ON SOME FUNCTIONS CONCERNING FUZZY PG-CLOSED SETS

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Abstract

In this paper we consider new weak and strong forms of fuzzy preirresoluteness and fuzzy pre-closureness via the concept of fuzzy pg-closed sets which we call ap-Fp-irresolute functions, ap-Fp-closed functions and contra fuzzy pre-irresolute functions and we use it to obtain a characterization of fuzzy pre-$T_{1/2}$ spaces.

1. Introduction and Preliminaries

In 1991 A. S. Binshahan [4] introduced and investigated the notions of fuzzy preopen and fuzzy preclosed sets. Since then the advent of these notions, several research papers with interesting results in di®erent respects came to existence ([1],[3],[5],[6],[7],[10,[12]).

In this paper we shall introduce new generalizations of fuzzy pre-closedness called ap-fuzzy pre-closedness by using fuzzy pg-closed sets and study some of their basic properties. This de¯nition enables us to obtain conditions under which functions and inverse functions preserve fuzzy pg-closed sets, also in this paper we present a new generalization of fuzzy pre-irresoluteness called contra fuzzy pre-irresolute.We de¯ne this last class of functions by the requirement that the inverse image of each fuzzy pre-open set in the co-domain is fuzzy preclosed in the domain. This notion is a stronger form of ap-fuzzy pre-irresolute and ap-fuzzy preclosed functions. Finally, we also characterize the class of fuzzy pre-$T_{1/2}$ spaces in terms of ap-fuzzy pre-irresolute and ap-fuzzy preclosed functions.

Throughout this paper (X,τ), (Y,σ) and (Z,γ) (or simply X,Y and Z) represent non-empty fuzzy topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let A be a fuzzy subset of X then $Cl(A)$ and $Int(A)$ denote the closure of A and the interior of A. respectively.

1.1 DEFINITION : The fuzzy set A of X is called fuzzy preopen [4] if $A \cdot IntCl(A)$.

The fuzzy subset B of X is called fuzzy preclosed if, its complement B^c is fuzzy preopen in X .

By $FPO(X)$ (rep. $FPC(X)$), we denote the family of all fuzzy pre-open(resp. fuzzy preclosed) sets of X.

1.2 DEFINITION : The intersection of all fuzzy preclosed sets containing A is called the preclosure of A and is denoted by pCl(A) [4].

1.3 DEFINITION : The fuzzy interior of A [4] is denoted by pInt(A), and is de¯ned by the union of all fuzzy preopen sets of X which are contained in A.

1.4 DEFINITION : Recall that a fuzzy set A is called fuzzy regular open(resp.regular closed) if $A = IntCl(A)$ (resp. $A = ClInt(A)$)[2].

1.5 DEFINITION : A function $f : (X, \tau) \to (Y, \sigma)$ is said to be :

(i) fuzzy preirresolute [8] if, $f^{-1}(V)$ is fuzzy preopen set in (X, τ) for each fuzzy open set V of (Y, σ),

(ii) fuzzy M-preclosed [11](resp.fuzzy M-preopen[11]) if, for every fuzzy preclosed (resp.fuzzy preopen) set B of (X, τ), $f(B)$ is fuzzy preclosed(resp. fuzzy preopen) set in (Y, σ),

(iii) fuzzy contra preclosed if $f(A)$ is fuzzy preopen set in (Y, σ) for every fuzzy closed set A of (X, τ)[9].

2. ap-Fp-IRRESOLUTE AND ap-Fp-CLOSED FUNCTIONS

2.1 DEFINITION : A fuzzy set A of (X, τ) is said to be fuzzy pre-generalized closed (briefly,Fpg-closed) if, $pCl(A) \cdot H$ whenever $A \cdot H$ and H is fuzzy preopen in X.A fuzzy set B is said to be fuzzy pre-generalized open (briefly, Fpg-open) in X if its complement B^c is Fpg-closed set in X.

2.2 REMARK : (i) Every fuzzy preclosed set is Fpg-closed but not conversely.

(ii) The concept of fuzzy g-closed sets and Fpg-closed sets are independent. For,

2.3 EXAMPLE : Let $X = \{a, b\}$, fuzzy sets A, B, E, H and K are defined as :

$$A(a) = 0.3, A(b) = 0.4 ;$$
$$B(a) = 0.6, B(b) = 0.5 ;$$
$$E(a) = 0.2, E(b) = 0.2 ;$$
$$H(a) = 0.3, H(b) = 0.7 ;$$
$$K(a) = 0.3, K(b) = 0.8 .$$

Let $\tau = \{0, A, 1\}$ and $\sigma = \{0, H, 1\}$. Then,

(i) B is fuzzy pg-closed but not fuzzy preclosed set in (X, τ).

(ii) E is fuzzy pg-closed but not fuzzy g-closed in (X, τ) and K is fuzzy g-closed but not fuzzy pg-closed set in (X, σ).

2.4 DEFINITION : A function $f : (X, \tau) \to (Y, \sigma)$ is said to be approximately fuzzy preirresolute (brie°y, ap-Fp- irresolute) if, $pCl(A) \cdot f^{i\,1}(H)$ whenever H is a fuzzy preopen subset of (Y, σ), A is a fuzzy pg-closed subset of (X, τ) and $A \cdot f^{i\,1}(H)$.

2.5 DEFINITION : A function $f : (X, \tau) \to (Y, \sigma)$ is said to be approximately fuzzy preclosed (brie°y, ap-Fp-closed)if $,f(B) \cdot pInt(A)$ whenever A is a fuzzy pg-closed subset of $(Y, \sigma),B$ is a fuzzy preclosed subset of (X, τ) and $f(B) \cdot A$.

2.6 THEOREM : (i)A function $f : (X, \tau) \to (Y, \sigma)$ is an ap-Fp-irresolute if, $f^{i\,1}(A)$ is fuzzy preclosed in (X, τ) for every $A \in FPO(Y)$.

(ii)$f : (X, \tau) \to (Y, \sigma)$ is ap-Fp-closed if, $f(B) \in FPO(Y)$ for every fuzzy preclosed subset B of (X, τ).

PROOF : (i) Let $H \cdot f^{i\,1}(A)$ where $A \in FPO(Y)$ and H is a fuzzy pg-closed subset of (X, τ). Therefore $pCl(H) \cdot pCl(f^{i\,1}(A)) = f^{i\,1}(A)$. Thus, f is ap-Fp-irresolute.

(ii) Let $f(B) \cdot A$, where B is a fuzzy preclosed subset of (X, τ) and A a fuzzy pg-open subset of $(Y, \sigma) \cdot$ Therfore, pInt(f(B))$\cdot pInt(A).Then,$f(B)$\cdot pInt(A).Thus, f is ap_{i} Fp_{i} closed.$

Clearly, fuzzy preirresolute functions are ap-Fp-irresolute.Also,fuzzy M-preclosed functions are ap-Fp- closed.The converse implication do not hold as it is shown in the following example.

2.7 EXAMPLE : Let $X = \{a, b\}$ and $Y = \{x, y\}$. Fuzzy sets E,A,B and H be de¯ned as :

E(a) = 0 , E(b) = 0.8 ;

A(a) = 0.3 , A(b) = 0.4 ;

H(x) = 0.3 , H(y) = 0.4 ;

B(x) = 0 , B(y) = 0.8 .

Let $\tau = \{0, E, 1\}$ and $\sigma = \{0, H, 1\}$, $_{i} = \{0, A, 1\}$ and $\nu = \{0, B, 1\}$. Then, the function $f : (X, \tau) \to (Y, \sigma)$ de¯ned by f(a) = x, f(b) = y is ap-Fp-irresolute but not fuzzy preirresolute since Z(x) = 0.1 and Z(y) = 0.1 are fuzzy preopen in Y but $f^{i\,1}(Z)$ is not fuzzy preopen in (X, τ), and the function $h : (X, _{i}) \to (Y, \nu)$ de¯ned by h(a) = x , h(b) = y , is ap-Fp-closed but not fuzzy M-preclosed.

2.8 REMARK : Converse of the Theorem 2.6 do not hold for .

2.9 EXAMPLE : The function f de⁻ned in Example 2.3 is ap-Fp-irresolute. Fuzzy set $K(x) = 0.9, K(y) = 0.9 are fuzzy preopen in Y but$ f$^{i\ 1}(K)$ is not fuzzy preclosed in (X, τ). The function function h de⁻ned in Example 2.7 is ap-Fp-closed and the fuzzy set $W(a) = 0.1, W(b) = 0.1$ is fuzzy preclosed set in (X, τ), but $h(W)$ is not fuzzy preopen set in (Y, ν).

In the following theorem, we get under certain conditions the converse of the Theorem 2.6 is true.

2.10 THEOREM : Let $f : (X, \tau)\ !\ (Y, \sigma)$ be a function. Then,
(i) Let all fuzzy subsetsof (X, τ) be fuzzy clopen, then f is ap-Fp-irresolute if and only if, $f^{i\ 1}(H)$ is fuzzy preclosed in (X, τ) for every $H\ 2\ FPO(Y)$.
(ii)Let all fuzzy subsets of (Y, σ) be fuzzy clopen, then f is ap-Fp-closed if and only if, $f(B)\ 2\ FPO(Y)$ for every fuzzy preclosed subset B of (X, τ).

PROOF : (i) Assume f is ap-Fp-irresolute. Let A be an arbitrary fuzzy subset of (X, τ) such that $A\ \cdot\ Q$ where $Q\ 2\ FPO(X)$. Then by hypothesis $pCl(A)\ \cdot\ pCl(Q)\ =\ Q$. Therefore all fuzzy subsets of (X,τ) are fuzzy pg-closed (and hence all are fuzzy pg-open). So for any $H\ 2\ FPO(Y)$, $f^{i\ 1}(H)$ is fuzy pg-closed in X . Since f is ap-Fp-irresolute, $pCl(f^{i\ 1}(H))\ \cdot\ f^{i\ 1}(H)$. Therefore, $pCl(f^{i\ 1}(H)\ =\ f^{i\ 1}(H)$, i.e., $f^{i\ 1}(H)$ is fuzzy preclosed in (X, τ).

The converse is clear by Theorem 2.6.

(ii) Assume f is ap-Fp-closed. Reasoning as in (i), we obtain that all fuzzy subsets of (Y, σ) are fuzzy pg-open.Therefore,for any fuzzy preclosed subset B of (X, τ), f(B) is fuzzy pg-open in Y.Since f is ap-Fp-closed, $f(B)\ \cdot\ pInt(f(B)).Therefore,$f(B) = pInt(f(B)),\quad $i.e., f(B) is fuzzy preopen.$

The converse is clear by Theorem 2.6.

As an immediate consequence of Theorem 2.10, we have the following. (ii)

2.11 COROLLARY : Let $f : (X, \tau)\ !\ (Y, \sigma)$ be a function. Then,

(i) Let all the fuzzy subsets of (X, τ) be fuzzy clopen, then f is ap-Fp-irresolute if and only if f is fuzzy preirresolute.

(ii) Let all the fuzzy subsets of (Y, σ) be fuzzy clopen, the f is ap-Fp-closed if and only if f is fuzzy M-preclosed.

3. CONTRA FUZZY PRE-IRRESOLUTE FUNCTIONS

3.1 DEFINITION : A function $f : (X, \tau) \to (Y, \sigma)$ is called :

(i) contra fuzzy preirresolute if $f^{-1}(A)$ is fuzzy preclosed in (X, τ) for each $A \in FPO(Y)$,

(ii) contra fuzzy M-preclosed if $f(B) \in FPO(Y)$ for each fuzzy pre-closed set B of (X, τ).

3.2 REMARK : The concepts of contra fuzzy preirresolutenes and fuzzy preirresoluteness are independent notions. For,

3.3 EXAMPLE : Let $X = \{a, b\}$, $Y = \{x, y\}$ and $Z = p, q$. Define fuzzy sets A,E and B as :

$\qquad$ A(a) = 0.7 , A(b) = 0.8 ;
E(x) = 0.4 , E(y) = 0.3 ;
B(p) = 0.3 , B(q) = 0.3 .

Let $\tau = \{0, A, 1\}$, $\mu = \{0, E, 1\}$ and $\sigma = \{0, B, 1\}$. Then the function $f : (X, \tau) \to (Y, \mu)$ defined by f(a) = x, f(b) = y is contra fuzzy preir-resolute but not fuzzy preirresolute and the function $h : (Y, \mu) \to (Z, \sigma)$ defined by h(x) = p and h(y) = q is fuzzy preirresolute but not contra fuzzy preirresolute.

In ths same manner one can prove that, contra fuzzy M-preclosed functions and fuzzy M-preclosed functions are indepedent notions.

3.4 DEFINITION : A function $f : (X, \tau) \to (Y, \sigma)$ is called contra fuzzy precontinuous if $f^{-1}(A)$ is fuzzy preclosed set in (X, τ) for each fuzzy open set A of (Y, σ).

3.5 REMARK : Every contra fuzzy preirresolute function is contra fuzzy precontinuous, but not conversely. For,

3.6 EXAMPLE : Let $X = \{a, b\}$ and $Y = \{x, y\}$. Fuzzy sets A and B are de¯ned as :

$$A(a) = 0.7 \ , \ A(b) = 0.8 \ ;$$
$$B(x) = 0 \ , \ B(y) = 0.6 \ .$$

Let $\tau = \{0, A, 1\}$ and $\sigma = \{0, B, 1\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ de¯ned by f(a) = x, f(b) = y is contra fuzzy precontinuous but not contra fuzzy preirresolute.

The following result can be easily veri¯ed. Its proof is straightforward.

3.7 THEOREM : Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following conditions are equivalent :
(i) f is contra fuzzy preirresolute,
(ii) The inverse image of each fuzzy preclosed set in Y is fuzzy preopen in X.

3.8 REMARK : By Theorem 2.6, we have that every contra fuzzy preirresolute function is ap-Fp-irresolute and every contra fuzzy preclosed function is ap-Fp-closed, the converse implication do not hold (see Remark 2.8)

Now, we de¯ne the following.
3.9 DEFINITION : A function $f : (X, \tau) \to (Y, \sigma)$ is called perfectly contra fuzzy preirresolute if the inverse image of every fuzzy preopen set in Y is fuzzy pre-clopen in X

Every perfectly contra fuzzy preirresolute function is contra fuzzy preirresolute and fuzzy preirresolute. But the converse may not be true. For,

3.10 EXAMPLE : The functions f and h de¯ned in Example 3.3 are contra fuzzy preirresolute and fuzzy preirresolute respectively but not perfectly contra fuzzy preirresolute.

Clearly, the following diagram holds and none of its implications is reversible.

$$
\begin{array}{ccc}
p.c.f.p.i. & \Rightarrow & c.f.p.i. \\
\Downarrow & & \Downarrow \\
f.p.i. & \Rightarrow & ap - Fp - i
\end{array}
$$
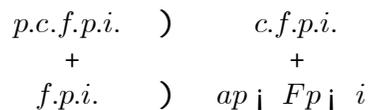
Where contra f.p.i = contra fuzzy preirresolute, ap-Fp-i.= ap-Fp-irresolute, perfectly contra f.p.i.= perfectly contra fuzzy preirresolute.

The following theorem is a decomposition of perfectly contra fuzzy preir-resoluteness.

3.11 THEOREM : For a function $f : (X, \tau) \rightarrow (Y, \sigma)$ the following conditions are equivalent:
(i) f is perfectly contra fuzzy preirresolute.
(ii) f is both contra fuzzy preirresolute and fuzzy preirresolute.

3.12 THEOREM : If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is ap-Fp-irresolute and fuzzy M-preclosed, then for every Fpg-closed set A of (X, τ), $f(A)$ is Fpg-closed in (Y, σ).

PROOF: Let A be a Fpg-closed subset of (X, τ). Let $f(A) \leq B$ where $B \in FPO(Y)$. Then $A \leq f^{-1}(B)$ holds. Since f is ap-Fp-irresolute $pCl(A) \leq f^{-1}(B)$ and hence $f(pCl(A)) \leq B$. Therefore, we have, pCl(f(A))$\leq pCl(f(pCl(A))) = f(pCl(A)) \leq B$. Hence $f(A)$ is fuzzy pg-closed in (Y, σ).

However the following theorem holds. The easy proof of it is omitted.

3.13 THEOREM : A function $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \gamma))$ be two functions such that $gof : (X, \tau) \rightarrow (Z, \gamma)$. Then,
(i) gof is contra fuzzy preirresolute, if g is fuzzy preirresolute and f is contra fuzzy preirresolute.
(ii) gof is contra fuzzy preirresolute, if g is contra fuzzy preirresolute and f is fuzzy preirresolute.

In an analogous way, we have the following.

3.14 THEOREM : Let function $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (X, \tau) \rightarrow (Z, \gamma)$ be two functions such that $gof : (X, \tau) \rightarrow (Z, \gamma)$. Then,
(i) gof is ap-Fp-closed, if f is fuzzy M-preclosed and g is ap-Fp-closed.
(ii) gof is ap-Fp-closed, if f is ap-Fp-closed and g is fuzzy M-preopen and g^{-1} preserves Fpg-closed sets.
(iii) gof is ap-Fp-irresolute, if f is ap-Fp-irresolute and g is fuzzy preir-resolute.

PROOF : (i) Suppose B is an arbitrary fuzzy preclosed subset in (X, τ) and A is a Fpg-open subset of (Z, γ) for which $(gof)(B) \leq A$. Then $f(B)$

is fuzzy preclosed in (Y,σ) because f is fuzzy M-preclosed. Since g is ap-Fp-closed, $g(f(B)) \cdot pInt(A)$. This implies that gof is ap-Fp-closed.

(ii) Suppose B is an arbitrary fuzzy preclosed subset of (X,τ) and A is a fuzzy preopen subset of (Z,γ) for which $(gof)(B) \cdot A$. Hence $f(B) \cdot g^{-1}(A)$. Then $f(B) \cdot pInt(g^{-1}(A))$ since $g^{-1}(A)$ is Fpg-open and f is ap-Fp-closed. Thus, $(gof)(B) = g(f(B)) \cdot g(pInt(g^{-1}(A))) \cdot pInt(gg^{-1}(A)) \cdot pInt(A)$. This shows that gof is ap-Fp-closed.

(iii) Suppose H is an arbitrary Fpg-closed subset of (X,τ) and $E \in FPO(Z)$ for which $H \cdot (gof)^{-1}(E)$. Then $g^{-1}(E) \in FPO(Y)$ since g is fuzzy preirresolute. Since f is ap-Fp-irresolute, $pCl(A) \cdot f^{-1}(g^{-1}(E)) = (gof)^{-1}(E)$. This proved that gof is ap-Fp-irresolute.

In the following result we o®er a new characterization of the class of fuzzy pre-$T_{1/2}$ spaces by using the concepts of ap-Fp-irresolute functions and ap-Fp- closed functions.

3.15 DEFINITION : A fuzzy topological space (X,τ) is said to be fuzzy pre-$T_{1/2}$ space if every Fpg-closed set is fuzzy preclosed in it.

3.16 THEOREM : Let (X,τ) be a fuzzy topological space. Then the following statements are equivalent:
(i)(X,τ) is fuzzy pre-$T_{1/2}$ space,
(ii)For every space (Y,σ) and every function $f : (X,\tau) \to (Y,\sigma)$, f is ap-Fp-irresolute.

PROOF: (i)$\to$ (ii) . Let E be a Fpg-closed subset of (X,τ) and suppose that $E \cdot f^{-1}(H)$ where $H \in FPO(Y)$. Since (X,τ) is fuzzy pre-$T_{1/2}$ space, E is fuzzy preclosed (i.e., $E = pCl(E)$). Therefore, $pCl(E) \cdot f^{-1}(H)$. Then f is ap-Fp-irresolute.

(ii)$\to$ (i). Let B be a Fpg-closed subset of (X,τ) and let Y be the fuzzy set of X with the topology $\sigma = 0, B, 1$. Finally, let $f : (X,\tau) \to (Y,\sigma)$ be the identiy function.By assumption f is ap-Fp-irresolute. Since B is Fpg-closed in (X,τ) and fuzzy preopen in (Y,σ) and $B \cdot f^{-1}(B)$, it follows that $pCl(B) \cdot f^{-1}(B) = B$. Hence B is fuzzy preclosed in (X,τ) and therefore, (X,τ) is a fuzzy pre-$T_{1/2}$ space.

3.17 THEOREM : Let (Y,σ) be a fuzzy topological space. Then the following statements are equivalent :

(i) (Y, σ) is a fuzzy pre-$T_{1/2}$ space,

(ii) For every space (X, τ) and every function $f : (X, \tau)$! (Y, σ), f is ap-Fp-closed.

Proof : Analogous to Theorem 3.16 making the obvious changes.

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