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GENERALIZED CONNECTIVITY *

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Abstract

In this paper, we introduce generalized connectivity in L-fuzzy topological spaces by Lukasiewicz logic and prove K. Fan's theorem.

Key words: *Lukasiewicz logic; L-fuzzy topology; quasi-coincident neighborhood system.*

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1. 1. Introduction and Preliminaries

Since Chang [1] introduced fuzzy theory into topology, many authors have discussed various aspects of fuzzy topology. In a Chang I-topology, the open sets were fuzzy, but the topology comprising those open sets was a crisp subset of the I-powerset I^X. On the other hand, fuzzification of openness was first initiated by Höhle [4] in 1980 and later developed to L-subsets of L^X independently by Kubiak [5] and Šostak [8] in 1985. In 1991, from a logical point of view, Ying [9] studied Höhle's topology and called it fuzzifying topology. In [3], Fang established fuzzy quasi-coincident neighborhood systems in I-fuzzy topological spaces.

Connectivity is an essential part of fuzzy topology, on which a lot of work has been done. In the framework of fuzzifying topologies, Ying [10] introduced connectivity and Fang [2] proved K. Fan's theorem. Considering the completeness and usefulness of theory of L-fuzzy topologies, in this paper, we will introduce connectivity in L-fuzzy topological spaces and give K. Fan's theorem.

Throughout this paper, X is a nonempty set and L is a completely distributive lattice with an order-reversing involution $'$ on it, and with a smallest element 0 and a largest element 1 ($0 \neq 1$). Obviously, L^X—all mappings from X to L—is also a completely distributive lattice. supp$A$ is the support of $A \in L^X$ and 1_U denotes the characteristic function of $U \in 2^X$, where 2^X is the powerset of X. An element $a \in L$ is said to be coprime (resp., prime) if $a \leq b \vee c$ (resp., $a \geq b \wedge c$) implies that $a \leq b$ or $a \leq c$ (resp., $a \geq b$ or $a \geq c$). The set of all coprimes (resp., primes) of L is denoted by $M(L)$(resp., $P(L)$).

Firstly, we display the Łukasiewicz logic and corresponding set-theoretical notations used in this paper. For any formula ϕ, the symbol $[\phi]$ means the truth value of ϕ and this truth value is in the unit interval $[0, 1]$. A formula ϕ is valid, we write $\models \phi$, if and only if $[\phi] = 1$ for every interpretation.

(1) $[\phi \wedge \psi] := \min\{[\phi], [\psi]\}$; $[\phi \to \psi] := \min\{1, 1 - [\phi] + [\psi]\}$.

(2) If X is the universe of discourse, then $[\forall x \in X \phi(x)] := \inf_{x \in X}[\phi(x)]$.

(3) $[\exists x \in X \phi(x)] := [\neg(\forall x \in X \neg\phi(x))] = \sup_{x \in X}[\phi(x)]$.

(4) $[\neg\phi] := [\phi \to 0] = 1 - [\phi]$.

(5) $[\phi \leftrightarrow \psi] := [\phi \to \psi] \wedge [\psi \to \phi]$.

Secondly, we give some definitions and results in L-fuzzy topological spaces.

Definition 1.1[5,8]. An L-fuzzy topology is a map $\eta : L^X \to [0,1]$ such that

(FCT1) $\eta(1) = \eta(0) = 1$;
(FCT2) $\eta(A \wedge B) \geq \eta(A) \wedge \eta(B)$ for all $A, B \in L^X$;
(FCT3) $\eta(\bigvee_{j\in J} A_j) \geq \bigwedge_{j\in J} \eta(A_j)$ for every family $\{A_j | j \in J\} \subseteq L^X$.

If η is an L-fuzzy topology on X, then we say the pair (L^X, η) is an L-fuzzy topological space (L-Ftop, for short). The value $\eta(A)$ can be interpreted as the degree of openness of $A \in L^X$. A continuous mapping between two L-Ftops (L^X, η) and (L^Y, δ) is a mapping $f : X \to Y$ such that $\eta(f_L^{\leftarrow}(B)) \geq \delta(B)$ for all $B \in L^Y$, where $f_L^{\leftarrow} : L^Y \to L^X$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$. $f : (L^X, \eta) \to (L^Y, \delta)$ is called a homeomorphism if and only if f is bijective and both f and f^{-1} are continuous.

Suppose that $\eta : L^X \to [0,1]$ is an L-fuzzy topology. Let $Q_e^\eta : L^X \to [0,1]$ be defined as follows:

$$Q_e^\eta(A) = \begin{cases} \bigvee_{eqB\leq A} \eta(B), & eqA, \\ 0, & e\neg qA. \end{cases}$$

for $e \in M(L^X)$ and $A \in L^X$, where eqA denotes $e \not\leq A'$. When $e \in M(L^X)$, we know that there exist $x \in X$ and $\lambda \in M(L)$ such that $e = x_\lambda$, where $x_\lambda \in L^X$ is defined by

$$x_\lambda(y) = \begin{cases} \lambda, & y = x, \\ 0, & \text{others.} \end{cases}$$

Hence, $e \not\leq A'$ means $x_\lambda \not\leq A'$, this is to say $\lambda \not\leq A'(x) = (A(x))'$. The set $Q = \{Q_e^\eta | e \in M(L^X)\}$ is called the induced fuzzy quasi-coincident neighborhood system by η. The value $Q_e^\eta(A)$ can be interpreted as the degree to which A is a quasi-coincident neighborhood of e. If no confusion arise, we omit the superscript η of Q_e^η.

Lemma 1.2[3] ($L = [0,1]$). $Q = \{Q_e | e \in M(L^X)\}$ defined above satisfied the following results:

(1) $Q_e(1_X) = 1$ and $Q_e(0_X) = 0$;
(2) $Q_e(A) > 0 \Rightarrow eqA$;
(3) $Q_e(A \wedge B) = Q_e(A) \wedge Q_e(B)$;
(4) $Q_e(A) = \bigvee\limits_{eqB\leq A} \bigwedge\limits_{aqB} Q_a(B)$;

(5) $\eta(A) = \bigwedge_{eqA} Q_e(A)$.

Definition 1.3[11] ($L = [0,1]$). Let (L^X, η) be an L-fuzzy topological space . If $\eta(A) = \inf_{r \in P(L)} \eta(1_{\sigma_r(A)})$ for all $A \in L^X$, then (L^X, η) is called an induced L-fuzzy topological space, where $\sigma_r(A) = \{x | A(x) \not\leq r\}$. If $\eta(\bar{\lambda}) = 1$ for all $\lambda \in L$, where $\bar{\lambda}$ is the constant function from X to L, then (L^X, η) is called a stratified L-fuzzy topological space.

Definition 1.4[11] ($L = [0,1]$). Let (L^X, η) be an L-fuzzy topological space on X.

(1) Define $[\eta] : 2^X \to [0,1]$ by $[\eta](U) = \eta(1_U)$. $[\eta]$ is called the fuzzifying background space of (L^X, η).

(2) Define $\phi_\eta : 2^X \to [0,1]$ by $\phi_\eta(U) = \sup_{r \in P(L)} \sup\{\eta(B) | B \in L^X, \sigma_r(B) = U\}$ for $U \in P(X)$. Then ϕ_η is the subbase of one fuzzifying topology and denote this fuzzifying topology by $\iota(\eta)$.

Lemma 1.5[11] ($L = [0,1]$). Let (X, τ) be a fuzzifying topological spaces. Then $\omega(\tau) : L^X \to [0,1]$ defined by $\omega(\tau)(A) = \inf_{r \in P(L)} \tau(\sigma_r(A))$ for $A \in L^X$ is an L-fuzzy topology on X.

Definition 1.6[10]**.** Let Γ be the class of fuzzifying topological spaces. A fuzzy unary predicate $I \in \mathcal{F}(\Gamma)$, called fuzzy connection, is given as follows:

$I(X,\tau) := \neg(\exists U)(\exists V)(S(U,V) \wedge (U \neq \emptyset) \wedge (C \neq \emptyset) \wedge (U \vee V = X))$,

i.e.,

$[I(X,\tau)] = 1 - \bigvee_{U,V \neq \emptyset,\ U \vee V = X} S(U,V) = 1 - \bigvee_{U \in \mathcal{P}(X) - \{\emptyset, X\}} \tau(U) \wedge \tau(U^c)$,

2. L-fuzzy connectivity

Definition 2.1. Let Σ denote all L-fuzzy topological spaces. A fuzzy unary predicate $Con \in \mathcal{F}(\Sigma)$, called L-fuzzy connection, is given as follows:

$Con(L^X, \eta) :=$

$\neg(\exists B)(\exists C)((B \in \eta) \wedge (C \in \eta) \wedge (B \neq 0_X) \wedge (C \neq 0_X) \wedge (B \vee C = 1_X) \wedge (B \wedge C = 0_X)),$

Hence

$$[Con(L^X, \eta)] = 1 - \bigvee_{(B,C) \in \mathcal{D}} \eta(B) \wedge \eta(C),$$

where $\mathcal{D} = \{(B, C) \in L^X \times L^X |\ B \neq 0_X,\ C \neq 0_X,\ B \vee C = 1_X$ and $B \wedge C = 0_X\}$.
The true value of $Con(L^X, \eta)$ can be interperted as the degree to which (L^X, η) is L-fuzzy connected.

Remark 2.2. It is easy to check that $[Con(L^X, \eta)] = 1 - \bigvee_{(B,C) \in \mathcal{D}} \eta(B') \wedge \eta(C')$. In Definition 2.1, if η is an Chang L-topology , then this definition is just the connectivity in [6]. When $L = \{0, 1\}$, Definition 2.1 will reduces to Defintion 1.6.

Rodabaugh [7] introduced a kind of connectivity in L-topological spaces. Let (L^X, η) be an L-fuzzy topological space. If we generalized Rodabaugh's connectivity for the L-Ftop setting as follows:

$RCon(L^X, \eta) :=$
$\neg(\exists B)(\exists C)((B \in \eta) \wedge (C \in \eta) \wedge (B \neq 0_X) \wedge (C \neq 0_X) \wedge (B \vee C > 0_X) \wedge (B \wedge C = 0_X)),$

i.e.,

$$[RCon(L^X, \eta)] = 1 - \bigvee_{(B,C) \in \mathcal{D}_{\mathcal{R}}} \eta(B) \wedge \eta(C),$$

where $\mathcal{D}_{\mathcal{R}} = \{(B, C) \in L^X \times L^X |\ B \neq 0_X,\ C \neq 0_X,\ B \vee C > 0_X$ and $B \wedge C = 0_X\}$.

From the generalization above, it is easy to check that $\models RCon(L^X, \eta) \to Con(L^X, \eta)$. We now see an example.

Example 2.3. Let $X = \{x, y\}$ and $L = [0, 1]$. Define B by $B(x) = \frac{1}{2}$ and $B(y) = 0$, and define C by $C(y) = \frac{1}{2}$ and $C(x) = 0$, respectively. Let

$\eta : L^X \to [0,1]$ be defined as follows:

$$\eta(A) = \begin{cases} 1, & A \in \{0_X,\ 1_X,\ \overline{\tfrac{1}{2}}\}, \\ \frac{1}{2}, & A \in \{B,\ C\}, \\ 0, & \text{others}. \end{cases}$$

Then η is an L-fuzzy topology on X. It is easy to verify that $[RCon(L^X, \eta)] = 1/2$ and $[Con(L^X, \eta)] = 1$.

Theorem 2.4. Let (L^X, η) be an L-fuzzy topological space. Then we have $\models Con(L^X, \eta) \to I(X, [\eta])$. Furthermore, if $1 \in M(L)$, then $\models Con(L^X, \eta) \leftrightarrow I(X, [\eta])$.

Proof. It needs to prove $[Con(L^X, \eta)] \leq [I(X, [\eta])]$, i.e., $\bigvee_{(B,C)\in\mathcal{D}} \eta(B) \wedge \eta(C) \geq \bigvee_{U\in\mathcal{P}(X)-\{\emptyset,X\}} [\eta](U) \wedge [\eta](U^c)$. Let $U \in \mathcal{P}(X) - \{\emptyset, X\}$. Then $1_U \vee 1_{U^c} = 1_X$ and $1_U \wedge 1_{U^c} = 0_X$. From the definition of $[\eta]$, we have

$$[\eta](U) \wedge [\eta](U^c) = \eta(1_U) \wedge \eta(1_{U^c}) \leq \bigvee_{(B,C)\in\mathcal{D}} \eta(B) \wedge \eta(C).$$

Therefore, $\bigvee_{U\in\mathcal{P}(X)-\{\emptyset,X\}} [\eta](U) \wedge [\eta](U^c) \leq \bigvee_{(B,C)\in\mathcal{D}} \eta(B) \wedge \eta(C)$.

In order to prove $[Con(L^X, \eta)] = [I(X, [\eta])]$, it suffices to show that $[Con(L^X, \eta)] \geq [I(X, [\eta])]$. This is to say $\bigvee_{(B,C)\in\mathcal{D}} \eta(B) \wedge \eta(C) \leq \bigvee_{U\in\mathcal{P}(X)-\{\emptyset,X\}} [\eta](U) \wedge [\eta](U^c)$. Let $(B, C) \in \mathcal{D}$. Since $1 \in M(L)$, we can get that $B = 1_{\text{supp}B}$ and $C = 1_{\text{supp}C}$. Hence $\text{supp}B \cap \text{supp}C = \emptyset$ and $\text{supp}B \cup \text{supp}C = X$. Therefore, $\eta(B) \wedge \eta(C) = \eta(1_{\text{supp}B}) \wedge \eta(1_{\text{supp}C}) = [\eta](\text{supp}B) \wedge [\eta](\text{supp}C) \leq \bigvee_{U\in\mathcal{P}(X)-\{\emptyset,X\}} [\eta](U) \wedge [\eta](U^c)$, as desired.

Theorem 2.5. Let (L^X, η) be an L-fuzzy topological space. If $1 \in M(L)$, then $\models I(X, \iota(\eta)) \to Con(L^X, \eta)$.

Proof. This can be obtained by $[\eta] \leq \iota(\eta)$ and Theorem 2.4.

Corollary 2.6. Let (X, τ) be a fuzzifying topological space. If $1 \in M(L)$, then $\models Con(L^X, \omega(\tau)) \leftrightarrow I(X, \tau)$.

Remark 2.7. In particular, if L is the unit interval $[0,1]$, then we have

$$\models Con([0,1]^X,\eta) \leftrightarrow I(X,[\eta]);$$

$$\text{Con }([0{,}1]^X,\omega(\tau)) \leftrightarrow I(X,\tau);\ I(X,\iota(\eta)) \rightarrow Con([0,1]^X,\eta).$$

If $1 \notin M(L)$, Theorem 2.4, 2.5 and Corollary 2.6. are not necessary valid. Now we see an example.

Example 2.8. Let $L = \{0,a,b,1\}$ be the diamond lattice, i.e, $a \vee b = 1$, $a \wedge b = 0$, $a' = b$ and $b' = a$, and let X be any nonempty set. Define $\tau : 2^X \rightarrow [0,1]$ by

$$\tau(U) = \begin{cases} 1, & U \in \{\emptyset, X\}, \\ 0, & \text{others.} \end{cases}$$

It is easy to verify that

$$\omega(\tau)(A) = \begin{cases} 1, & A \in \{0_X, 1_X, \bar{a}, \bar{b}\}, \\ 0, & \text{others.} \end{cases}$$

where $\bar{a}$ and $\bar{b}$ denote the constant mapping from X to L taking the value a and b, respectively. We know that $\bar{a} \wedge \bar{b} = 0_X$ and $\bar{a} \vee \bar{b} = 1_X$. Hence $[Con(L^X,\omega(\tau))] = 0$, but $[I(X,\tau)] = 1$.

Theorem 2.9. If $f : (L^X,\eta) \rightarrow (L^Y,\delta)$ is a continuous mapping, then $\models Con(L^X,\eta) \rightarrow Con(L^Y,\delta)$.

Proof. It suffices to show $[Con(L^X,\eta)] \leq [Con(L^Y,\delta)]$, i.e., $\bigvee_{(A,B)\in\mathcal{D}_\mathcal{X}} \eta(A) \wedge \eta(B) \geq \bigvee_{(C,D)\in\mathcal{D}_\mathcal{Y}} \delta(C) \wedge \delta(D)$. Let $(C,D) \in \mathcal{D}_\mathcal{Y}$ and define $A^* = f_L^{\leftarrow}(C)$ and $B^* = f_L^{\leftarrow}(D)$. Then we have $(A^*,B^*) \in \mathcal{D}_\mathcal{X}$. Since $f : (L^X,\eta) \rightarrow (L^Y,\delta)$ is continuous, $\delta(C) \leq \eta(A^*)$ and $\delta(D) \leq \eta(B^*)$. Therefore, $\delta(C) \wedge \delta(D) \leq \eta(A^*) \wedge \eta(B^*) \leq \bigvee_{(A,B)\in\mathcal{D}_\mathcal{X}} \eta(A) \wedge \eta(B)$. Hence $\bigvee_{(A,B)\in\mathcal{D}_\mathcal{X}} \eta(A) \wedge \eta(B) \geq \bigvee_{(C,D)\in\mathcal{D}_\mathcal{Y}} \delta(C) \wedge \delta(D)$, as desired.

Corollary 2.10. If $f : (L^X,\eta) \rightarrow (L^Y,\delta)$ is a homeomorphism, then $\models Con(L^X,\eta) \leftrightarrow Con(L^Y,\delta)$.

Corollary 2.11. Let $\{(L^{X_t}, \eta_t)\}_{t\in T}$ be a family of L-fuzzy topological spaces and (L^X, η) be the product space of $\{(L^{X_t}, \eta_t)\}_{t\in T}$. Then

$$\models Con(L^X, \eta) \to (\forall t \in T)(Con(L^{X_t}, \eta_t)).$$

Example 2.12. We consider the I-fuzzy unit interval $I(I)$ in I-topological spaces. For details about $I(I)$, please refer to [6]. It can be also regarded as I-fuzzy topology according to the characteristic function, i.e.,

$$I(I)(A) = \begin{cases} 1 & A \in I(I), \\ 0 & A \notin I(I). \end{cases}$$

The readers can easily check $[Con(I^X, I(I))] = 1$.

3. K. Fan's theorem

As is well know, in L-topology, there is a theorem, named K. Fan's theorem, which describes connectivity in a geometric manner. According to K. Fan's theorem, a Chang L-topological space (L^X, δ) is connected if and only if $\forall f : M(L^X) \to L^X$ with the property that $f(e)$ is a quasi-coincident neighborhood of e for all $e \in M(L^X)$, there is a finite subset $\{e_1, e_2, ..., e_n\} \subseteq M(L^X)$ such that

$$e_1 = a,\ e_n = b \text{ and } f(e_i) \wedge f(e_{i+1}) \neq 0_X,\ i = 1, 2, ..., n-1$$

whenever $a, b \in M(L^X)$ are fixed. In this section, we will generalize K. Fan's theorem to L-fuzzy topology. At first, we introduce some definitions.

Definition 3.1. Let (L^X, η) be an L-fuzzy topological space and let Ξ denote all mappings from $M(L^X)$ to L^X. A unary predicate $M \in \mathcal{F}(\Xi)$, called fuzzy quasi-coincident neighborhood map, is defined as follows:

$$\forall f \in \Xi,\ M(f) := (\forall e \in M(L^X))(f(e) \in Q_e).$$

Intuitively, the degree to which f is a fuzzy quasi-coincident neighborhood map is

$$[M(f)] = \bigwedge_{e \in M(L^X)} Q_e(f(e)).$$

Definition 3.2. (1) Let (L^X, η) be an L-fuzzy topological space. A unary predicate $P \in \mathcal{F}(M(L^X) \times M(L^X))$, called L-fuzzy point-connection, is defined as follows:

$P(a, b) := (\forall f)(M(f) \rightarrow$
$(\exists\{e_1, e_2, ..., e_n\} \subseteq M(L^X)((e_1 = a) \wedge (e_n = b) \wedge \bigwedge_{i=1}^{i=n-1}(f(e_i) \wedge f(e_{i+1}) \neq 0_X))$.

This is to say the degree to which a and b are connective is

$$[P(a, b)] = \bigwedge_{f \in \Xi} min\{1, 1 - [M(f)] + \sup_{\{e_i\}_{i=1}^{i=n}}^{e_1=a, e_n=b} \bigwedge_{i=1}^{i=n-1} [f(e_i) \wedge f(e_{i+1}) \neq 0_X]\},$$

where $\{e_i\}_{i=1}^{i=n} = \{e_1, e_2, ..., e_n\} \subseteq M(L^X)$.

(2) A unary predicate $K \in \mathcal{F}(\Sigma)$, called K. Fan connection, is defined as follows:

$$K(L^X, \eta) := (\forall (a, b) \in M(L^X) \times M(L^X))(P(a, b)),$$

i.e., the degree to which (L^X, η) is K. Fan connection is

$$[K(L^X, \eta)] = \bigwedge_{(a,b) \in M(L^X) \times M(L^X)} [P(a, b)].$$

Theorem 3.3 (K. Fan's theorem). For any $(L^X, \eta) \in \Sigma$, $\models K(L^X, \eta) \leftrightarrow Con(L^X, \eta)$.

Proof. According to Łukasiewicz logic, we need to show the truth value equality: $[K(L^X, \eta)] = [Con(L^X, \eta)]$. At first, we want to show $[K(L^X, \eta)] \leq [Con(L^X, \eta)]$. Let $\alpha > [Con(L^X, \eta)]$. By the definition of $[Con(L^X, \eta)]$, there exists $(B, C) \in \mathcal{D}$ such that $1 - \eta(B) \wedge \eta(C) < \alpha$, i.e., $\eta(B) > 1 - \alpha$ and $\eta(C) > 1 - \alpha$. Define $f_0 : M(L^X) \rightarrow L^X$ as follows:

$$f_0(e) = \begin{cases} B, & e \leq C', \\ C, & e \leq B'. \end{cases}$$

Then we have

$$\begin{aligned} Q_e(f_0(e)) &= \begin{cases} Q_e(B), & e \leq C', \\ Q_e(C), & e \leq B'. \end{cases} \\ &\geq \begin{cases} \eta(B), & e \leq C', \\ \eta(C), & e \leq B'. \end{cases} \\ &> 1 - \alpha \end{aligned}$$

Hence $[M(f_0)] = \bigwedge_{e\in M(L^X)} Q_e(f_0(e)) \geq 1-\alpha$, i.e., $1-[M(f_0)] \leq \alpha$. Since $B' \neq 0_X$ and $C' \neq 0_X$, we can take $a \in M(L^X)$ and $b \in M(L^X)$ such that $a \leq B'$ and $b \leq C'$. Since $\sup_{\{e_i\}_{i=1}^{i=n}}^{e_1=a,e_n=b} \bigwedge_{i=1}^{i=n-1} [f_0(e_i) \wedge f_0(e_{i+1}) \neq 0_X] \in \{0,1\}$, we can assert that

$$\sup_{\{e_i\}_{i=1}^{i=n}}^{e_1=a,e_n=b} \bigwedge_{i=1}^{i=n-1} [f_0(e_i) \wedge f_0(e_{i+1}) \neq 0_X] = 0.$$

In fact, let $\{e_1, e_2, ..., e_n\} \subseteq M(L^X)$ be any finite set with the property $e_1 = a$ and $e_n = b$, and let $i_0 = \max\{i \leq n | e_i \leq B'\}$.

Then we have $i_0 \leq n-1$ and $e_{i_0+1} \leq C'$. By the definition of f_0, we have $f_0(e_{i_0}) = C$ and $f_0(e_{i_0+1}) = B$. Hence $f_0(e_{i_0}) \wedge f_0(e_{i_0+1}) = C \wedge B = 0_X$. Thus $[f_0(e_{i_0}) \wedge f_0(e_{i_0+1}) \neq 0_X] = 0$. Therefore,

$$\sup_{\{e_i\}_{i=1}^{i=n}}^{e_1=a,e_n=b} \bigwedge_{i=1}^{i=n-1} [f_0(e_i) \wedge f_0(e_{i+1}) \neq 0_X] = 0,$$

as desired. So

$$\begin{aligned}
[K(L^X,\eta)] &= \bigwedge_{(c,d)\in M(L^X)\times M(L^X)} [P(c,d)] \leq [P(a,b)] \\
&= \bigwedge_{f\in\Xi} \min\{1, 1-[M(f)] + \sup_{\{e_i\}_{i=1}^{i=n}}^{e_1=a,e_n=b} \bigwedge_{i=1}^{i=n-1} [f(e_i) \wedge f(e_{i+1}) \neq 0_X]\} \\
&\leq \min\{1, 1-[M(f_0)] + \sup_{\{e_i\}_{i=1}^{i=n}}^{e_1=a,e_n=b} \bigwedge_{i=1}^{i=n-1} [f_0(e_i) \wedge f_0(e_{i+1}) \neq 0_X]\} \\
&= 1-[M(f_0)] \leq \alpha.
\end{aligned}$$

We complete the proof of $[K(L^X,\eta)] \leq [Con(L^X,\eta)]$ from the arbitrariness of α.

Secondly, we prove that $[K(L^X,\eta)] \geq [Con(L^X,\eta)]$. If $[K(L^X,\eta)] = 1$, then $[K(L^X,\eta)] \geq [Con(L^X,\eta)]$ is obvious. We assume that $[K(L^X,\eta)] < 1$. Let $[K(L^X,\eta)] < \alpha < 1$. Then there exist $(a,b) \in M(L^X)\times M(L^X)$ and $f_0 : M(L^X) \to L^X$ such that

$$\min\{1, 1-[M(f)] + \sup_{\{e_i\}_{i=1}^{i=n}}^{e_1=a,e_n=b} \bigwedge_{i=1}^{i=n-1} [f_0(e_i) \wedge f_0(e_{i+1}) \neq 0_X]\} < \alpha$$

This is to say that

$$1-[M(f_0)]+\sup_{\{e_i\}_{i=1}^{i=n}}^{e_1=a,e_n=b}\bigwedge_{i=1}^{i=n-1}[f_0(e_i)\wedge f_0(e_{i+1})\neq 0_X]<\alpha.$$

Hence we have

$$\sup_{\{e_i\}_{i=1}^{i=n}}^{e_1=a,e_n=b}\bigwedge_{i=1}^{i=n-1}[f_0(e_i)\wedge f_0(e_{i+1})\neq 0_X]=0$$

and $1-[M(f_0)]<\alpha$. In the following, we will call $c,d\in M(L^X)$ jointed by f_0 if

$$\sup_{\{e_i\}_{i=1}^{i=n}}^{e_1=c,e_n=d}\bigwedge_{i=1}^{i=n-1}[f_0(e_i)\wedge f_0(e_{i+1})\neq 0_X]=1.$$

Obviously, a and b are not jointed by f_0. Now we set

$$\mathcal{B}=\{e\in M(L^X)|\ a \text{ and } e \text{ can be jointed by } f_0\}$$

and

$$\mathcal{C}=\{e\in M(L^X)|\ a \text{ and } e \text{ can not be jointed by } f_0\}.$$

Let $B=\bigvee_{e\in\mathcal{B}}e$ and $C=\bigvee_{e\in\mathcal{C}}e$. It is obvious that $a\leq B$, $b\leq C$ and $B\vee C=1_X$. We can also assert that $B\wedge C=0_X$. In fact, if $B\wedge C\neq 0_X$, take $z\in M(L^X)$ such that $z\leq B\wedge C$. Clearly, $z\leq B$ and $z\leq C$. Since $1-[M(f_0)]<\alpha$, i.e., $[M(f_0)]=\bigwedge_{e\in M(L^X)}Q_e(f_0(e))>1-\alpha$, we have $Q_z(f_0(z))>1-\alpha>0$. Therefore, from Lemma 1.2 (2), we have $zqf_0(z)$, i.e., $z\not\leq(f_0(z))'$.

Hence $B\not\leq(f_0(z))'$. Then there exists $d\in\mathcal{B}$ such that $d\not\leq(f_0(z))'$. By $d\not\leq(f_0(d))'$, we obtain $d\not\leq(f_0(z))'\vee(f_0(d))'$. Hence $f_0(d)\wedge f_0(z)\neq 0_X$. Thus, we can get a and z can be jointed by f_0 since d and a can be jointed by f_0. Similarly, since $z\leq C$, there exists $m\in\mathcal{C}$ such that $m\not\leq(f_0(z))'\vee(f_0(m))'$. Then $f_0(z)\wedge f_0(m)\neq 0_X$. Therefore, m and z can be jointed by f_0. Thus m and a can be jointed by f_0 since z and a can be jointed by f_0. Therefore, $m\in\mathcal{B}$. This is a contradiction to $m\in\mathcal{C}$. So $B\wedge C=0_X$, as desired.

For B and C defined above, we want to prove $\eta(B')\geq 1-\alpha$ and $\eta(C')\geq 1-\alpha$. If not, then $\eta(B')<1-\alpha$ or $\eta(C')<1-\alpha$. For convenience, we assume that $\eta(B')<1-\alpha$. From Lemma 1.2 (5), we have $\eta(B')=\bigwedge_{eqB'}Q_e(B')<1-\alpha$. Then there exists $e\in M(L^X)$ such that eqB' and $Q_e(B')<1-\alpha$. Since $Q_e(f_0(e))>1-\alpha$, we know that $f_0(e)\not\leq B'$, i.e., $B\not\leq(f_0(e))'$.

Hence there exists $z \in \mathcal{B}$ such that $z \not\leq (f_0(e))'$. Moreover $z \not\leq (f_0(e))' \vee (f_0(z))'$. Thus e and a can be jointed by f_0. Therefore, $e \leq B$. This is contradict to eqB'. So $\eta(B') \geq 1 - \alpha$ and $\eta(C') \geq 1 - \alpha$, i.e., $\eta(B') \wedge \eta(C') \geq 1 - \alpha$. Hence $[Con(L^X, \eta)] \leq \alpha$. From the arbitrariness of α, we have $[K(L^X, \eta)] \geq [Con(L^X, \eta)]$. Thus the conclusion.

Question 3.4. In Theorem 2.4–Remark 2.7, we study some relationships on between $Con(L^X, \eta)$ and $I(X, \tau)$. We do not know whether there are some relationships between $K(L^X, \eta)$ defined in this paper and $K(X, \tau)$ defined in [2] and we leave it as an open question.

4. Conclusions

In this paper, we offer an application of Łukasiewicz logic to L-fuzzy topology. We introduce generalized connectivity in L-fuzzy topological spaces and prove K.Fan's theorem. One thing we want to point out that L-fuzzy connectivity defined in this paper is for the whole L-fuzzy topological space not for an arbitray fuzzy subsets. The K. Fan theorem gives us one approach to difine generalized connectivity for an arbitray fuzzy subsets in L-fuzzy topological space.

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References

[1] C. L. Chang, Fuzzy topological spaces, J.Math.Anal.Appl. 24, pp. 182–193, (1968).

[2] J. Fang, Y. Yue, K. Fan's theorem in fuzzifying topology, Information Sciences, 162, pp. 139-146, (2004).

[3] J. Fang, I-FTOP is isomorphic to I-FQN and I-AITOP, Fuzzy Sets and Systems 147, pp. 317–325, (2004).

[4] U. Höhle, Uppersemicontinuous fuzzy sets and applications, J. Math. Anal. Appl. 78, pp. 659–673, (1980).

[5] T. Kubiak, On fuzzy topologies (PhD Thesis, Adam Mickiewicz, Poznan (Poland), (1985).

[6] Y. M. Liu, M. K. Luo, Fuzzy Topology, World Scientific Publishing Co.Pte.Ltd, Singapore, (1997).

[7] S.E. Rodabaugh, Connectivity and the L-fuzzy unit interval, Rocky Mount. J. Math 12(1), pp. 113-121, (1982).

[8] A. P. Šostak, on a fuzzy topological structure, Rendiconti Circolo Matematico Palermo (Suppl. Ser. II) 11, pp. 89–103, (1985).

[9] M. Ying, A new approach to fuzzy topology (I), Fuzzy Sets and Systems 39, pp. 303–321, (1991).

[10] M. Ying, A new approach to fuzzy topology (II), Fuzzy Sets and Systems 47, pp. 221–232, (1992).

[11] Y. Yue, J. Fang, Generated I-fuzzy topological spaces, Fuzzy Sets and Systems, 154, pp. 103-117, (2005).

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