

Proyecciones
Vol. 25, N° 3, pp. 237-247, December 2006.
Universidad Católica del Norte
Antofagasta - Chile

L-FUZZY CLOSURE OPERATOR *

YUELI YUE
INSTITUTE OF TECHNOLOGY, CHINA
FU-GUI SHI
UNIVERSITY OF CHINA, CHINA

Received : May 2006. Accepted : October 2006

Abstract

*The aim of this paper is to study L -fuzzy closure operator in L -fuzzy topological spaces. We introduce two kinds of L -fuzzy closure operators from different point view and prove that both L -**TFCS**—the category of topological L -fuzzy closure spaces—and L -**PTFCS**—the category of topological pointwise L -fuzzy closure spaces—are isomorphic to L -**FCTOP**.*

Key words: *L -fuzzy co-topology; fuzzy remote neighborhood system; L -fuzzy closure operator*

*The project is supported by the National Natural Science Foundation of China (10371079) and the Base Research Foundation of Beijing Institute of Technology.

1. Introduction

Since Chang [2] introduced fuzzy set theory to topology, many researchers have tried successfully to generalize the theory of general topology to the fuzzy setting with crisp methods. However, in a completely different direction, Höhle [5] created the notion of a topology being viewed as an L -subset of a powerset. Then Kubiak [7] and Šostak [12] independently extended Höhle's notion to L -subsets of L^X . In [15], we established fuzzy remote neighborhood systems in L -fuzzy co-topology and prove that **TFRNS** is isomorphic to L -**FCTOP**.

It is well-known that closure operator (or closure system) plays an important role in topology and it is a very good way to characterize topology. Many authors [3, 7, 13] have studied closure operators in L -fuzzy topological spaces. But it is a pity that their closure operators are actually defined by the level L -topology of the L -fuzzy topology, not by L -fuzzy topology itself. In other words, their closure operators are still the closure operators in L -topologies. The aim of this paper is to study L -fuzzy closure operators in L -fuzzy topological spaces in different ways from [3, 7, 13]. We give two kinds of L -fuzzy closure operators and prove that both L -**TFCS**—the category of topological L -fuzzy closure spaces—and L -**PTFCS**—the category of topological pointwise L -fuzzy closure spaces—are isomorphic to L -**FCTOP**.

2. Preliminaries

An element a in a complete lattice L is said to be coprime if $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$. The set of nonzero \vee -irreducible elements (or coprimes) of L is denoted by $c(L)$. We say a is wedge below b , in symbols, $a \triangleleft b$ or $b \triangleright a$, if for every arbitrary subset $D \subseteq L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. A complete lattice is said to be completely distributive if every element in L is the supremum of all the elements wedge below it. For more details about completely distributive lattice, please refer to [4].

By the definition of complete distributivity it is easy to see that a complete lattice L is completely distributive if and only if the operator $\bigvee : \text{Low}(L) \rightarrow L$ taking every lower set to its supremum has a left adjoint β , and in the case, $\beta(a) = \{b \mid b \triangleleft a\}$ for all $a \in L$. Hence the wedge below relation has the interpolation property in a completely distributive lattice, this is to say, $a \triangleleft b$ implies there is some $c \in L$ such that $a \triangleleft c \triangleleft b$. $\{a \in L \mid a \triangleleft b\}$ is called the greatest minimal family of b , in symbol $\beta(b)$.

Moreover, for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \triangleleft b'\}$ which is called the greatest maximal family of b .

Throughout this paper L is a completely distributive lattice and M is a completely distributive lattice with an order reversing involution $'$. L^X is the set of all L -fuzzy sets on X . The set of nonzero \vee -irreducible elements (or coprime) in L^X is denoted by $c(L^X)$. Let $x_\lambda|A$ denote the set $\{B \in L^X \mid x_\lambda \not\leq B \geq A\}$ for $x_\lambda \in c(L^X)$ and $A \in L^X$. For undefined notions about categories, please refer to [1], [6] and [11].

Definition 2.1^[7,12]. An L -fuzzy co-topology is a mapping $\eta : L^X \rightarrow M$ such that

- (FCT1) $\eta(1_X) = \eta(0_X) = 1$;
- (FCT2) $\eta(A \vee B) \geq \eta(A) \wedge \eta(B)$ for all $A, B \in L^X$;
- (FCT3) $\eta(\bigwedge_{j \in J} A_j) \geq \bigwedge_{j \in J} \eta(A_j)$ for every family $\{A_j \mid j \in J\} \subseteq L^X$.

The pair (L^X, η) is called an L -fuzzy co-topological space (L -FCTop, for short). A mapping $F : (L^X, \eta) \rightarrow (L^Y, \eta_1)$ is said to be continuous with respect to η and η_1 if $\eta(F_L^-(B)) \geq \eta_1(B)$ for all $B \in L^Y$, where $F_L^-(B)(x) = B(F(x))$ (following the notation in [14]). The category of L -FCTops is denoted by L -**FCTOP**.

Definition 2.2^[15]. A fuzzy remote neighborhood system is a set $\mathcal{R} = \{R_{x_\lambda} \mid x_\lambda \in c(L^X)\}$ of mappings $R_{x_\lambda} : L^X \rightarrow M$ such that

- (FRN1) $R_{x_\lambda}(1_X) = 0, R_{x_\lambda}(0_X) = 1$;
- (FRN2) $R_{x_\lambda}(A) > 0 \Rightarrow x_\lambda \not\leq A$;
- (FRN3) $R_{x_\lambda}(A \vee B) = R_{x_\lambda}(A) \wedge R_{x_\lambda}(B)$.

\mathcal{R} will be called a topological fuzzy remote neighborhood system if it also satisfies the following equation:

$$(FRN4) \quad R_{x_\lambda}(A) = \bigvee_{B \in x_\lambda|A} \bigwedge_{y_\mu \not\leq B} R_{y_\mu}(B),$$

and (L^X, \mathcal{R}) is called a topological fuzzy remote neighborhood space (TFRNS, for short). A fuzzy continuous mapping between topological fuzzy remote neighborhood spaces (L^X, \mathcal{R}) and (L^Y, \mathcal{S}) is a mapping $F : L^X \rightarrow L^Y$ such that $S_{F_L^-(x_\lambda)}(B) \leq R_{x_\lambda}(F_L^-(B))$ for all $x_\lambda \in c(L^X)$ and $B \in L^Y$. The category of TFRNSs and fuzzy continuous mappings is denoted by **TFRNS**.

Lemma 2.3^[15]. Let $\eta : L^X \rightarrow M$ be an L -fuzzy co-topology. Then we have

(1) $\mathcal{R}^\eta = \{R_{x_\lambda}^\eta | x_\lambda \in c(L^X)\}$ is a topological fuzzy remote neighborhood system, where $R_{x_\lambda}^\eta$ is defined by

$$R_{x_\lambda}^\eta(A) = \begin{cases} \bigvee_{B \in x_\lambda | A} \eta(B), & x_\lambda \not\leq A, \\ 0, & x_\lambda \leq A. \end{cases}$$

for $x_\lambda \in c(L^X)$ and $A \in L^X$.

(2) If η and ζ are two L -fuzzy co-topologies which determine the same topological fuzzy remote neighborhood system, then $\eta = \zeta$.

Lemma 2.4^[15]. Let $\mathcal{R} = \{R_{x_\lambda} | x_\lambda \in c(L^X)\}$ be a fuzzy remote neighborhood system and $\eta : L^X \rightarrow M$ be defined by $\eta(u) = \bigwedge_{x_\lambda \not\leq A} R_{x_\lambda}(A)$ for all $A \in L^X$. Then η is an L -fuzzy co-topology. Furthermore, if \mathcal{R} and \mathcal{S} are two topological fuzzy remote neighborhood systems which determine the same L -fuzzy co-topology, then $\mathcal{R} = \mathcal{S}$.

Lemma 2.5^[15]. **TFRNS** is isomorphic to **L -FCTOP**.

3. L -fuzzy closure operator

In this section, we study L -fuzzy closure operator in a different direction from [3, 7, 13].

Lemma 3.1. Let (L^X, η) be an L -fuzzy co-topological space and define $C_A^\eta : L^X \rightarrow M$ by

$$C_A^\eta(B) = \begin{cases} 0, & A \not\leq B, \\ \eta(B) \wedge \bigwedge_{A \leq D \not\leq B} \eta(D)', & A \leq B. \end{cases}$$

Then $\{C_A^\eta\}_{A \in L^X}$ satisfying the following properties.

- (1) $C_{0_X}^\eta(0_X) = 1$;
- (2) $C_A^\eta(B) > 0 \Rightarrow A \leq B$;
- (3) $C_A^\eta(B) \wedge C_D^\eta(E) \leq C_{A \vee D}^\eta(B \vee E)$;
- (4) $\bigwedge_{t \in T} C_{A_t}^\eta(B_t) \leq C_{\bigwedge_{t \in T} B_t}^\eta(\bigwedge_{t \in T} B_t)$;
- (5) $C_A^\eta(B) \wedge C_B^\eta(E) \leq C_A^\eta(E)$;
- (6) $C_A^\eta(B) = \bigwedge_{A \leq D \leq B} C_D^\eta(B)$;
- (7) $C_A^\eta(A) = \eta(A)$;

$$(8) C_A^\eta(B) = \begin{cases} 0, & A \not\leq B, \\ C_B^\eta(B) \wedge \bigwedge_{A \leq D \not\leq B} (C_D^\eta(D))', & A \leq B. \end{cases}$$

Proof. We prove (3).

$$\begin{aligned} C_A^\eta(B) \wedge C_D^\eta(E) &= \eta(B) \wedge \bigwedge_{A \leq F \not\leq B} \eta(F)' \wedge \eta(E) \wedge \bigwedge_{D \leq G \not\leq E} \eta(G)' \\ &\leq \eta(B \vee E) \wedge \bigwedge_{A \leq F \not\leq B} \eta(F)' \wedge \bigwedge_{D \leq G \not\leq E} \eta(G)' \\ &\leq \eta(B \vee E) \wedge \bigwedge_{A \vee D \leq H \not\leq B \vee E} \eta(H)' \\ &= C_{A \vee D}^\eta(B \vee E). \end{aligned}$$

Remark 3.2. (1) The value $C_A^\eta(B)$ can be interpreted as the degree to which B is the closure of A . When $M = \{0, 1\}$, that is to say η is an L -co-topology, and define $\bar{A} = B$ when $C_A^\eta(B) = 1$, then the operator $- : L^X \rightarrow L^X$ is just the closure operator induced by η .

(2) The readers can easily show that C_A^η can be written as follows:

$$C_A^\eta(B) = \begin{cases} 0, & A \not\leq B, \\ \bigwedge_{x_\lambda \not\leq B} R_{x_\lambda}^\eta(B) \wedge \bigwedge_{x_\lambda \leq B} (R_{x_\lambda}^\eta(A))', & A \leq B. \end{cases}$$

Definition 3.3. An L -fuzzy closure operator is a set $\mathcal{C} = \{C_A | A \in L^X\}$ of mappings $C_A : L^X \rightarrow M$ such that:

- (FC1) $C_{0_X}(0_X) = 1$;
- (FC2) $C_A(B) \wedge C_D(E) \leq C_{A \vee D}(B \vee E)$;
- (FC3) $\bigwedge_{t \in T} C_{A_t}(B_t) \leq C_{\bigwedge_{t \in T} B_t}(\bigwedge_{t \in T} B_t)$.

\mathcal{C} is called a topological L -fuzzy closure operator if it also satisfies the following condition

$$(FC4) C_A(B) = \begin{cases} 0, & A \not\leq B, \\ C_B(B) \wedge \bigwedge_{A \leq D \not\leq B} (C_D(D))', & A \leq B. \end{cases}$$

The pair (L^X, \mathcal{C}) is called topological L -fuzzy closure space. A fuzzy continuous mapping between L -fuzzy closure spaces (L^X, \mathcal{C}^1) and (L^Y, \mathcal{C}^2) is a mapping $F : X \rightarrow Y$ such that $C_{F_L^-(B)}^1(F_L^-(B)) \geq C_B^2(B)$ for all $B \in L^Y$. The category of topological L -fuzzy closure spaces and continuous mappings is denoted by $L\text{-TFCS}$.

Lemma 3.4. Let $\mathcal{C} = \{C_A | A \in L^X\}$ be an L -fuzzy closure operator. Then $\eta^{\mathcal{C}} : L^X \rightarrow M$ defined by $\eta^{\mathcal{C}}(A) = C_A(A)$ is an L -fuzzy co-topology.

Lemma 3.5. (1) Let η be an L -fuzzy co-topology. Then $\eta = \eta^{\mathcal{C}^\eta}$.
 (2) If \mathcal{C} is topological L -fuzzy closure operator, then $\mathcal{C} = \mathcal{C}^{\eta^{\mathcal{C}}}$.

Lemma 3.6. (1) $F : (L^X, \mathcal{C}^1) \rightarrow (L^Y, \mathcal{C}^2)$ is continuous, then $F : (L^X, \eta^{\mathcal{C}^1}) \rightarrow (L^Y, \eta^{\mathcal{C}^2})$ is continuous.
 (2) $F : (L^X, \eta^1) \rightarrow (L^Y, \eta^2)$ is continuous, then $F : (L^X, \mathcal{C}^{\eta^1}) \rightarrow (L^Y, \mathcal{C}^{\eta^2})$ is continuous.

Theorem 3.7. L -TFCS is isomorphic to L -FCTOP.

Question 3.8. We know that B is the closure of A if and only if B is the smallest closed set containing A . In fact, the L -fuzzy closure operator studied in Lemma 3.1 is just defined according to this kind of definition. We can also define the closure of A from other ways, such as from the direction of adherent point of A ,

$$C_A^{\eta^*}(B) = \bigwedge_{x_\lambda \not\leq B} R_{x_\lambda}^\eta(A) \wedge \bigwedge_{x_\lambda \leq B} (R_{x_\lambda}^\eta(A))'.$$

From Remark 3.2 (2), we know that there is some difference in this form and that in Lemma 3.1. We wonder whether $C_A^{\eta^*}(B)$ is equal to $C_A^\eta(B)$ or not.

4. Pointwise L -fuzzy closure operator

In this section, we give one definition of Pointwise L -fuzzy closure operator and study the relationship between this kind of closure operator and fuzzy remote neighborhood system.

Lemma 4.1. Let $\eta : L^X \rightarrow M$ be an L -fuzzy co-topology and define $C_{x_\lambda}^\eta : L^X \rightarrow M$ by $C_{x_\lambda}^\eta(A) = \bigwedge_{B \in x_\lambda | A} \eta(B)'$. Then $\mathcal{C}^\eta = \{C_{x_\lambda}^\eta | x_\lambda \in c(L^X)\}$ satisfies:

- (1) $C_{x_\lambda}^\eta(1_X) = 1, C_{x_\lambda}^\eta(0_X) = 0;$

- (2) $C_{x_\lambda}^\eta(A) < 1 \Rightarrow x_\lambda \not\leq A$;
- (3) $C_{x_\lambda}^\eta(A \vee B) = C_{x_\lambda}^\eta(A) \vee C_{x_\lambda}^\eta(B)$;
- (4) $C_{x_\lambda}^\eta(A) = \bigwedge_{B \in x_\lambda|A} \bigvee_{y_\mu \leq B} C_{y_\mu}^\eta(B)$.

Proof. From the definition of $C_{x_\lambda}^\eta$, we know that $C_{x_\lambda}^\eta(A) = R_{x_\lambda}^\eta(A)'$. By (FRN1)–(FRN4), we have (1)–(4).

Remark 4.2. The value $C_{x_\lambda}^\eta(A)$ can be interpreted as the degree to which x_λ is an adherent point of A .

Definition 4.3. A pointwise L -fuzzy closure operator is a set $\mathcal{C} = \{C_{x_\lambda} | x_\lambda \in c(L^X)\}$ of mappings $C_{x_\lambda} : L^X \rightarrow M$ such that

- (FPC1) $C_{x_\lambda}(1_X) = 1, C_{x_\lambda}(0_X) = 0$;
- (FPC2) $C_{x_\lambda}(A) < 1 \Rightarrow e \not\leq A$;
- (FPC3) $C_{x_\lambda}(A \vee B) = C_{x_\lambda}(A) \vee C_{x_\lambda}(B)$.

and (L^X, \mathcal{C}) is called an pointwise L -fuzzy closure space (L -PFCS, for short). \mathcal{C} will be called a topological L -fuzzy closure operator if it also satisfies the following equation:

$$(FPC4) \quad C_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda|A} \bigvee_{y_\mu \leq B} C_{y_\mu}(B),$$

and (L^X, \mathcal{C}) is called a topological pointwise L -fuzzy closure space (L -PTFCS, for short). A fuzzy continuous mapping between topological L -fuzzy closure spaces (L^X, \mathcal{C}^1) and (L^Y, \mathcal{C}^2) is a mapping $F : X \rightarrow Y$ such that $C_{x_\lambda}^1(A) \leq C_{F_L^{-1}(x_\lambda)}^2(F_L^{-1}(A))$ for all $x_\lambda \in c(L^X)$ and $A \in L^X$. The category of L -PTFCS and continuous mappings is denoted by L -PTFCS.

Remark 4.4. When $L = \{0, 1\}$, the definition of pointwise L -fuzzy closure operator is just the definition of fuzzifying closure operator in [11].

It is easy to verify the next two theorems:

Theorem 4.5. (1) Let (L^X, \mathcal{C}) be a topological pointwise L -fuzzy closure space and define $R_{x_\lambda}^{\mathcal{C}} : L^X \rightarrow M$ by $R_{x_\lambda}^{\mathcal{C}}(A) = \mathcal{C}_{x_\lambda}(A)'$. Then $\mathcal{R}^{\mathcal{C}} = \{R_{x_\lambda}^{\mathcal{C}} | x_\lambda \in c(L^X)\}$ is a topological fuzzy remote neighborhood system.

(2) Let (L^X, \mathcal{R}) be a topological fuzzy remote neighborhood space and define $C_{x_\lambda}^{\mathcal{R}} : L^X \rightarrow M$ by $C_{x_\lambda}^{\mathcal{R}}(A) = R_{x_\lambda}(A)'$. Then $\mathcal{C}^{\mathcal{R}} = \{C_{x_\lambda}^{\mathcal{R}} | x_\lambda \in c(L^X)\}$

is a topological pointwise L -fuzzy closure space.

Theorem 4.6. (1) If $F : (L^X, \mathcal{C}^1) \rightarrow (L^Y, \mathcal{C}^2)$ is continuous, then $F : (L^X, \mathcal{R}^{\mathcal{C}^1}) \rightarrow (L^Y, \mathcal{R}^{\mathcal{C}^2})$ is continuous.

(2) If $F : (L^X, \mathcal{R}^1) \rightarrow (L^Y, \mathcal{R}^2)$ is continuous, then $F : (L^X, \mathcal{C}^{\mathcal{R}^1}) \rightarrow (L^Y, \mathcal{C}^{\mathcal{R}^2})$ is continuous.

From Theorem 4.5 and Theorem 4.6, we have one main theorem in this paper.

Theorem 4.7. L -PTFCS is isomorphic to TFRNS. Hence, L -PTFCS is isomorphic to L -FCTOP.

Theorem 4.8. Let $\mathcal{C} = \{C_{x_\lambda} | x_\lambda \in M(L^X)\}$ be a pointwise L -fuzzy closure operator. Then the following statements are equivalent:

- (1) $C_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda | A} \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B)$;
- (2) $C_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda | A} (C_{x_\lambda}(B) \vee \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B))$.
- (3) $C_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda | A} (C_{x_\lambda}(A) \vee \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B))$.
- (4) $C_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda | A} (C_{x_\lambda}(B) \vee \bigvee_{y_\mu \not\leq B} C_{y_\mu}(A))$.

Proof. (1) \Leftrightarrow (2) \Leftrightarrow (3) is trivial. We only prove (1) \Leftrightarrow (4). (1) \Rightarrow (4) is trivial. Now suppose (4) holds, i.e.,

$$C_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda | A} (C_{x_\lambda}(B) \vee \bigvee_{y_\mu \not\leq B} C_a(A)).$$

Let

$$\begin{aligned} t \in \alpha(C_{x_\lambda}(A)) &= \alpha\left(\bigwedge_{B \in x_\lambda | A} (C_{x_\lambda}(B) \vee \bigvee_{y_\mu \not\leq B} C_a(A))\right) \\ &= \bigcup_{B \in x_\lambda | A} \alpha(C_{x_\lambda}(B) \vee \bigvee_{y_\mu \not\leq B} C_{y_\mu}(A)). \end{aligned}$$

Then there exists some $B \in x_\lambda | A$ such that

$$(1) t \in \alpha(C_{x_\lambda}(B)); \quad (2) \forall y_\mu \not\leq B, t \in \alpha(C_{y_\mu}(A)).$$

It is clear that the meet of fuzzy sets containing A and fulfilling (1) and (2) is still of such kind. So we can define B_t to be the minimal fuzzy

set containing A and fulfilling (1) and (2) , i.e., $t \in \alpha(C_{x_\lambda}(B_t))$ and $t \in \alpha(C_{y_\mu}(A))$ for all $y_\mu \not\leq B_t$. Thus, $\forall y_\mu \not\leq B_t$, it follows from $t \in \alpha(C_{y_\mu}(A))$ that there exists $W_{y_\mu} \in y_\mu|A$ such that

$$(3) t \in \alpha(C_{y_\mu}(W_{y_\mu})), \quad (4) \forall z_\gamma \not\leq W_{y_\mu}, t \in \alpha(C_{z_\gamma}(A)).$$

It is easy to check that $B_t \wedge W_{y_\mu}$ satisfies (1) and (2). Hence, by the minimality of B_t , it follows that $B_t \leq B_t \wedge W_{y_\mu}$. Therefore $B_t \leq W_{y_\mu}$. Then we can get that $\forall y_\mu \not\leq B_t, C_{y_\mu}(B_t) \leq C_{y_\mu}(W_{y_\mu})$. Then $t \in \alpha(C_{y_\mu}(B_t))$ Thus, $t \geq \bigvee_{y_\mu \not\leq B_t} C_{y_\mu}(B_t)$. Therefore, $\bigwedge_{B \in x_\lambda|A} \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B) \leq t$. From the arbitrariness of t , we have $C_{x_\lambda}(A) \geq \bigwedge_{B \in x_\lambda|A} \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B)$. Since $C_{x_\lambda}(A) \leq \bigwedge_{B \in x_\lambda|A} \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B)$ is obvious, we have $C_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda|A} \bigvee_{y_\mu \not\leq B} C_{y_\mu}(B)$, as desired.

It is well-known that derived operator and closure operator have close relationship in L -topology. In the following discussion, we study the relationship between pointwise L -fuzzy derived operator and pointwise L -fuzzy closure operator.

Lemma 4.9^[7]. Let $A \in L^X, x_\lambda \in c(L^X)$ and define $A - x_\lambda \in L^X$ by

$$A - x_\lambda(y) = \begin{cases} A(y), & x \neq y, \\ \bigvee_{\lambda \not\leq \gamma, x_\gamma \leq A} \gamma, & x = y. \end{cases}$$

Then

- (1) $\bigvee_{t \in T} (A_t - x_\lambda) = \bigvee_{t \in T} A_t - x_\lambda$;
- (2) $x_\lambda \not\leq A \Rightarrow A - x_\lambda = A$.

Definition 4.10. A topological pointwise L -fuzzy derived operator is a set $\mathcal{D} = \{D_{x_\lambda} | x_\lambda \in c(L^X)\}$ of mappings $D_{x_\lambda} : L^X \rightarrow M$ such that

- (FD1) $D_{x_\lambda}(0_X) = 0$;
- (FD2) $D_{x_\lambda}(A) < 1 \Rightarrow x_\lambda \not\leq A - x_\lambda$;
- (FD3) $D_{x_\lambda}(A \vee B) = D_{x_\lambda}(A) \vee D_{x_\lambda}(B)$;
- (FD4) $D_{x_\lambda}(A) = \bigwedge_{B \in x_\lambda|A-x_\lambda} \bigvee_{y_\mu \not\leq B} D_{y_\mu}(B)$.

(L^X, \mathcal{D}) is called a topological pointwise L -fuzzy derived space (L -TFDS, for short).

It is easy to verify the following two theorems.

Theorem 4.11. (1) Let (L^X, \mathcal{C}) be a topological pointwise L -fuzzy closure space and define $D_{x_\lambda}^{\mathcal{C}} : L^X \rightarrow M$ by $D_{x_\lambda}^{\mathcal{C}}(A) = C_{x_\lambda}(A - x_\lambda)$. Then $\mathcal{D}^{\mathcal{C}} = \{R_{x_\lambda}^{\mathcal{C}} | x_\lambda \in c(L^X)\}$ is a topological pointwise L -fuzzy derived operator.

(2) Let (L^X, \mathcal{D}) be a topological pointwise L -fuzzy derived operator and define $C_{x_\lambda}^{\mathcal{D}} : L^X \rightarrow M$ by

$$C_{x_\lambda}^{\mathcal{D}}(A) = \begin{cases} 1, & x_\lambda \leq A, \\ D_{x_\lambda}(A), & x_\lambda \not\leq A. \end{cases}$$

Then $\mathcal{C}^{\mathcal{D}} = \{C_{x_\lambda}^{\mathcal{D}} | x_\lambda \in c(L^X)\}$ is a topological pointwise L -fuzzy closure operator.

Theorem 4.12. (1) Let \mathcal{C} be a topological pointwise L -fuzzy closure operator. Then $\mathcal{C} = \mathcal{C}^{\mathcal{D}^{\mathcal{C}}}$.

(2) Let \mathcal{D} be a topological pointwise L -fuzzy closure operator. Then $\mathcal{D} \geq \mathcal{D}^{\mathcal{C}^{\mathcal{D}}}$.

References

- [1] J. Adamek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories, JohnWiley and Sons, Inc., (1990).
- [2] C. L. Chang, Fuzzy topological spaces, J.Math.Anal.Appl. 24, pp. 182–193, (1968).
- [3] K. C. Chattopadhyay, S. K. Samanta, Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness, Fuzzy Sets and Systems 54, pp. 207–212, (1993).
- [4] Gierz, et al., A Compendium of Continuous Lattices, Springer, Berlin, (1980).
- [5] U. Höhle, Upper semicontinuous fuzzy sets and applications, J.Math.Anal.Appl. 78, pp. 659–673, (1980).
- [6] U. Höhle, A.P. Šostak, Axiomatic foundations of fixed-basis fuzzy topology, Chapter 3 in: U. Höhle, S.E. Rodabaugh (Eds), Mathematics of Fuzzy Sets-Logic, Topology, and Measure Theory, Kluwer Academic Publishers (Boston/Dordrecht/London), pp. 123–272, (1999).

- [7] Y. C. Kim, Initial L -fuzzy closure spaces, *Fuzzy Sets and Systems* 133, pp. 277–297, (2003).
- [8] T. Kubiak, On fuzzy topologies, Ph.D. Thesis, Adam Mickiewicz, Poznan, Poland, (1985).
- [9] Y. Liu, M. Luo, *Fuzzy Topology*, World Scientific Publishing Co.Pte.Ltd, Singapore, (1997).
- [10] S. E. Rodabaugh, Powerset operator foundations for poslat fuzzy set theories and topologies, Chapter 2 in [6].
- [11] S. E. Rodabaugh, Categorical foundations of variable-basis fuzzy topology, Chapter 4 in [6].
- [12] A. P. Šostak, On a fuzzy topological structure, *Rendiconti Ciecoco Matematico Palermo (Suppl.Ser.II)* 11, pp. 89–103, (1985).
- [13] A.P. Šostak,, Basic structures of fuzzy topology, *J. Math. Sci.* 78 (6), pp. 662–701, (1996).
- [14] R. Lowen, L. Xu, Alternative characterizations of FNCS, *Fuzzy Sets and Systems*, 104, pp. 381-391, (1999).
- [15] Y. Yue , J. Fang , Categories isomorphic to the Kubiak-Šostak extension of **TML**, *Fuzzy Sets and Systems*, 157, pp. 832-842 (2006).

Yueli Yue

Department of Mathematics
Beijing Institute of Technology
Beijing, 100081,
P. R. China
e-mail : yueliyue@163.com

Fu-Gui Shi

Department of Mathematics
Ocean University of China,
Qingdao, 266071,
P. R. China
e-mail : fuguishi@bit.edu.cn