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COUNTABLE S*-COMPACTNESS IN L-SPACES

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Abstract

In this paper, the notions of countable S^*-compactness is introduced in L-topological spaces based on the notion of S^*-compactness. An S^*-compact L-set is countably S^*-compact. If $L = [0, 1]$, then countable strong compactness implies countable S^*-compactness and countable S^*-compactness implies countable F-compactness, but each inverse is not true. The intersection of a countably S^*-compact L-set and a closed L-set is countably S^*-compact. The continuous image of a countably S^*-compact L-set is countably S^*-compact. A weakly induced L-space $(X, \mathcal{T})$ is countably S^*-compact if and only if $(X, [\mathcal{T}])$ is countably compact.

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1. Introduction

The concept of compactness is one of most important concepts in general topology. The concept of compactness in [0, 1]-fuzzy set theory was first introduced by C.L. Chang in terms of open cover [1]. Goguen was the first to point out a deficiency in Chang's compactness theory by showing that the Tychonoff Theorem is false [5]. Since Chang's compactness has some limitations, Gantner, Steinlage and Warren introduced α-compactness [3], Lowen introduced F-compactness, strong compactness and ultra-compactness [9], Liu introduced Q-compactness [7], Li introduced strong Q-compactness [6] which is equivalent to strong F-compactness in [10], and Wang and Zhao introduced N-compactness [16, 21].

In [15], Shi introduced a new notion of fuzzy compactness in L-topological spaces, which is called S^*-compactness. Ultra-compactness implies S^*-compactness. S^*-compactness implies F-compactness. If $L = [0, 1]$, then strong compactness implies S^*-compactness.

There has been many papers about countable fuzzy compactness of L-sets (see [11, 12, 14, 18, 19, 20] etc.). They were based on the concepts of N-compactness, Chang's compactness, strong compactness and F-compactness respectively.

In this paper, based on the S^*-compactness, we shall introduce the notion of countable S^*-compactness and research its properties.

2. Preliminaries

Throughout this paper $(L, \bigvee, \bigwedge, ')$ is a completely distributive de Morgan algebra. X is a nonempty set. L^X is the set of all L-fuzzy sets on X. The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$.

An element a in L is called prime if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. An element a in L is called co-prime if a' is a prime element [4]. The set of nonunit prime elements in L is denoted by $P(L)$. The set of nonzero co-prime elements in L is denoted by $M(L)$. The set of nonzero co-prime elements in L^X is denoted by $M(L^X)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [2]. In a completely distributive de Morgan algebra L, each member b is a sup of $\{a \in L \mid a \prec b\}$. In the sense of [8, 17], $\{a \in L \mid a \prec b\}$ is the greatest minimal family of b, in symbol $\beta(b)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For an L-set $A \in L^X$, $\beta(A)$ denotes the greatest minimal family of A and $\beta^*(A) = \beta(A) \bigcap M(L^X)$.

For $a \in L$ and $A \in L^X$, we use the following notations in [15].

$$A_{[a]} = \{x \in X \mid A(x) \geq a\}, \quad A_{(a)} = \{x \in X \mid a \in \beta(A(x))\},$$
$$A^{(a)} = \{x \in X \mid A(x) \not\leq a\}.$$

An L-topological space (or L-space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\underline{0}$, $\underline{1}$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Each member of $\mathcal{T}$ is called an open L-set and its complement is called a closed L-set.

Definition 2.1. [[8, 17]] For a topological space (X, τ), let $\omega_L(\tau)$ denote the family of all lower semi-continuous maps from (X, τ) to L, i.e., $\omega_L(\tau) = \{A \in L^X \mid A^{(a)} \in \tau, a \in L\}$. Then $\omega_L(\tau)$ is an L-topology on X, in this case, $(X, \omega_L(\tau))$ is called topologically generated by (X, τ).

Definition 2.2. [[8, 17]] An L-space $(X, \mathcal{T})$ is called weakly induced if $\forall a \in L$, $\forall A \in \mathcal{T}$, it follows that $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in $\mathcal{T}$.

Lemma 2.3. [[15]] Let $(X, \mathcal{T})$ be a weakly induced L-space, $a \in L, A \in \mathcal{T}$. Then $A_{(a)}$ is an open set in $[\mathcal{T}]$.

Definition 2.4. [[20]] An L-space $(X, \mathcal{T})$ is called countably ultra-compact if $\iota_L(\mathcal{T})$ is countably compact, where $\iota_L(\mathcal{T})$ is the topology generated by $\{A^{(a)} \mid A \in \mathcal{T}, a \in L\}$.

Definition 2.5. [[11]] Let $(X, \mathcal{T})$ be an L-space, $A \in L^X$. A is called countably N-compact if for every $a \in M(L)$, every countable a-R-neighborhood family of G has a finite subfamily which is an a^--R-neighborhood family of G.

Definition 2.6. [[19]] Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$. G is called countably strong compact if for every $a \in M(L)$, every countable a-R-neighborhood family of G has a finite subfamily which is an a-R-neighborhood family of G.

Definition 2.7. Let $(X, \mathcal{T})$ be an L-space, $a \in L\backslash\{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq \mathcal{T}$ is called a Q_a-open cover of G if $a \leq \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right)$.

It is obvious that for $a \in M(L)$, the notion of Q_a-open cover in Definition 2.7 is the corresponding notion in [15].

Definition 2.8. [[12]] Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$. G is called countably F-compact if for any $a \in M(L)$ and for any $b \in \beta^*(a)$, every constant a-sequence in G has a cluster point in G with height b.

Definition 2.9. [[15]] Let $(X, \mathcal{T})$ be an L-space, $a \in M(L)$ and $G \in L^X$. A family $\mathcal{U} \subseteq \mathcal{T}$ is called a β_a-open cover of G if for any $x \in X$, it follows that $a \in \beta \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right)$.

When $L = [0, 1]$, $\mathcal{U}$ is a β_a-open cover of $\underline{1}$ if and only if $\mathcal{U}$ is an a-shading of $\underline{1}$ in the sense of [3]. $\mathcal{U}$ is a β_a-open cover of G if and only if $\mathcal{U}'$ is an a'-R-neighborhood family of G.

3. Countable S^*-compactness

Definition 3.1. Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then G is called countably S^*-compact if for any $a \in M(L)$, each countable β_a-open cover of G has a finite subfamily which is a Q_a-open cover of G. $(X, \mathcal{T})$ is said to be countably S^*-compact if $\underline{1}$ is countably S^*-compact.

Obviously we have the following theorem.

Theorem 3.2. S^*-compactness implies countably S^*-compactness.

From Theorem 3.2, we know that an L-set with finite support is S^*-compact. Moreover in an L-space $(X, \mathcal{T})$ with a finite L-topology, each L-set is S^*-compact.

Definition 3.3. Let $\mathcal{A} \subset L^X$, $G, H \in L^X$ and $a \in M(L)$.

(1) H is called Q_a-nonempty in G if there exists $x \in X$ such that $G(x) \wedge A(x) \not\leq a'$.

(2) H is called weak Q_a-nonempty in G if there exists $x \in X$ such that $a' \notin \alpha(G(x) \wedge A(x))$.

(3) $\mathcal{A}$ is said to have a Q_a-nonempty intersection in G if $\bigwedge \mathcal{U}$ is Q_a-nonempty in G.

(4) $\mathcal{A}$ is said to have a weak Q_a-nonempty intersection in G if $\bigwedge \mathcal{U}$ is weak Q_a-nonempty in G.

(5) If each finite subfamily of $\mathcal{A}$ has Q_a-nonempty intersection in G, then $\mathcal{A}$ is said to have finite Q_a-intersection property in G.

It is obvious that if $\mathcal{A}$ has a Q_a-nonempty intersection in G, then it also has a weak Q_a-nonempty intersection in G.

It is easy to prove the following theorem.

Theorem 3.4. For an L-space $(X, \mathcal{T})$ and $G \in L^X$, the following conditions are equivalent:

(1) G is countably S^*-compact.

(2) Each countable family of closed L-sets with finite Q_a-intersection property in G has weakly Q_a-nonempty intersection in G.

(3) For each decreasing sequence $F_1 \supset F_2 \supset \cdots$ of closed L-sets which are Q_a-nonempty in G, $\{F_i \mid i = 1, 2, \cdots\}$ has a weakly Q_a-nonempty intersection in G.

Theorem 3.5. If G is countably S^*-compact and H is closed, then $G \wedge H$ is countably S^*-compact.

Proof. Suppose that $\mathcal{U}$ is a countable β_a-open cover of $G \wedge H$. Then $\mathcal{U} \cup \{H'\}$ is a countable β_a-open cover of G. By countable S^*-compactness of G, we know that $\mathcal{U} \cup \{H'\}$ has a finite subfamily $\mathcal{V}$ which is a Q_a-open cover of G. Take $\mathcal{W} = \mathcal{V} \backslash \{H'\}$. Then $\mathcal{W}$ is Q_a-open cover of $G \wedge H$. This shows that $G \wedge H$ is countably S^*-compact. □

Theorem 3.6. If G is countably S^*-compact in $(X, \mathcal{T}_1)$ and $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is continuous, then $f_L^{\rightarrow}(G)$ is countably S^*-compact in $(Y, \mathcal{T}_2)$.

Proof. Let $\mathcal{U} \subseteq \mathcal{T}_2$ be a countable β_a-open cover of $f_L^{\rightarrow}(G)$. Then for any $y \in Y$, we have that $a \in \beta\left(f_L^{\rightarrow}(G)'(y) \vee \bigvee_{A \in \mathcal{U}} A(y)\right)$. Hence for any $x \in X$, $a \in \beta\left(G'(x) \vee \bigvee_{A \in \mathcal{U}} f_L^{\leftarrow}(A)(x)\right)$. This shows that $f_L^{\leftarrow}(\mathcal{V}) = \{f_L^{\leftarrow}(A) \mid A \in \mathcal{U}\}$ is a countable β_a-open cover of G. By countable S^*-compactness of G, we know that $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ such that $f_L^{\leftarrow}(\mathcal{V})$ is a Q_a-open cover of G. By the following equation we can obtain that $\mathcal{V}$ is a Q_a-open cover of $f(G)$.

$$\begin{aligned}
f_L^{\rightarrow}(G)'(y) \vee \left(\bigvee_{A \in \mathcal{V} A(y)}\right) &= \left(\bigwedge_{x \in f^{-1}(y)} G'(x)\right) \vee \left(\bigvee_{A \in \mathcal{V} A(y)}\right) \\
&= \bigwedge_{x \in f^{-1}(y)} \left(G'(x) \vee \left(\bigvee_{A \in \mathcal{V} A(f(x))}\right)\right) \\
&= \bigwedge_{x \in f^{-1}(y)} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}(f^{\leftarrow}(A))(x)}\right).
\end{aligned}$$

Therefore $f_L^{\rightarrow}(G)$ is countably S^*-compact. □

Theorem 3.7. If $(X, \mathcal{T})$ is a weakly induced L-space, then $(X, \mathcal{T})$ is countably S^*-compact if and only if $(X, [\mathcal{T}])$ is countably compact.

Proof. Let $(X,[\mathcal{T}])$ be countably compact. For $a \in M(L)$, let $\mathcal{U}$ be a countable β_a-open cover of $\underline{1}$ in $(X,\mathcal{T})$. Then by Lemma 2.2, $\{A_{(a)} \mid A \in \mathcal{U}\}$ is a countable open cover of $(X,[\mathcal{T}])$. By countable compactness of $(X,[\mathcal{T}])$, we know that there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}_{(a)} = \{A_{(a)} \mid A \in \mathcal{V}\}$ is an open cover of $(X,[\mathcal{T}])$. Obviously $\mathcal{V}$ is a β_a-open cover of $\underline{1}$ in $(X,\mathcal{T})$, of course it is also a Q_a-open cover of $\underline{1}$ in $(X,\mathcal{T})$. This shows that $(X,\mathcal{T})$ is countably S^*-compact.

Conversely let $(X,\mathcal{T})$ be countably S^*-compact and $\mathcal{W}$ be a countable open cover of $(X,[\mathcal{T}])$. Then for each $a \in \beta^*(1)$, $\mathcal{W}$ is a countable β_a-open cover of $\underline{1}$ in $(X,\mathcal{T})$. By countable S^*-compactness of $(X,\mathcal{T})$, we know that there exists a finite subfamily $\mathcal{V}$ of $\mathcal{W}$ such that $\mathcal{V}$ is a Q_a-open cover of $\underline{1}$ in $(X,\mathcal{T})$. Obviously $\mathcal{V}$ is an open cover of $(X,[\mathcal{T}])$. This shows that $(X,[\mathcal{T}])$ is compact. □

Corollary 3.8. Let (X,τ) be a crisp topological space. Then $(X,\omega_L(\tau))$ is countably S^*-compact if and only if (X,τ) is countably compact.

4. A comparison of different notions of countable compactness

In [13], a characterization of F-compactness was presented by means of Q_a-open cover. Analogously we can present the characterization of countable F-compactness as follows:

Lemma 4.1. Let $(X,\mathcal{T})$ be an L-space, $G \in L^X$. Then G is countably F-compact if and only if for all $a \in M(L)$, for all $b \in \beta^*(a)$, each countable Q_a-open cover Φ of G has a finite subfamily $\mathcal{B}$ such that $\mathcal{B}$ is a Q_b-open cover of G.

Theorem 4.2. Countable S^*-compactness implies countable F-compactness.

Proof. Let $(X,\mathcal{T})$ be an L-space and $G \in L^X$ be countably S^*-compact. To prove that G is countably F-compact, suppose that $\mathcal{U}$ is a countable Q_a-open cover of G. Obviously for any $b \in \beta^*(a)$, $\mathcal{U}$ is a countable β_b-open cover of G. By countable S^*-compactness of G we know that $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ which is a Q_b-open cover of G. By Lemma 4.1 we know that G is countably F-compact. □

In general, countable F-compactness needn't imply countable S^* -compactness. This can be seen from Example 6.2 in [12].

When $L = [0,1]$, since each β_a-open cover of G is Q_a-open cover of G and $\mathcal{U}$ is a β_a-open cover of G if and only if $\mathcal{U}$ is an a-shading of G, we can obtain the following:

Theorem 4.3. When $L = [0,1]$, countable strong compactness implies countable S^*-compactness, hence countable N-compactness implies countable S^*-compactness.

In general, countable S^*-compactness needn't imply countable strong compactness. This can be seen from Example 6.4 in [12].

Theorem 4.4. If $(X, \mathcal{T})$ is a countably ultra-compact L-space, then it is countably S^*-compact.

Proof. By countable ultra-compactness of $(X, \mathcal{T})$ we know that $(X, \iota(\mathcal{T}))$ is countably compact. This shows that $(X, \omega_L \circ \iota_L(\mathcal{T}))$ is countably S^*-compact from Corollary 3.8. Further from $\omega_L \circ \iota_L(\mathcal{T}) \supseteq \mathcal{T}$ we can obtain the proof. $\square$

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