K$.

The valuation is called *trivial* if $G = \{1\}$. The *balls* $B(\alpha, g) := \{\lambda \in K : |\lambda - \alpha| < g\}$ where $\alpha \in K,\ g \in G$, induce a topology on K making it into a topological field; we assume the valued field $(K, |\ |)$ to be equipped with this topology. One introduces the notion of a Cauchy net in K in a natural way. $(K, |\ |)$ is called *complete* if each Cauchy net converges. We will need the following criterion on metrizability of K.

Proposition 1.1. ([3] 1.4.1) $(K, |\ |)$ *is metrizable if and only if* G *has a coinitial sequence.*

A linearly ordered set X without smallest element is called a *G-module* if there exists an action $(g, x) \mapsto gx$ of G on X that is increasing in both variables and such that Gx is coinitial in X for all $x \in X$.

Let E be a vector space over a valued field $(K, |\ |)$, and let X be a G-module, augmented with and element 0_X satisfying $0_X < x,\ 0 \cdot x = 0 \cdot 0_X = 0_X = g \cdot 0_X$ for all $x \in X,\ g \in G$. For simplicity we will write 0 instead of 0_X.

An *X-norm* is a map $\|\ \| :\ E \to X \cup \{0\}$ satisfying

(i) $\|x\| = 0$ if and only if $x = 0$

(ii) $\|\lambda x\| = |\lambda|\, \|x\|$

(iii) $\|x + y\| \leq \max(\|x\|,\ \|y\|)$

for all $x, y \in E,\ \lambda \in K$.

The space $(E, \|\ \|)$ is as usual called a *normed space* (more precisely, an *X-normed space*). Notice that the subset $\|E\| \backslash \{0\} := \{\|x\| :\ x \in E,\ x \neq 0\}$ of X is a G-module in its own right. The X-norm induces naturally a topology on E and the notion of a Cauchy net. $(E, \|\ \|)$ is called a *Banach space* if E and K are complete. It is easily seen that if $(K, |\ |)$ is metrizable then so is $(E, \|\ \|)$.

Let $(E, \|\ \|)$ be a Banach space over K. A system $\{e_1, e_2, \ldots\} \subset E \backslash \{0\}$ is called a *topological base of E* if each $x \in E$ has a unique expansion as a convergent sum

$$x = \sum_{n=1}^{\infty} \lambda_n e_n. \qquad (\lambda_n \in K)$$

If, in addition

$$\|x\| = \max_n \|\lambda_n e_n\|$$

it is called an *orthogonal base*.

Since in this paper we are concerned with the (algebraic) dimension of E, we recall that this is the cardinality of an algebraic base of E (in the sense that each $x \in E$ can uniquely be represented as a *finite* linear combination of its elements).

2. The main result

We prove first a well-known general fact about infinite dimensional vector spaces.

Lemma 2.1. *Let E be a vector space of infinite dimension d over a field K. Let κ be the cardinality (finite or not) of K, and let ε be the cardinality of E. Then $\varepsilon = d\kappa$.*

Proof. Let δ be an ordinal with cardinality d and let $\{e_\nu : \ \nu \in \delta\}$ be an (algebraic) base of E.

For every $n \geq 1$ the cardinality of the set of elements of the form $\alpha_1 e_{\nu_1} + \alpha_2 e_{\nu_2} + \ldots \alpha_n e_{\nu_n}$ with $\alpha_i \neq 0$ $(i = 1, \ldots, n)$ is equal to $((\kappa - 1)d)^n$. Therefore

$$\varepsilon = \sum_{n=0}^{\infty} ((\kappa - 1)d)^n = d(\sum_{n=0}^{\infty} (\kappa - 1)^n),$$

since $d^n = d$.

If $\kappa < \aleph_0$ then $\sum_{n=0}^{\infty} (\kappa - 1)^n = \aleph_0$ and $\varepsilon = \ d\, \aleph_0 = d = d\kappa$.

If $\kappa \geq \aleph_0$ then $\sum_{n=0}^{\infty} (\kappa - 1)^n = \kappa$ and $\varepsilon = \ d\kappa$.

From now on K shall be an infinite field. As customary, we will often use the aleph notation for infinite cardinalities.

Lemma 2.2. *Let K be a valued field with value group $G \neq \{1\}$. Let X be a G-module, let K_0 be a subfield of K, $K_0 \neq K$. Then, for each $s, \ t \in X$ there exists a $\lambda \in K \backslash K_0$ such that $|\lambda| s < t$.*

Proof. Let $G_0 := |K_0^*|$.

(i) Suppose $G_0 s$ is coinitial in X. Then choose $\mu \in K \backslash K_0$. There is a $\lambda_0 \in K_0^*$ such that $|\lambda_0| s < |\mu|^{-1} t$, i.e. $|\lambda_0 \mu| s < t$. Choose $\lambda := \lambda_0 \mu$. Clearly, $\lambda \notin K_0$ (otherwise, $\mu = \lambda \lambda_0^{-1} \in K_0$, a contradiction).

(ii) Suppose $G_0 s$ is not coinitial in X. Then there is a $v \in X$ such that $g_0 s \geq v$ for all $g_0 \in G_0$. In this case, choose $\lambda \in K^*$ such that $|\lambda| s < t$ and $|\lambda| s < v$. Then $|\lambda| \notin G_0$, so $\lambda \notin K_0$.

Let E be an infinite-dimensional Banach space with a topological base $e_1, e_2, \ldots$ over a nontrivially valued field K. We assume K to be metrizable. We can identify E with a subspace of K^N as follows.

$$E = \{(\xi_1, \xi_2, \ldots) \in K^N : |\xi_n| \|e_n\| \to 0\}$$

(where the $\|e_n\|$ are in the G-module $\|E\| \backslash \{0\}$).

We want to prove the following.

Theorem 2.3. *Let E be an infinite-dimensional Banach space with topological base $e_1, e_2, \ldots$ over a metrizable K. Then the dimension of E is equal to its cardinality.*

Proof. Let $\aleph_\kappa$ be the cardinality of K, let $\aleph_\varepsilon$ be the cardinality of E, let d be the dimension of E. It is enough to prove that $d \geq \aleph_\kappa$, since by 2.1 $\aleph_\varepsilon = d\, \aleph_\kappa$. Therefore we shall assume, by contradiction, that $d < \aleph_\kappa$. Let δ be an ordinal with cardinality d and $\{f_\nu : \nu \in \delta\}$ an algebraic base of E.

For every $\nu \in \delta$ we write $f_\nu = \sum_{i=0}^{\infty} \alpha_i^\nu e_i$, and let $M := \{\alpha_i^\nu : \nu \in \delta,\ i \in N\}$. Therefore the cardinality of M is less than or equal to $d\,\aleph_0$. We also fix a sequence $t_1, t_2, \ldots$ in $\|E\| \backslash \{0\}$ such that $t_n \to 0$; we will use it to construct a chain $K_0 \subset K_1 \subset K_2, \ldots$ of subfields of K. In fact, let K_0 be the subfield of K generated by M. Then, observing that the cardinality of the prime field of K is at most $\aleph_0$, we conclude that the cardinality of K_0 is at most $d\, \aleph_0 = d$. Since by assumption $d < \aleph_\kappa$, there exists a $\xi_1 \in K \backslash K_0$, and by 2.2 we can assume that $|\xi_1| \|e_1\| < t_1$. We define $K_1 := K_0(\xi_1)$; once again K_1 has no more than $d\, \aleph_0 = d$ elements, and we can pick $\xi_2 \in K \backslash K_1$ such that $|\xi_2| \|e_2\| < t_2$. Recursively we obtain a sequence $K_0 \subset K_1 \subset K_2 \subset \ldots$ of subfields, where $K_n = K_{n-1}(\xi_n)$ and $|\xi_n| \|e_n\| < t_n$ for all n.

We define the vector $\xi := (\xi_n)_{n \in N}$, note that ξ belongs to E by construction. Write ξ as a finite linear combination of the vectors in the algebraic base

$$\xi = \sum_{j=1}^{n} \eta_j f_{\nu_j}. \qquad (*)$$

Let $K_\infty = \bigcup_n K_n$, and consider K as a K_∞- vector space. Then K_∞, as a subspace of K, has a complement W. Let $\varphi : K \to K_\infty$ be the K_∞-linear map such that $\varphi|_{K_\infty}$ is the identity map and $\varphi|_W = 0$.

Then $\psi : K^N \to (K_\infty)^N$ defined by the formula
$\psi(\alpha_1, \alpha_2, \ldots) = (\varphi(\alpha_1), \varphi(\alpha_2), \ldots)$ is a K_∞-linear map that is the identity on $(K_\infty)^N$. Note that, since $M \subset K_0 \subset K_\infty$, the vectors ξ as well as f_ν, for any $\nu \in \delta$, belong to $(K_\infty)^N$. Now $\eta_j f_{\nu_j} = \eta_j(\alpha_i^{\nu_j})_{i\in N} = (\eta_j \alpha_i^{\nu_j})_{i\in N}$, and therefore $\psi(\eta_j f_{\nu_j}) = (\varphi(\eta_j)\alpha_i^{\nu_j})_{i\in N}$.
Then it follows from $(*)$ that

$$\psi(\xi) = \xi = \sum_{j=1}^{n} \eta_j f_{\nu_j} = \sum_{j=1}^{n} \varphi(\eta_j) f_{\nu_j}$$

and, by linear independence of the set of base vectors f_ν, we obtain that $\eta_j = \varphi(\eta_j)$, hence $\eta_j \in K_\infty$. But then, there exists an $m \in N$ such that all of the $\eta_1, \eta_2, \ldots, \eta_n$ belong to K_m. Then all coordinates of $\sum_{j=1}^{n} \eta_j f_{\nu_j}$ lie in K_m. Therefore for all $i \in N$ we have that $\xi_i \in K_m$, and this contradiction shows that $d \geq \aleph_\kappa$, which finishes the proof.

We can even say more.

Theorem 2.4. *Let E be an infinite-dimensional Banach space with topological base $e_1, e_2, \ldots$ over a metrizable K. Let $\aleph_\kappa$ be the cardinality of K, let d be the dimension of E. Then $d = \aleph_\kappa^{\aleph_0}$.*

Proof. By 1.1 there exist $\alpha_1, \alpha_2, \ldots \in K^*$ such that $|\alpha_n| \to 0$. Choose $s \in \|E\| \backslash \{0\}$ and put $t_n := |\alpha_n| s$. Then $t_n \to 0$, and E contains the set

$$B = \{(\xi_n)_{n\in N} : \ \xi_n \in B_n\}$$

where $B_n := \{\mu \in K : |\mu| \|e_n\| \leq t_n\}$.
All B_n are balls in K about 0. Now we claim that $\aleph_\beta$, the cardinality of B_n, is equal to $\aleph_\kappa$. In fact, choose $1 < |\lambda_1| < |\lambda_2| < \ldots \quad , |\lambda_j| \to \infty$. Then for each element $\alpha \in K$ we can choose $m \in N$ such that $\mu' = \lambda_m^{-1}\alpha$ belongs to B_n. Therefore $\alpha = \lambda_m \mu'$, which implies that $\aleph_\kappa \leq \aleph_0 \cdot \aleph_\beta = \aleph_\beta$. The other inequality being trivial, we obtain $\aleph_\beta = \aleph_\kappa$. In particular all the balls B_n have the same cardinality. Since $E \supset B$ and the cardinality of B is $\aleph_\kappa^{\aleph_0}$, we have $\aleph_\varepsilon \geq \aleph_\kappa^{\aleph_0}$. As $E \subset K^N$, the opposite inequality is trivial.

Remark 2.5. From set theory (see [1]) we know, assuming the Axiom of Choice and the Generalized Continuum Hypothesis, that if $\kappa \neq 0$ is a limit ordinal with cofinality $\aleph_0$ then $\aleph_\kappa^{\aleph_0} = \aleph_{\kappa+1} = 2^{\aleph_\kappa}$ and that in all other cases $\aleph_\kappa^{\aleph_0} = \aleph_\kappa$.

3. The case for non-metrizable K

Theorem 3.1. *Let E be an infinite-dimensional Banach space with topological base $e_1, e_2, \ldots$ over a non-metrizable K. Then the dimension of E is $\aleph_0$ and the cardinalities of E and K are equal.*

Proof. Let $x \in E$ have the expansion $\sum_{n=0}^{\infty} \lambda_n e_n$; we will show that this is in fact a finite sum. In fact, assume $\|\lambda_{n_1} e_{n_1}\| > \|\lambda_{n_2} e_{n_2}\| > \ldots$, $\|\lambda_{n_i} e_{n_i}\| \to 0$. Since $\|E\| \backslash \{0\}$ is a G-module, there are $\mu_1, \mu_2, \ldots \in K^*$ such that $|\mu_k| \|\lambda_{n_1} e_{n_1}\| < \|\lambda_{n_k} e_{n_k}\|$ for all k. It follows that $\mu_k \lambda_{n_1} e_{n_1} \to 0$, hence $|\mu_k| \to 0$, so that G has a coinitial sequence, conflicting 1.1.
We see that E is the space of all finite linear combinations of $e_1, e_2, \ldots$, which is algebraically isomorphic to $\bigoplus_N K$ and the conclusion follows.

Remark. At first sight it may seem strange that a Banach space can have countable dimension! But one has to keep in mind that, due to non-metrizability, the Baire Category Theorem does not apply to E.

4. Appendix

As promised in the Introduction we compute the dimension of l^2. In fact we prove more.

Proposition 4.1. *Let E be a Banach space over* **R** *or* **C** *with a topological base $e_1, e_2, \ldots$. Then the dimension of E is the power of the continuum.*

Proof. Let L be either **R** or **C**, with cardinality c. For a set I, let $l^\infty(I)$ be the L-vector space of all bounded functions $I \to L$. By 2.1 we only have to prove that dim $E \geq c$. To this end we may assume by scalar multiplication, that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Then the formula

$$(\xi_1, \xi_2, \ldots) \mapsto \sum_{n=1}^{\infty} \xi_n e_n$$

defines a linear injection $l^\infty(\mathbf{N}) \to E$. Since $\mathbf{Q}$ is countable, the spaces $l^\infty(\mathbf{N})$ and $l^\infty(\mathbf{Q})$ are isomorphic. For each $t \in \mathbf{R}$, let $f_t \in l^\infty(\mathbf{Q})$ be defined by

$$f_t(q) = \begin{cases} 1 & \text{if} \quad q \in Q, \quad q \geq t \\ 0 & \text{if} \quad q \in Q, \quad q < t \end{cases}$$

It is easily seen that the f_t ($t \in \mathbf{R}$) are linearly independent. Then dim $E \geq \dim l^{\infty}(\mathbf{N}) = \dim l^{\infty}(\mathbf{Q}) \geq c$ and we are done.

Remark. It is not hard to see that the techniques used in Section 2, appropriately modified, may furnish another proof of the Proposition above.

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