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A SPECTRAL EXPANSION FOR SCHRÖDINGER OPERATOR

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Abstract

In this paper we consider the Schrödinger operator L generated in $L^2(\mathbf{R}_+)$ *by*

$y'' + q(x)\, y = \mu y, \quad x \in \mathbf{R}_+ := [0, \infty)$

subject to the boundary condition

$y'(0) - hy(0) = 0,$

where,q *is a complex valued function summable in* $[0, \infty$ *and* $h \neq 0$ *is a complex constant,* μ *is a complex parameter. We have assumed that*

$\sup_{x \in \mathbf{R}_+} \{\exp(\varepsilon\sqrt{x})\, |q(x)|\} < \infty, \ \ \varepsilon > 0,$

holds which is the minimal condition that the eigenvalues and the spectral singularities of the operator L *are finite with finite multiplicities. Under this condition we have given the spectral expansion formula for the operator* L *using an integral representation for the Weyl function of* L*. Moreover we also have investigated the convergence of the spectral expansion.*

AMS *Subject Class : 47A10*

Key Words : *Spectrum, Weyl Function, Spectral Expansion.*

1. Introduction

Let L denote the operator generated in $L^2(\mathbf{R}_+)$ by differential equation

$$-y'' + q(x)\,y = \mu y, \quad x \in \mathbf{R}_+$$

(1.1)

and the boundary condition

$$y'(0) - hy(0) = 0, \tag{1.2}$$

here q is a complex valued function, and $h \neq 0$ is a complex number, μ is a complex parameter. The spectral analysis of L was first investigated by Naimark [6].In this paper, he has proved that some of the poles of the resolvent′s kernel of L are not the eigenvalues of the operator. Moreover he has proved that these poles are on the continuous spectrum. (Schwartz named these poles as spectral singularities [7]. Furthermore Naimark has shown that if the condition

$\int\limits_0^\infty e^{\varepsilon x}\,|q(x)|\,dx < \infty, \quad \varepsilon > 0$(1.3)

holds, then L has a finite number of eigenvalues and spectral singularities each of them is of finite multiplicity.

The effect of spectral singularities in the spectral expansion of the operator L in terms of principal functions under the condition (1.3) has been investigated in [3].

The spectral expansion in terms of principal vectors for the non-selfadjoint discrete Dirac operatos has been given in [1] using the integral representation for the Weyl function of the operator.

Suppose that the condition

$$\sup_{x\in\mathbf{R}_+}\left\{\exp\left(\varepsilon\sqrt{x}\right)|q(x)|\right\} < \infty, \quad \varepsilon > 0, \tag{1.4}$$

holds. Under the condition (1.4) the operator L has a finite number of eigenvalues and spectral singularities each of them with finite multiplicity (see [2]) .

In this paper, using the similar technique used in [1], we have obtained the spectral expansion of the operator L using an integral representation for the Weyl function of L under the condition (1.4). Moreover we have investigated the convergence of the spectral expansion.

2. Preliminaries

We denote the solutions of (1.1) satisfying the initial conditions

$$w\left(0,\sqrt{\mu}\right)=1,\ w'\left(0,\sqrt{\mu}\right)=h,$$

$$\varphi\left(0,\mu\right)=1,\ \varphi'\left(0,\mu\right)=0, \tag{2.1}$$

$$\theta\left(0,\mu\right))=0,\ \theta'\left(0,\mu\right))=-1$$

by $w\left(x,\sqrt{\mu}\right),\ \varphi\left(x,\mu\right),\ \theta\left(x,\mu\right),$ respectively, the set of all finite functions in $L^2\left(\mathbf{R}_+\right)$ by C_0^∞ and the $w-$Fourier transform of an arbitrary element $f\ \in C_0^\infty$ by $E_f\left(\sqrt{\mu}\right)=\int\limits_0^\infty f\left(x\right)w\left(x,\sqrt{\mu}\right)dx$. Moreover $\mathbf{C}$ denotes the complex plane. Marchenko has obtained an analytical expression of the generalized spectral function of the operator L by using Weyl function [4].

We give some important results which we need.

Theorem 2.1 ([4]) . *Suppose the potential of the equation* (1.1) *satisfies the condition*

$$r_0\leq \operatorname{Im} q\leq r_1. \tag{2.2}$$

Then (1.1) *has a unique solution given by*

$$\psi\left(x,\mu\right)=\theta\left(x,\mu\right)+m\left(\mu\right)w\left(x,\sqrt{\mu}\right)\in L^2\left(\mathbf{R}_+\right) \tag{2.3}$$

for all μ *which is not in the strip*

$$r_0\leq \operatorname{Im}\mu\leq r_1, \tag{2.4}$$

where $m\left(\mu\right)$ *is analytic outside the strip* (2.4) .

The function $m\left(\mu\right)$ given in Theorem 2.1 is the Weyl function, and the function $\psi\left(x,\mu\right)$ is the Weyl solution of the operator L. Weyl function and Weyl solution may be obtained as follows:

$$\psi_1\left(x,\mu\right)=\frac{\psi\left(x,\mu\right)}{1+hm\left(\mu\right)},\quad m_1\left(\mu\right)=\frac{m\left(\mu\right)}{1+hm\left(\mu\right)}, \tag{2.5}$$

moreover $m\left(\mu\right)$ satisfies

$$\lim_{\left|\mu\right|\to\infty} m\left(\mu\right)=0 \tag{2.6}$$

outside the strip (2.4) ([4]) .

Theorem 2.2 ([4]) . *If the condition* (2.2) *holds, then for all* $f \in C_0^\infty$, *the generalized spectral function* R *of the operator* L *is given by the formula*

$$(R, E_f(\sqrt{\mu})) = \frac{1}{2\pi i} \lim_{\delta \to 0} \left\{ \int_{-\infty + i(r_1+\varepsilon)}^{\infty + i(r_1+\varepsilon)} e^{-\delta\mu^2} m_1(\mu) E_f(\sqrt{\mu}) \, d\mu \right.$$

$$\left. + \int_{\infty + i(r_0-\varepsilon)}^{-\infty + i(r_0-\varepsilon)} e^{-\delta\mu^2} m_1(\mu) E_f(\sqrt{\mu}) \, d\mu \right\} \tag{2.7}$$

$$- \sum [m_1(\mu) E_f(\sqrt{\mu})],$$

where $\varepsilon > 0$ *is a sufficiently small number and residues are computed for all poles of* $m_1(\mu)$ *which is not in the strip* $r_0 - \varepsilon \le \operatorname{Im} \mu \le r_1 + \varepsilon$.

It is known that to every boundary value problem (1.1) – (1.2) there corresponds a functional
$R \in Z'$, f, $g \in C_0^\infty$, such that

$$(R, E_f(\sqrt{\mu}) E_g(\sqrt{\mu})) = \int_0^\infty f(x) g(x) \, dx, \tag{2.8}$$

where Z' denotes the space of all linear, continuous functionals defined on Z which is the set of all even, entire functions of exponential type, summable on $\mathbf{R}$. (2.8) is called Marchenko-Parseval equality, and R is the generalized spectral function of L.

Theorem 2.3. ([5]) . *Suppose that the condition*

$$\int_0^\infty x |q(x)| \, dx < \infty \tag{2.9}$$

holds. If $\mu = \lambda^2$ *then the equation* (1.1) *has the solution*

$$e(x, \lambda) = e^{i\lambda x} + \int_x^\infty K(x,t) e^{i\lambda t} dt \tag{2.10}$$

which is analytic in $\operatorname{Im}\lambda > 0$ *and continuous on* $\operatorname{Im}\lambda \geq 0$, satisfying $\lim_{x\to\infty} e(x,\lambda)\, e^{-i\lambda x} = 1$ for $\operatorname{Im}\lambda \geq 0$. *Furthermore* $K(x,t)$ *satisfies the inequality*

$|K(x,t)| \leq \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)\exp\sigma_1(x)$,

where $\sigma(x) = \int\limits_x^\infty |q(t)|\, dt,\ \sigma_1(x) = \int\limits_x^\infty t\,|q(t)|\, dt.$

Now let us denote the eigenvalues and the spectral singularities and continuous spectrum of L by $\sigma_d(L)$, $\sigma_{ss}(L)$ and $\sigma_c(L)$, respectively.

Clearly $\sigma_d(L) = \{\mu : \mu = \lambda^2,\ \operatorname{Im}\lambda > 0,\ d(\lambda) = 0\}$,

$\sigma_{ss}(L_\lambda) = \{\mu :\ \mu = \lambda^2,\ \lambda \in \mathbf{R},\ d(\lambda) = 0\ \}\setminus\{0\}$,

$\sigma_c(L) = [0,\infty,$

where

$$d(\lambda) = e_x(0,\lambda) - he(0,\lambda)\,, \tag{2.11}$$

[6]. Therefore we have the following.

Theorem 2.4. *If the condition* (1.4) *holds then the operator L has a finite number of eigenvalues and spectral singularities with finite multiplicities.*

For the proof see for ex. [2].

Let us introduce the following notations

$$\widehat{e}(x,\mu) = e(x,\sqrt{\mu})\,,\ \widehat{d}(\mu) = d(\sqrt{\mu})\,, \tag{2.12}$$

$\mu \in \mathbf{C}\setminus\mathbf{R}_+$ (i.e.: the complex plane except for nonnegative real numbers).

Remark 2.5. *If the condition* (1.4) *holds, then* $|q(x)|$ *will be bounded. From Theorem 2.1 Weyl's limit point case is satisfied for the operator L. So all solutions of* (1.1) *belonging to* $L^2(\mathbf{R}_+)$ *for* μ *outside the strip* (2.4) *are linearly dependent with the solution* $\psi(x,\mu)$. *From* (2.5) $\psi_1(x,\mu) \in L^2(\mathbf{R}_+)$ *for* μ *outside the strip* (2.4) *and* $m(\mu) \neq \frac{-1}{h}$. *Therefore considering* (2.10) *and* (2.12) *we arrive at* $\widehat{e}(x,\mu) \in L^2(\mathbf{R}_+)$ *for* $\mu \in C\setminus R_+$. *So there exists a constant* $c \neq 0$ *such that*

$$\psi_1(x,\mu) = c\widehat{e}(x,\mu)\,, \tag{2.13}$$

moreover we get

$$m_1(\mu) = \frac{-\widehat{e}(\mu)}{\widehat{d}(\mu)} \tag{2.14}$$

by (2.3) *and* (2.13) *, where* $\widehat{e}(\mu) = \widehat{e}(0, \mu)$ *, for* μ *outside the strip* (2.4) *and* $m(\mu) \neq \frac{-1}{h}$.

Remark 2.6. *If the condition* (1.4) *holds, then we can omit the summing factor* $e^{-\delta\mu^2}$ *in* (2.7) *. In this case the function* $m_1(\mu)$ *is analytic and bounded in* $\mathbf{C}\backslash\mathbf{R}_+$ *except for the finite number of poles. So the formula* (2.7) *may be reduced into the form ([1])*

$$(R, E_f(\sqrt{\mu})) = -\frac{1}{2\pi i}\int_\gamma m_1(\mu) E_f(\sqrt{\mu})\, d\mu, \tag{2.15}$$

where γ *is the contour enclosing the spectrum of the operator* L*, and* $E_f(\sqrt{\mu})$ *is the* $w-$*Fourier transform of* $f \in C_0^\infty$*. Taking* $E_g(\sqrt{\mu})$ *as the* $w-$*Fourier transform of* $g \in C_0^\infty$*. We know that Parseval's equality is in the form of*

$$(R, E_f(\sqrt{\mu})\, E_g(\sqrt{\mu})) = -\frac{1}{2\pi i}\int_\gamma m_1(\mu) E_f(\sqrt{\mu})\, E_g(\sqrt{\mu})\, d\mu,$$

(2.16)

by the formulas (2.8) *and* (2.15) *. From* (2.12) , (2.14) and Theorem.2.4, $m_1(\mu)$ *can be extended as a finite meromorphic function from outside the strip* (2.4) *onto* $\mathbf{C}\backslash\mathbf{R}_+$*. By Theorem* 2.4 *we have that* L *has a finite number of spectral singularities;* $\mu_1, ...\mu_k$*, and eigenvalues;* $\mu_{k+1}, ..., \mu_m$ *each of them having finite multiplicity* $\alpha_1, ..., \alpha_k$ *and* $\alpha_{k+1}, ..., \alpha_m$*, respectively. From* (2.14) *, if* (1.4) *holds* $m_1(\mu)$ *is a meromorphic function in* $\mathbf{C}\backslash\mathbf{R}_+$ *and has a finite number of poles* $\mu_1, ..., \mu_m$ *with finite multiplicities. In the neighborhood of the spectral singularities* $\mu_1, ..., \mu_k$*,* $m_1(\mu)$ *satisfies the conditions*

$$|m_1(u + iv)| \leq \frac{c_i}{|v|^{\alpha_i}} \tag{2.17}$$

for $a_i \leq u \leq a_{i+1}$*, where* $\mu = u + iv,\ \alpha_i,\ 0 < \alpha_i < \infty$*, is the multiplicity of* $\mu_i,\ i = 1, ..., k$*, and*

$$\bigcup_{i=1}^{k} [a_i, a_{i+1}] = [0, N]. \tag{2.18}$$

3. Integral Representation of $m_1(\mu)$.

To derive an expansion formula in terms of eigenfunctions of the operator L, for (2.16), we will transform the integration over γ enclosing the spectrum, to the integration over spectrum. In order to do this we need the integral representation of the function $m_1(\mu)$ similar to that of given in [1].

Let us suppose that the following conditions hold:

(a) the operator L has no eigenvalues,

(b) there exists only one spectral singularity of the operator L with finite multiplicity α.

Clearly $m_1(\mu)$ is an analytic function in $\mathbf{C}\backslash\mathbf{R}_+$ and in the neighborhood of the spectral singularity, the following condition holds

$$|m_1(\mu)| \le \frac{c}{|\operatorname{Im}\mu|^{\alpha}}, \tag{3.1}$$

for c is a constant and $0 < \alpha < \infty$.

Lemma 3.1. *If (a) and (b) are satisfied, then there exist continuous complex functions $\sigma_1(t)$ and $\sigma_2(t)$ defined on the segment $[0,N]$ and a function $n(\mu)$ analytic in $\mathbf{C}\backslash\mathbf{R}_+$, continuous up to the boundary for which*

$$m_1(\mu) = \int_0^N \frac{\sigma_1(t)}{\mu - t}dt + (\alpha+1)!\int_0^N \frac{\sigma_2(t)}{(\mu-t)^{\alpha+2}}dt + n(\mu). \tag{3.2}$$

Proof. Let us consider the following functions ;

$$M(\mu) = \begin{cases} m_1(\mu); & \mu \in [0,N] \\ 0\ ; & \mu \in (N,\infty) \\ \frac{m_1(\mu)}{2}; & \mu \in \mathbf{C}\backslash\mathbf{R}_+ \end{cases}, \tag{3.3}$$

$$n(\mu) = \begin{cases} 0; & \mu \in [0,N] \\ m_1(\mu); & \mu \in (N,\infty) \\ \frac{m_1(\mu)}{2}; & \mu \in \mathbf{C}\backslash\mathbf{R}_+ \end{cases}. \tag{3.4}$$

Therefore we can write the function $m_1(\mu)$ as follows:

$$m_1(\mu) = M(\mu) + n(\mu), \tag{3.5}$$

here $M(\mu)$ and $n(\mu)$ tend to zero as $|\mu|$ tends to infinity by (2.6), moreover $M(\mu)$ is analytic in $C\backslash[0,N]$ and in the neighborhood of the spectral singularity; $|M(\mu)| \leq \frac{c}{|\operatorname{Im}\mu|^{\alpha}}$ holds, here $c>0$ is a constant, $0<\alpha<\infty$. Therefore using the method similar to that of [1], we obtain the representation (3.2).

Now we assume that the following condition holds.

(c) the operator L has a finite number of spectral singularities each of them is of finite multiplicity.

Lemma 3.2. *If the conditions* (a) *and* (c) *are satisfied, then there exist continuous complex functions* $\rho_i(t)$, *defined on* $[a_i, a_{i+1}]$, $i=1,...,k$, *respectively, a piecewise continuous function* $\sigma(t)$ defined on $[0,N]$ and a function $n(\mu)$ *analytic in* $\mathbf{C}\backslash\mathbf{R}_+$, *continuous up to the boundary such that*

$$m_1(\mu) = \int_0^N \frac{\sigma(t)}{\mu - t}dt + \sum_{i=1}^{k}(\alpha_i+1)! \int_{a_i}^{a_{i+1}} \frac{\rho_i(t)}{(\mu-t)^{\alpha_i+2}}dt + n(\mu). \tag{3.6}$$

Proof. From $(3.3)-(3.5)$ proof can be obtained easily.

Finally we assume that the following condition holds.

(d) the operator L has a finite number of spectral singularities $\mu_i \in [a_i, a_{i+1}]$,
with multiplicities α_i, $0<\alpha_i<\infty$, $i=1,...,k$, and a finite number of eigenvalues μ_j, with multiplicities α_j, $0<\alpha_j<\infty$, $j=k+1,...,m$, respectively.

Now we arrive at the following.

Theorem 3.3. *If the condition* (1.4) *holds, then there exists continuous complex functions* $\rho_i(t)$ *defined on* $[a_i, a_{i+1}]$, $i=1,...,k$, *a piecewise continuous complex function* $\sigma(t)$ *defined on* $[0,N]$ *and a function* $n(\mu)$ *analytic in* $\mathbf{C}\backslash\mathbf{R}_+$, *continuous up to the boundary such that*

$$m_1(\mu) = \frac{1}{\Phi(\mu)}\int_0^N \frac{\sigma(t)}{\mu-t}dt + \sum_{i=1}^{k}\frac{(\alpha_i+1)!}{\Phi(\mu)}\int_{a_i}^{a_{i+1}}\frac{\rho_i(t)}{(\mu-t)^{\alpha_i+2}}dt + \frac{n(\mu)}{\Phi(\mu)}, \tag{3.7}$$

where

$$\Phi(\mu) = \begin{cases} \prod_{j=k+1}^{m} (\mu-\mu_j)^{\alpha_j}; & \mu \in C_R \\ 1; & \mu \in \mathbf{C}\backslash C_R \end{cases}, \tag{3.8}$$

and $C_R = \{\mu : |\mu| \le R\}$ *so that* $\mu_j \in C_R,\ j = k+1, ..., m.$

Proof. It is clear that under the condition (1.4) the operator L satisfies (d). From Lemma (3.1) and Lemma (3.2) we reach to the desired result.

4. Expansion in Terms of the Principal Functions

We will call the functions $w(x, \sqrt{\mu}),\ \mu > 0$, the principal functions of the continuous spectrum. Note that these functions do not belong to $L^2(\mathbf{R}_+)$. In order to find an expansion formula for finite functions in terms of principal functions of the operator L, we introduce the principal functions of the spectral singularities by

$$w^{(s)}(x, \sqrt{\mu_i}) = \left(\frac{d}{d\mu}\right)^s \{w(x, \sqrt{\mu})\}_{\mu=\mu_i}, \tag{4.1}$$

$s = 0, 1, ..., \alpha_i - 1,\ i = 1, ..., k,$ and principal functions of the discrete spectrum by

$$w^{(s)}(x, \sqrt{\mu_i}) = \left(\frac{d}{d\mu}\right)^s \{w(x, \sqrt{\mu})\}_{\mu=\mu_i} \in L^2(\mathbf{R}_+), \tag{4.2}$$

$s = 0, 1, ..., \alpha_i - 1,\ i = k+1, ..., m$, where $w(x, \sqrt{\mu_i})$ is an eigenfunction and $w^{(1)}(x, \sqrt{\mu_i}), ..., w^{(\alpha_i - 1)}(x, \sqrt{\mu_i})$ are associated functions corresponding to the eigenvalues $\mu_i,\ i = k+1, ..., m$. Obviously $w^{(s)}(x, \sqrt{\mu_i}) \notin L^2(\mathbf{R}_+),\ i = 1, ..., k,\ s = 0, 1, ..., \alpha_i - 1.$

Remark 4.1. *From Theorem* (2.3) *we easily get that*

$$\sup_{x \in \mathbf{R}_+} \frac{\left| w^{(s)}(x, \sqrt{\mu}) \right|}{(1+x)^s} < \infty, \quad \mu > 0, \quad s = 0, 1, 2, ...,$$

here $w^{(s)} = \left(\frac{\partial}{\partial \mu}\right)^{(s)} w.$

Let us recall the Hilbert spaces ([6])

$$H_n = \left\{ f : \int_0^\infty (1+x)^{2n} |f(x)|^2 \, dx < \infty \right\},$$
$$H_{-n} = \left\{ g : \int_0^\infty (1+x)^{-2n} |g(x)|^2 \, dx < \infty \right\}$$

with respective norms

$\|f\|_n^2 = \int_0^\infty (1+x)^{2n} |f(x)|^2 \, dx, \quad \|g\|_{-n}^2 = \int_0^\infty (1+x)^{-2n} |g(x)|^2 \, dx,$

$n = 0, 1, \ldots.$ Clearly

$\mathrm{H}_0 = L^2(\mathbf{R}_+), \quad H_n L^2(\mathbf{R}_+) H_{-n}, \quad n = 1, 2, \ldots.$

It is known that H_{-n} is isomorphic to the dual of H_n $\left(H_{-n} \sim H_n^{'}\right)$.

Let us choose s_0 such that $s_0 = \max\{\alpha_1, \ldots, \alpha_k\}$. In this case

$$H_{(s_0+2)} \underset{\neq}{\subset} L^2(\mathbf{R}_+) \underset{\neq}{\subset} H_{-(s_0+2)}. \tag{4.3}$$

We obtain for the principal functions of the spectral singularities that

$$\frac{\partial^s}{\partial \mu^2} \{w(x, \sqrt{\mu})\}_{\mu=\mu_i} \in H_{-(s+1)}$$

for $s = 0, 1, \ldots, \alpha_i - 1,\ i = 1, \ldots, k$, hence, we get that

$$\frac{\partial^s}{\partial \mu^2} \{w(x, \sqrt{\mu})\}_{\mu=\mu_i} \in H_{-s_0}$$

for $s = 0, 1, \ldots, \alpha_i - 1,\ i = 1, \ldots, k$, by the choice of s_0.

Now, we need the following two lemmas ([6]).

Lemma 4.2. *There is a number $c > 0$ such that for every finite function $f \in L^2(\mathbf{R}_+)$*

$\int_0^\infty \left|E_f(\sqrt{\mu})\right|^2 \sqrt{\mu} d\mu \leq c \int_0^\infty |f(x)|^2 \, dx$

holds.

Lemma 4.3. *If*

$$\int_0^\infty (1+x)^{2n} |f(x)|^2 \, dx < \infty$$

holds, then the function $E_f\left(\sqrt{\mu}\right)$ has a derivative of order $(n-1)$ which is absolutely continuous on every finite interval of the half axis $\mu>0$. There exists a number $c_n>0$ such that

$$\int\limits_{-\infty}^{\infty}\left|\left(\frac{d}{d\lambda}\right)^{n}\lambda E_f\left(\lambda\right)\right|^{2}d\lambda\leq c_n\int\limits_{0}^{\infty}\left(1+x\right)^{2n}\left|f\left(x\right)\right|^{2}dx$$

holds.

Let us suppose the following notations:
$\mathrm{H}_{\pm}:=H_{\pm(s_0+2)}$,

$$\Phi_j\left(\mu\right):=\prod_{l=k+1j\neq l}^{m}\left(\mu-\mu_l\right)^{\alpha_l},\tag{4.4}$$

$j=k+1,...,m,\ \mu\in\mathbf{C}\backslash\mathbf{R}_{+}$.

$$\widetilde{w}^{(\alpha_i+1-s_i)}\left(x,\sqrt{t}\right):=\left(\frac{d}{d\mu}\right)^{(\alpha_i+1-s_i)}\left\{\frac{w\left(x,\sqrt{\mu}\right)}{\Phi\left(\mu\right)}\right\}_{\mu=t},\tag{4.5}$$

$$\psi^{(\alpha_j-1-s_j)}\left(t,\mu_j;x\right):=\left(\frac{d}{d\mu}\right)^{\alpha_j-1-s_j}\left\{\frac{w\left(x,\sqrt{\mu}\right)}{\left(\mu-t\right)\Phi_j\left(\mu\right)}\right\}_{\mu=\mu_j},\tag{4.6}$$

$$\widetilde{\psi}^{(\alpha_j-1-s_j)}\left(t,\mu_j,\alpha_i;x\right)$$

$$:=\left(\frac{d}{d\mu}\right)^{(\alpha_j-1-s_j)}\left\{\frac{w\left(x,\sqrt{\mu}\right)}{\left(\mu-t\right)^{\alpha_i+2}\Phi_j\left(\mu\right)}\right\}_{\mu=\mu_j},\tag{4.7}$$

$i=1,...,k,\ j=k+1,...,m$, where Φ is the function defined by (3.8).

$$\left(f,\widetilde{w}^{(\alpha_i+1-s_i)}\left(\sqrt{t}\right)\right):=\int\limits_{0}^{\infty}f\left(y\right)\widetilde{w}^{(\alpha_i+1-s_i)}\left(y,\sqrt{t}\right)dy,\tag{4.8}$$

$$\left(f,\psi^{(\alpha_j-1-s_j)}\left(t,\mu_j\right)\right):=\int\limits_{0}^{\infty}f\left(y\right)\psi^{(\alpha_j-1-s_j)}\left(t,\mu_j;y\right)dy,$$

$$\left(f,\widetilde{\psi}^{(\alpha_j-1-s_j)}\left(t,\mu_j;\alpha_i\right)\right):=\int\limits_{0}^{\infty}f\left(y\right)\widetilde{\psi}^{(\alpha_j-1-s_j)}\left(t,\mu_j,\alpha_i;y\right)dy.$$

Theorem 4.4. Under the condition (1.4), the following spectral expansion formula in terms of the principal functions of L holds

$$\mathrm{f}(x)=\int_0^N \frac{\sigma(t)}{\Phi(t)} w\left(x,\sqrt{t}\right) E_f\left(\sqrt{t}\right) dt$$

$$+\sum_{i=1}^{k}\sum_{s_i=0}^{\alpha_i+1} C_{\alpha_i+1}^{s_i}\int_{a_i}^{a_{i+1}} \rho_i(t)\left(f,\widetilde{w}^{(\alpha_i+1-s_i)}\left(\sqrt{t}\right)\right) w^{(s_i)}\left(x,\sqrt{t}\right) dt$$

$$+\sum_{j=k+1}^{m}\sum_{s_j=0}^{\alpha_j-1} \frac{1}{(\alpha_j-1)!} C_{\alpha_j-1}^{s_j}\times$$

$$\times\int_0^N \sigma(t)\left(f,\psi^{(\alpha_j-1-s_j)}(t,\mu_j)\right) w^{(s_j)}\left(x,\sqrt{t}\right) dt$$

$$+\sum_{i=1}^{k}\sum_{j=k+1}^{m}\sum_{s_j=0}^{\alpha_j-1}\frac{(\alpha_i+1)!}{(\alpha_j-1)!} C_{\alpha_j-1}^{s_j}\times \tag{4.9}$$

$$\times\int_{a_i}^{a_{i+1}} \rho_i(t)\left(f,\widetilde{\psi}^{(\alpha_j-1-s_j)}(t,\mu_j;\alpha_j)\right) w^{(s_j)}\left(x,\sqrt{t}\right) dt$$

$$+\frac{1}{2\pi i}\int_N^\infty \frac{1}{\Phi(\mu)}\left\{m_1(\mu+i0)-m_1(\mu-i0)\right\} w\left(x,\sqrt{\mu}\right) E_f\left(\sqrt{\mu}\right) d\mu$$

for all $f\in H_+$, where C_α^s are the binomial coefficients. The integrals in (4.9) *converge in the norm of H_-.*

Proof. Substituting the representation for $m_1(\mu)$ given by (3.7) in the Parseval equality (2.16), we obtain that

$$(f,g)=\frac{1}{2\pi i}\int_0^N \sigma(t)\int_\gamma \frac{E_f\left(\sqrt{\mu}\right)E_g\left(\sqrt{\mu}\right)}{(\mu-t)\,\Phi(\mu)} d\mu dt$$

$$+\frac{1}{2\pi i}\sum_{i=1}^{k}(\alpha_i+1)!\int_{a_i}^{a_{i+1}}\rho_i(t)\int_\gamma \frac{E_f\left(\sqrt{\mu}\right)E_g\left(\sqrt{\mu}\right)}{(\mu-t)^{\alpha_i+2}\,\Phi(\mu)} d\mu dt \tag{4.10}$$

$$+\frac{1}{2\pi i}\int_N^{\infty}\{m_1(\mu+i0)-m_1(\mu-i0)\}\frac{E_f(\sqrt{\mu})\,E_g(\sqrt{\mu})\,d\mu}{\Phi(\mu)},$$

where

$$\int_\gamma n(\mu)\,E_f(\sqrt{\mu})\,E_g(\sqrt{\mu})\,d\mu$$

$$=\int_N^{\infty}\{m_1(\mu+i0)-m_1(\mu-i0)\}\,E_f(\sqrt{\mu})\,E_g(\sqrt{\mu})\,d\mu$$

by (3.4). If we evaluate the contour integrals in (4.10), using Cauchy Residue theory and taking care of the notations (4.4) – (4.8), we get the expansion (4.9) for $f \in C_0^{\infty} \subset H_+$. Since C_0^{∞} is dense in H_+ we have the expansion for $f \in H_+$.

Now we define the following operators;

$$T_0 f(x) \;:\; = \int_0^N \frac{\sigma(t)}{\Phi(t)} w\left(x,\sqrt{t}\right) E_f\left(\sqrt{t}\right) dt,$$

$$\mathrm{T}_{\widetilde{w}} f(x) := \sum_{i=1}^{k}\sum_{s_i=0}^{\alpha_i+1} C_{\alpha_i+1}^{s_i}\times$$

$$\times\int_{a_i}^{a_{i+1}} \rho_i(t)\left(f,\widetilde{w}^{(\alpha_i+1-s_i)}\left(\sqrt{t}\right)\right) w^{(s_i)}\left(x,\sqrt{t}\right) dt,$$

$$\mathrm{T}_{\psi} f(x) := \sum_{j=k+1}^{m}\sum_{s_j=0}^{\alpha_j-1} \tfrac{1}{(\alpha_j-1)!} C_{\alpha_j-1}^{s_j}\times$$

$$\times\int_0^N \sigma(t)\left(f,\psi^{(\alpha_j-1-s_j)}(t,\mu_j)\right) w^{(s_j)}\left(x,\sqrt{t}\right) dt,$$

$$\mathrm{T}_{\widetilde{\psi}} f(x) := \sum_{i=1}^{k}\sum_{j=k+1}^{m}\sum_{s_j=0}^{\alpha_j-1} \tfrac{(\alpha_j+1)!}{(\alpha_j-1)!} C_{\alpha_j-1}^{s_j}\times$$

$$\times\int_{a_i}^{a_{i+1}} \rho_i(t)\left(f,\widetilde{\psi}^{(\alpha_j-1-s_j)}(t,\mu_j;\alpha_j)\right) w^{(s_j)}\left(x,\sqrt{t}\right) dt,$$

$$\mathrm{T}_N f(x) := \tfrac{1}{2\pi i}\int_N^{\infty} \tfrac{1}{\Phi(\mu)}\{m_1(\mu+i0)-m_1(\mu-i0)\}\,w(x,\sqrt{\mu})\,E_f(\sqrt{\mu})\,d\mu.$$

We now show that each of the operators T_0, $T_{\widetilde{w}}$, T_ψ, $T_{\widetilde{\psi}}$ and T_N is continuous from H_+ onto H_-.

T_0 is continuous from $L^2(\mathbf{R}_+)$ onto itself by Lemma 4.2. Using isomorphism
$H_{-s_0} \sim H'_{s_0}$ we get that T_0 is continuous from H_{s_0} onto H_{-s_0} or from H_+ onto H_- by inclusions (4.3). Hence there exists a constant c_0 such that

$$\|T_0 f\|_- \leq c_0 \|f\|_+ \; 4.11 \tag{4.11}$$

for any $f \in H_+$.

$T_{\widetilde{w}}$ includes the principal functions of the continuous spectrum.Since

$$\|T_{\widetilde{w}} f(x)\|_- = \int_0^\infty \left| \frac{T_{\widetilde{w}} f(x)}{(1+x)^{s_0+2}} \right|^2 dx$$

$$\leq \sum_{i=1}^{k} \sum_{s_i=0}^{\alpha_i+1} C_{\alpha_i+1}^{s_i} \int_0^\infty \int_{a_i}^{a_{i+1}} \left| \frac{\rho_i(t) w^{(s_i)}(x,\sqrt{t})}{(1+x)^{s_0+2}} \right|^2 dt dx \times$$

$$\times \int_{a_i}^{a_{i+1}} \left| \left(f, \widetilde{w}^{(\alpha_i+1-s_i)}\left(\sqrt{t}\right) \right) \right|^2 dt.$$

We see that first double integral is convergent from (4.5), Remark 4.1, Lemma 4.2 and Lemma 4.3. Therefore we get from Lemma 4.2 and Lemma 4.3 that

$$\|T_{\widetilde{w}} f\|_- \leq c_{\widetilde{w}} \|f\|_+ , \tag{4.12}$$

where $c_{\widetilde{w}} > 0$ is a constant.

T_ψ and $T_{\widetilde{\psi}}$ are continuousfrom $L^2(\mathbf{R}_+)$ onto itself by (4.2), (4.6) and (4.7), so we can write that

$$\|T_\psi\| \leq c_\psi \|f\| , \quad \left\|T_{\widetilde{\psi}}\right\| \leq c_{\widetilde{\psi}} \|f\|$$

where $c_\psi > 0$, and $c_{\widetilde{\psi}} > 0$ are constants. Using (4.3) and isomorphism $H_{-s_0} \sim H'_{s_0}$ we get that T_ψ and $T_{\widetilde{\psi}}$ are continuous from H_+ onto H_-. Hence we have

$$\|T_\psi f\|_- \leq c_\psi \|f\|_+ , \quad \left\|T_{\widetilde{\psi}} f\right\|_- \leq c_{\widetilde{\psi}} \|f\|_+ \tag{4.13}$$

for the constants given above.

Finally we consider the operator T_N. From (2.10) and Theorem 2.3, using a similar proof to that of the well- known Lemma 4.2 we get that

$\|T_N f\| \leq c_N \|f\|$,

here $c_N > 0$ is a constant. From the inclusions (4.3) and the isomorphism we find that

$$\|T_N f\|_- \leq c_N \|f\|_{+.}\,, \tag{4.14}$$

where $c_N > 0$ is a constant.

We get the proof of the theorem by (4.11), (4.12), (4.13) and (4.14).

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