THE NATURAL VECTOR BUNDLE OF THE SET OF MULTIVARIATE DENSITY FUNCTIONS *

E. MARCHI

and
P. M. MORILLAS
UNSL - CONICET, ARGENTINA

Abstract

We find a description of the set of multivariate density functions with given marginals and introduce an associated vector bundle. The interest for the probability theory is restricted to the nonnegative elements in the sets of the derived vector bundle. The fiber is the space of all correlation measures among a multivariate density function and its unidimensional marginals.

1. Introduction

An important branch in probability and statitics is the study of distributions with given marginals. This study was initialized by Wassily Hoeffding (1940, 1941) and Maurice Fréchet (1951). Since then, its theoretical and applied development have been remarkable (for an account of the first steps in this development see Dall'Aglio (1991)). When the marginals are unidimensional, the most important tool in this study are copulas. Copulas were introduced by Abe Sklar (1959) in response to a query of Fréchet. Sklar proved that if H is the joint distribution function of n random variables $X_1, ..., X_n$, and $F_1, ..., F_n$ are the distribution functions of $X_1, ..., X_n$, respectively, then there exists an n-copula C such that $H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n))$. The n-copula is uniquely determined on $range(F_1) \times ... \times range(F_n)$, so that C can be thought of as a description of the way in which a joint distribution function is related to its one-dimensional marginals. Excellent references about copulas are Joe (1997) and Nelsen (1999).

Independently of the above mentioned development and using a geometrical approach, Ezio Marchi (1969, 1972) studied the set of discrete multivariate density functions with given unidimensional marginals which it motivates a great variety of new concepts of cooperative equilibrium in game theory (see Marchi (1976, 1986a, 1986b, 1988, 1992, 1996) and Marchi, García Jurado and Prada (1991)). He first studied the bivariate case concerning to some problems about bimatrix zero sum games. Then, he studied the multivariate case introducing a natural vector bundle.

In particular, Marchi (1972) showed by considering finite sets X_i, $i = 1, ..., n$, that the Cartesian product $\widetilde{X}_N$ of the sets of density functions $\widetilde{X}_i$ defined on X_i, $i = 1, ..., n$, is naturally embedded in the set $\widetilde{X_N}$ of the density functions defined on the Cartesian product X_N of the sets X_i. Moreover, he derived $\widetilde{X_N}$ as a vector bundle having as the underlined manifold the image of the sets $\widetilde{X}_i$ under the natural embedding. The interest for the probability theory is restricted to the nonnegative elements in the sets of the derived vector bundle. Specifically, the base of the vector bundle contains the set $\widetilde{X}_N$ and the space contains the set of tensor products of elements in $\widetilde{X}_i$, $i = 1, ..., n$. If $g_i \in \widetilde{X}_i$, $i = 1, ..., n$, then the set of functions in $\widetilde{X_N}$ with unidimensional marginals g_i consists of the nonnegative elements of the fiber in the vector bundle at the tensor product $\otimes_{i=1}^{n} g_i$. Moreover, any element of this set can be written as the sum of $\otimes_{i=1}^{n} g_i$ and an element of the type fiber of the vector bundle. So the fiber plays an

important role, namely that of being the space of all correlation measures among a multivariate density function and its unidimensional marginals.

In this paper we extend the results obtained by Marchi (1972) considering σ-finite measure spaces (X_i, μ_i), $i = 1, ..., n$. It is worth noting that in the papers just mentioned, Marchi, in order to prove the results, considered extreme points and introduced basis which are not available in the general setting of arbitrary measure spaces. Here we use decompositions for the different involved spaces simplifying considerably the proofs of the different results.

The rest of the paper is organized as follows. In section 2 we obtain a decomposition of $L^1(X_N, \mu_N)$ where (X_N, μ_N) denotes the product measure space corresponding to the measure spaces (X_i, μ_i), $i = 1, ..., n$. We express $L^1(X_N, \mu_N)$ as a topological sum of tensor products of some subspaces of $L^1(X_i, \mu_i)$, $i = 1, ..., n$. In section 3 we obtain a description of the set of density functions in $L^1(X_N, \mu_N)$ with given unidimensional marginals and, in section 4, we introduce the vector bundle. Finally, some concluding remarks are given in section 5.

2. A decomposition of $L^1(X_N, \mu_N)$.

If A is a finite set, then $|A|$ denotes the number of elements of A. We denote with **N** and

R the set of natural numbers and the set of real numbers, respectively. If $n \in \mathbf{N}$, then $N = \{1, ..., n\}$. To avoid ambiguities we assume that $n \geq 2$. If $K = \{k_1, ..., k_m\} \subseteq N$ with $k_1 < ... < k_m$, then $(x_K) = (x_{k_1}, ..., x_{k_m})$. If $S_1, ..., S_n$ are sets, then $S_K = S_{k_1} \times ... \times S_{k_m}$.

If S is a subset of a vectorial space, then $\langle S \rangle$ is the linear space generated by S. If E is a subset of a normed vectorial space X, then clE is the closure of E in X. If X_i, $i \in N$, are normed vectorial spaces with norms $\|.\|_i$, then X_N is equipped with the product topology defined by any of the equivalent norms $\|(x_N)\|_N = max(\|x_i\|_i : i \in N)$, or $\|(x_N)\|_N = \sum_{i=1}^n \|x_i\|_i$ for all $(x_N) \in X_N$. The symbol $\oplus$ denotes the topological sum.

In this paper we use some notions from measure and integral theory. For Fubini's theorem to be valid, all the measure spaces are considered to be σ-finite. If $f \in L^1(X, \mu)$, then $\|f\|_{(X,\mu)}$ is the norm of f. We consider the subspace of $L^1(X, \mu)$, $L^1_0(X, \mu) = \{f \in L^1(X, \mu) : \int_X f d\mu = 0\}$. If (X_i, μ_i), $i \in N$, are σ-finite measure spaces and (X_N, μ_N) is the corresponding product measure space, we set $\int_{X_K} f d\mu_K = f$ for $f \in L^1(X_N, \mu_N)$ and $K = \emptyset$.

Theorem 2.1. *2.1. Let $g \in L^1(X, \mu)$ be such that $\int_X g d\mu = 1$. Then $L^1(X, \mu) = L_0^1(X, \mu) \oplus \langle g \rangle$.*

Proof. Let σ be the continuous function defined by

$$(2.1) \qquad \begin{aligned} \sigma : L_0^1(X, \mu) \times \langle g \rangle &\rightarrow L^1(X, \mu) \\ (z, cg) &\rightarrow z + cg.. \end{aligned}$$

The function σ is one to one and onto. In effect, if $f \in L^1(X, \mu)$, we can write $f = z + cg$ with $c = \int_X f d\mu$ and $z = f - cg \in L_0^1(X, \mu)$. This shows that σ is onto. Suppose now that

$$z_1 + c_1 g = z_2 + c_2 g,$$

with $z_1, z_2 \in L_0^1(X, \mu)$, $c_1, c_2 \in \mathbf{R}$. Integrating the previous equality on X, and taking into account that $\int_X z_1 d\mu = \int_X z_2 d\mu = 0$ and $\int_X g d\mu = 1$, we obtain $c_1 = c_2$, and then $z_1 = z_2$. Thus σ is one to one.

By the open map theorem, since $L_0^1(X, \mu)$ and $\langle g \rangle$ are closed subspaces of $L^1(X, \mu)$, then σ is a topological isomorphism, or which is the same, the subspaces $L_0^1(X, \mu)$ and $\langle g \rangle$ decompose $L^1(X, \mu)$ in a topological sum.
$\square$

Given n real functions $f_i : X_i \rightarrow \mathbf{R}$, $i \in N$, where each X_i is an arbitrary set, the tensor product of the functions f_i is the function $\otimes_{i=1}^n f_i : X_N \rightarrow \mathbf{R}$ given by $\otimes_{i=1}^n f_i(x_1, ..., x_n) = f_1(x_1)...f_n(x_n)$. Given n closed subspaces $E_i \subseteq L^1(X_i, \mu_i)$, $i \in N$, the tensor product of the spaces E_i is $\otimes_{i=1}^n E_i = cl \langle \{\otimes_{i=1}^n f_i : f_i \in E_i, i \in N\} \rangle$. For $i \in N$, let $E_i, F_i \subseteq L^1(X_i, \mu_i)$ be closed subspaces, $f_i \in E_i$, $g_i \in F_i$ and $K = \{k_1, ..., k_m\} \subseteq N$ with $k_1 < ... < k_m$, then we set

$$\otimes_{k \in K} f_k \otimes_{k' \in N \setminus K} g_{k'} = g_1 \otimes ... \otimes g_{k_1 - 1} \otimes f_{k_1} \otimes g_{k_1 + 1} \otimes ... \otimes g_{k_m - 1} \otimes \\ \otimes f_{k_m} \otimes g_{k_m + 1} \otimes ... \otimes g_n,$$

and

$$\otimes_{k \in K} E_k \otimes_{k' \in N \setminus K} F_{k'} = F_1 \otimes ... \otimes F_{k_1 - 1} \otimes E_{k_1} \otimes F_{k_1 + 1} \otimes ... \otimes F_{k_m - 1} \otimes \\ \otimes E_{k_m} \otimes F_{k_m + 1} \otimes ... \otimes F_n.$$

We have the equality

(2.2) $$L^1(X_N, \mu_N) = \otimes_{i=1}^n L^1(X_i, \mu_i).$$

In effect, the set of simple integrable functions is dense in $L^1(X_N, \mu_N)$ and any simple integrable function can be approximated in $L^1(X_N, \mu_N)$ by finite linear combinations of characteristic functions of the type $\chi_{E_N} = \otimes_{i=1}^n \chi_{E_i}$ with $E_i \subseteq X_i$, $\mu_i(E_i) < \infty$ (this last fact can be deduced from the definition of product measure).

From Theorem 2.1 and (2.2) we obtain a decomposition of $L^1(X_N, \mu_N)$ given in the next theorem.

Theorem 2.2. *2.2. Let $g_i \in L^1(X_i, \mu_i)$ with $\int_{X_i} g_i = 1$, $i \in N$. Then*

(2.3) $$L^1(X_N, \mu_N) = \oplus_{K \subseteq N} \otimes_{k \in K} \langle g_k \rangle \otimes_{k' \in N \setminus K} L_0^1(X_{k'}, \mu_{k'}).$$

For a detailed proof of the above theorem see Morillas (2003).

3. Density functions in $L^1(X_N, \mu_N)$ with given unidimensional marginals.

If $\emptyset \subset K \subseteq N$, then the set of *density functions* in $L^1(X_K, \mu_K)$ is

$$\widetilde{X_K} = \{f \in L^1(X_K, \mu_K) : f \geq 0$$

and

$$\int_{X_K} f d\mu_K = 1\}.$$

For $i \in N$, consider the continuous linear function

$$s_i : L^1(X_N, \mu_N) \rightarrow L^1(X_i, \mu_i)$$
$$f \rightarrow \int_{X_{N \setminus \{i\}}} f d\mu_{N \setminus \{i\}}.$$

If $g_i \in \widetilde{X_i}$, $i \in N$, then the set of *density functions in $L^1(X_N, \mu_N)$ with unidimensional marginals g_i* is

$$U(g_N) = \{f \in \widetilde{X_N} : s_i(f) = g_i, i \in N\}.$$

In this section we are going to give an analytical description of $U(g_N)$. In what follows we will suppose that for $i \in N$, $g_i, h_i \in L^1(X_i, \mu_i)$ and $\int_{X_i} g_i d\mu_i = \int_{X_i} h_i d\mu_i = 1$. Next we will consider some functions introduced

by Marchi (1972) for the case in which X_i are finite sets. We begin with the continuous linear function

$$s_N : L^1(X_N, \mu_N) \rightarrow \times_{i=1}^n L^1(X_i, \mu_i)$$
$$f \rightarrow (s_1(f), ..., s_n(f))$$

By Theorem 2.1, $L^1(X_i, \mu_i) = L_0^1(X_i, \mu_i) \oplus \langle h_i \rangle$. Thus, if we consider the functions associated with this topological sum, σ_i given by (2.1) and the projection p_i given by

$$p_i : L_0^1(X_i, \mu_i) \times \langle h_i \rangle \rightarrow L_0^1(X_i, \mu_i)$$
$$(z, ch_i) \rightarrow z,$$

then the function
$$u_i : L^1(X_i, \mu_i) \rightarrow L_0^1(X_i, \mu_i)$$
$$f \rightarrow p_i \sigma_i^{-1}(f),$$

is a continuous linear transformation. If $f \in L^1(X_N, \mu_N)$, then

(3.1)
$$s_i(f) = u_i(s_i(f)) + \left(\int_{X_i} s_i(f) d\mu_i \right) h_i$$
$$= u_i(s_i(f)) + \left(\int_{X_N} f d\mu_N \right) h_i$$

Consider now the function

$$p_{h_i} : L^1(X_i, \mu_i) \rightarrow L^1(X_i, \mu_i)$$
$$f \rightarrow u_i(f) + h_i.$$

Let
(3.2)
$$M_i : L^1(X_i, \mu_i) = range(p_{h_i})\mu_i)$$
$$= \{f \in L^1(X_i, \mu_i) : \int_{X_i} f d\mu_i = 1\}$$
$$= \{z + h_i : z \in L_0^1(X_i, \mu_i)\},$$

and

(3.3)
$$\mathcal{M} = \{f \in L^1(X_N, \mu_N) : \int_{X_N} f d\mu_N = 1\}.$$

From (3.1), if $f \in \mathcal{M}$ then

(3.4)
$$p_{h_i} s_i(f) = s_i(f).$$

Let
$$p_{h_N} : \times_{i=1}^n L^1(X_i, \mu_i) \rightarrow \times_{i=1}^n L^1(X_i, \mu_i)$$
$$(f_N) \rightarrow (p_{h_1}(f_1), ..., p_{h_n}(f_n)),$$

$$(3.5) \qquad \begin{aligned} \Psi : \times_{i=1}^{n} L^1(X_i, \mu_i)(f_n)), &\to L^1(X_N, \mu_N) \\ (f_N) &\to \otimes_{i=1}^{n} f_i, \end{aligned}$$

and

$$\begin{aligned} \pi_{h_N} : L^1(X_N, \mu_N) &\to L^1(X_N, \mu_N) \\ f &\to \Psi p_{h_N} s_N(f) \end{aligned}$$

Next theorem asserts that $U(g_N)$ consists of the nonnegative elements of $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N))$.

Theorem 3.1. *3.1.* $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) = \{f \in L^1(X_N, \mu_N) : s_i(f) = g_i, i \in N\}$.

Proof. Suppose that $f \in \mathcal{M}$. Using (3.4) we obtain

$$(3.6) \qquad \begin{aligned} \pi_{h_N} &= \Psi p_{h_N} s_N(f)). \\ &= \Psi p_{h_N} (s_1(f), ..., s_n(f)) \\ &= \Psi (s_1(f), ..., s_n(f)) \\ &= \Psi (s_N(f)) \end{aligned}$$

By (3.6), if $f \in \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N))$, then

$$\Psi(s_N(f)) = \Psi(g_N).$$

Taking into account that $\int_{X_i} s_i(f) d\mu_i = \int_{X_N} f d\mu_N = 1$ and $\int_{X_i} g_i d\mu_i = 1$, for all $i \in N$, from the previous equality it results that if $i \in N$, then

$$\begin{aligned} s_i(f) &= s_i \left(\Psi \left(s_N \left(f \right) \right) \right) \\ &= s_i(\Psi(g_N)) \\ &= g_i. \end{aligned}$$

Suppose now that for each $i \in N$, $s_i(f) = g_i$. Then,

$$int_{X_N} f d\mu_N = \int_{X_i} s_i(f) d\mu_i = \int_{X_i} g_i d\mu_i = 1$$

and

$$\Psi(s_N(f)) = \Psi(g_N).$$

Thus $f \in \mathcal{M}$ and, by (3.6), $\pi_{h_N}(f) = \Psi(g_N)$. $\qquad \square$

Remark 3.2. Observe that if $f \in L^1(X_N, \mu_N)$, by (3.1)

$$
\begin{aligned}
\pi_{h_N}(f) &= \pi_{h_N}(f) s_N(f) \\
&= \Psi p_{h_N}(s_1(f), ..., s_n(f)) \\
(3.7) \quad &= \quad \begin{aligned}[t] & \Psi p_{h_N}(u_1(s_1(f)) + \left(\int_{X_N} f d\mu_N\right) h_1, ..., u_n(s_n(f)) \\ & + \left(\int_{X_N} f d\mu_N\right) h_n) \end{aligned} \\
&= \Psi(u_1(s_1(f)) + h_1, ..., u_n(s_n(f)) + h_n) \\
&= \sum_{K \subseteq N} \otimes_{k \in K} h_k \otimes_{k' \in N \setminus K} u_{k'}(s_{k'}(f)).
\end{aligned}
$$

By (3.1), if $s_i(f) = \left(\int_{X_N} f d\mu_N\right) h_i$, $i \in N$, then $u_i(s_i(f)) = 0$ and, by (3.7), $\pi_{h_N}(f) = \Psi(h_N)$.

On the other hand, since $\int_{X_k} h_k d\mu_k = 1$ and $u_k(s_k(f)) \in L_0^1(X_k, \mu_k)$, $k \in N$, from (3.7) we deduce that

$$
s_i(\pi_{h_N}(f)) = u_i(s_i(f)) + h_i.
$$

If $\pi_{h_N}(f) = \Psi(h_N)$, then for each $i \in N$, $s_i(\pi_{h_N}(f)) = h_i$; and from the above equality, $u_i(s_i(f)) = 0$, so, by (3.1), $s_i(f) = (\int_{X_N} f d\mu_N) h_i$.

From the previous observations we have,

$$
(3.8) \quad \pi_{h_N}^{-1}(\Psi(h_N)) = \{ f \in L^1(X_N, \mu_N) : s_i(f) = \left(\int_{X_N} f d\mu_N\right) h_i, i \in N \}.
$$

In the sequel we will need the following results which proofs can be found in Morillas (2003).

Lemma 3.2. *3.3.* Let $g_i \in L^1(X_i, \mu_i)$ with $\int_{X_i} g_i d\mu_i = 1$, $i \in N$. The following identities hold:

1. $\otimes_{i=1}^n \langle g_i \rangle = \langle \otimes_{i=1}^n g_i \rangle$.

2. $\otimes_{k \in N \setminus \{i\}} \langle g_k \rangle \otimes L_0^1(X_i, \mu_i) = \{ \otimes_{k \in N \setminus \{i\}} g_k \otimes z_i : z_i \in L_0^1(X_i, \mu_i) \}$.

3. If $K \subseteq N$, $|K| \geq 2$, then
   $\otimes_{k \in N \setminus K} \langle g_k \rangle \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'}) =$
   $= cl \{ \otimes_{k \in N \setminus K} g_k \otimes \sum_{l=1}^m \otimes_{k' \in K} z_{k'}^l : z_{k'}^l \in L_0^1(X_{k'}, \mu_{k'}), k' \in K, l \in \{1, ..., m\}, m \in \mathbf{N} \}$.

Lemma 3.3. *3.4. Let $g_i \in L^1(X_i, \mu_i)$ with $\int_{X_i} g_i d\mu_i = 1$, $i \in N$. Let $\emptyset \subset K \subseteq N$, $\emptyset \subseteq J \subseteq N$ and $f_{N \setminus K} \in \otimes_{k \in N \setminus K} \langle g_k \rangle \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'})$. The following assertions hold:*

1. *If $J = K$, then $\int_{X_{N \setminus J}} f_{N \setminus K} d\mu_{N \setminus J} \in \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'})$.*

2. *If $J \neq K$ and $|K| \geq |J|$, then $\int_{X_{N \setminus J}} f_{N \setminus K} d\mu_{N \setminus J} = 0$.*

Now are in conditions to give an analytical description of the set $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N))$.

Theorem 3.4. *3.5. $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) = \{ f \in L^1(X_N, \mu_N) : f = \Psi(g_N) + r, r \in \oplus_{K \subseteq N, |K| \geq 2} \otimes_{k \in N \setminus K} \langle g_k \rangle \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'}) \}$.*

Proof. Suppose that $f = \Psi(g_N) + r$ where $r \in \oplus_{K \subseteq N, |K| \geq 2} \otimes_{k \in N \setminus K} \langle g_k \rangle \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'})$. If $i \in N$, using Lemma 3.4 (2) with $J = \{i\}$, it results that $s_i(f) = g_i$. So, by Theorem 3.1, $f \in \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N))$.

Suppose now that $f \in \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N))$. By Theorem 2.2,

$$(3.9) \qquad f = c\Psi(g_N) + \sum_{K \subseteq N, |K| \geq 1} f_{N \setminus K},$$

with $f_{N \setminus K} \in \otimes_{k \in N \setminus K} \langle g_k \rangle \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'})$, for $K \subseteq N$. Using Lemma 3.4 (2) with $J = \emptyset$, it results $\int_{X_N} f d\mu_N = c$. Since $f \in \mathcal{M}$, then $c = 1$. By Lemma 3.3 (2), for each $i \in N$, there exists $z_i \in L_0^1(X_i, \mu_i)$ such that $f_{N \setminus \{i\}} = \otimes_{k \in N \setminus \{i\}} g_k \otimes z_i$. If $i \in N$, by Lemma 3.2. with $J = \{i\}$ and (3.9) with $c = 1$, we have

$$s_i(f) = g_i + z_i.$$

By Theorem 3.1, since $f \in \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N))$, then $s_i(f) = g_i$, and consequently $z_i = 0$. Thus, for $i \in N$, $f_{N \setminus \{i\}} = 0$, resulting for f the expression that we like. $\square$

Remark 3.6. Using (3.8) and a similar argument to that used in the previous proof, we obtain that

$$\pi_{h_N}^{-1}(\Psi(h_N)) = \{f \in L^1(X_N, \mu_N) : f = \left(\int_{X_N} f d\mu_N\right) \Psi(h_N) + r,$$
$$r \in \oplus_{K \subseteq N, |K| \geq 2} \otimes_{k \in N \setminus K} \langle h_k \rangle \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'})\}.$$

By Theorem Theorem 3.5,

$$(3.10) \qquad \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) - \Psi(g_N) =$$
$$\oplus_{K \subseteq N, |K| \geq 2} \otimes_{k \in N \setminus K} \langle g_k \rangle \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'}).$$

Corollary 3.1. Let $g_i' \in L^1(X_i, \mu_i)$ be such that $\int_{X_i} g_i' d\mu_i = 1$, $i \in N$. Then $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) - \Psi(g_N) = \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N')) - \Psi(g_N')$.

Proof. If $r \in \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) - \Psi(g_N)$, then there exists $f \in \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N))$ such that $r = f - \Psi(g_N)$. By Theorem 3.1, $s_i(f) = g_i$, so $f' = r + \Psi(g_N')$ satisfies $s_i(f') = g_i'$. Consequently, using again Theorem 3.1, $f' \in \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N'))$, and thus $r \in \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N')) - \Psi(g_N')$. This shows that $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) - \Psi(g_N) \subseteq \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N')) - \Psi(g_N')$. Since (g_N) and (g_N') are arbitrary, this shows that $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) - \Psi(g_N) = \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N')) - \Psi(g_N')$. $\qquad\square$

By the previous corollary and (3.10),

$$\oplus_{K \subseteq N, |K| \geq 2} \otimes_{k \in N \setminus K} \langle g_k \rangle \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'}) =$$
$$= \oplus_{K \subseteq N, |K| \geq 2} \otimes_{k \in N \setminus K} \langle g_k' \rangle \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'}).$$

In what follows we will denote with $\mathcal{R}$ the set given by

$$(3.11) \qquad \mathcal{R} = \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) - \Psi(g_N)$$
$$= \oplus_{K \subseteq N, |K| \geq 2} \otimes_{k \in N \setminus K} \langle g_k \rangle \otimes_{k' \in K} L_0^1(X_{k'}, \mu_{k'})$$

The previous results tell us that the Banach space $\mathcal{R}$ ($\mathcal{R}$ is a closed subspace of $L^1(X_N, \mu_N)$) do not depend of the functions g_i nor of the functions h_i.

By the results of this section, when each $g_i \in \widetilde{X}_i$, by Theorem 3.1, $U(g_N)$ consists of the nonnegative elements of $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N))$ whose analytical description is given by Theorem 3.5. Concretely,

$$U(g_N) = \{f \in \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) : f \geq 0\}$$
$$= \{\Psi(g_N) + r : r \in \mathcal{R} \, and \, r \geq -\Psi(g_N)\}.$$

4. The natural vector bundle associated with $\widetilde{X_N}$

In this section we are going to use some notions about differentiable manifolds (see e.g. Larotonda (1980)). For simplicity, we are going to use the term manifold as a synonym of C^∞-manifold.

In order to obtain the vector bundle we introduce some differentiable structures.

Lemma 4.1. *4.1. Let q be the continuous linear transformation defined by*

(4.1)
$$q : L^1(X, \mu) \to \mathbf{R}$$
$$f \to \int_X f d\mu.$$

Then q is a submersion.

Proof. Clearly q is C^∞. Since for all $f \in L^1(X, \mu)$, $Dq(f) = q$ is onto, then $Ker(Dq(f)) = L_0^1(X, \mu)$ and, by Theorem 2.1, $L_0^1(X, \mu)$ has a topological supplement, then $Dq(f)$ is a topological epimorphism. Thus q is a submersion. $\qquad\square$

Proposition 4.2. *4.2. The set $\mathcal{M}$ given in (3.3) is a closed submanifold of $L^1(X_N, \mu_N)$ pure of model $L_0^1(X_N, \mu_N)$.*

Proof. If q is given by **(??)** with (X_N, μ_N) instead of (X, μ), then $\mathcal{M} = q^{-1}(\{1\}) \neq \emptyset$ and the proposition follows from Lemma **??**. $\qquad\square$

In a similar manner it can be proved that M_i (see (3.2) is a closed submanifold of $L^1(X_i, \mu_i)$ pure of model $L_0^1(X_i, \mu_i)$. We also consider the product manifold $M_N = \times_{i=1}^n M_i = range(p_{h_N})$ pure of model $\times_{i=1}^n L_0^1(X_i, \mu_i)$.

Lemma 4.3. *4.3. Let Ψ be the continuous symmetric multilinear function given by (3.5). Then $\Psi|M_N : M_N \to L^1(X_N, \mu_N)$ is a regular immersion.*

Proof. Let $(f_N), (f'_N) \in M_N$ such that $\Psi(f_N) = \Psi(f'_N)$. For each $i \in N$,

$$\int_{X_{N \setminus \{i\}}} \Psi(f_N) d\mu_{N \setminus \{i\}} = \int_{X_{N \setminus \{i\}}} \Psi(f'_N) d\mu_{N \setminus \{i\}},$$

or which is the same, $f_i = f'_i$. Then, $(f_N) = (f'_N)$. This shows that $\Psi|M_N$ is one to one.

Since

$$\|\Psi(f_N) - \Psi(f'_N)\|_{(X_N, \mu_N)} =$$
$$= \left\|\Psi(f_N) - f_i \otimes_{k \in N \setminus \{i\}} f'_k + f_i \otimes_{k \in N \setminus \{i\}} f'_k - \Psi(f'_N)\right\|_{(X_N, \mu_N)}$$
$$\leq \|f_i\|_{(X_i, \mu_i)} \left\|\otimes_{k \in N \setminus \{i\}} f_k - \otimes_{k \in N \setminus \{i\}} f'_k\right\|_{(X_{N \setminus \{i\}}, \mu_{N \setminus \{i\}})} +$$
$$+ \|f_i - f'_i\|_{(X_i, \mu_i)} \left\|\otimes_{k \in N \setminus \{i\}} f'_k\right\|_{(X_{N \setminus \{i\}}, \mu_{N \setminus \{i\}})},$$

it results that Ψ is continuous.

If $(f_N), (f'_N) \in M_N$, we have

$$\int_{X_{N \setminus \{i\}}} \otimes_{k \in N \setminus \{i\}} f_k d\mu_{N \setminus \{i\}} = \int_{X_{N \setminus \{i\}}} \otimes_{k \in N \setminus \{i\}} f'_k d\mu_{N \setminus \{i\}} = 1$$

for each $i \in N$. Therefore

$$\|f_i - f'_i\|_{(X_i, \mu_i)} =$$
$$= \int_{X_i} \left\lfloor f_i \int_{X_{N \setminus \{i\}}} \otimes_{k \in N \setminus \{i\}} f_k d\mu_{N \setminus \{i\}} \right.$$
$$\left. - f'_i \int_{X_{N \setminus \{i\}}} \otimes_{k \in N \setminus \{i\}} f'_k d\mu_{N \setminus \{i\}} \right| d\mu_i$$
$$= \int_{X_i} \int_{X_{N \setminus \{i\}}} |\Psi(f_N) - \Psi(f'_N)| \, d\mu_{N \setminus \{i\}} d\mu_i$$
$$= \|\Psi(f_N) - \Psi(f'_N)\|_{(X_N, \mu_N)}.$$

Thus, $\Psi|M_N$ is a topological isomorphism from M_N to $\Psi(M_N)$.

Since Ψ is a continuous symmetric multilinear function then it is C^∞ and its differential is given by

$$D\Psi(g_N)(z_N) = \sum_{i=1}^{n} \Psi(z_i, g_{N \setminus \{i\}}),$$

for all $(g_N), (z_N) \in \times_{i=1}^n L^1(X_i, \mu_i)$. Let $(g_N) \in M_N, (z_N) \in \times_{i=1}^n L_0^1(X_i, \mu_i)$ be such that $D\Psi(g_N)(z_N) = 0$. For each $k \in N$,

$$z_k = \sum_{i=1}^n \int_{X_{N\setminus\{k\}}} \Psi(z_i, g_{N\setminus\{i\}}) d\mu_{N\setminus\{k\}}$$
$$= \int_{X_{N\setminus\{k\}}} D\psi(g_N)(z_N) d\mu_{N\setminus\{k\}}$$
$$= 0.$$

Then $(z_N) = 0$. Therefore, for each $(g_N) \in M_N$, $D\Psi(g_N)|\times_{i=1}^n L_0^1(X_i, \mu_i)$ is one to one.

We also have, from Lemma 3.3 and Theorem 2.2, that for each $(g_N) \in M_N$, the set

$$range(D\Psi(g_N)| \times_{i=1}^n L_0^1(X_i, \mu_i))$$
$$= \{\sum_{i=1}^n \Psi(z_i, g_{N\setminus\{i\}}) : z_i \in L_0^1(X_i) \text{ for } i \in N\}$$
$$= \oplus_{i=1}^n \otimes_{k \in N\setminus\{i\}} \langle g_k \rangle \otimes L_0^1(X_i, \mu_i),$$

is a direct subspace of $L^1(X_N, \mu_N)$.

Consequently, for each $(g_N) \in M_N$, $D\Psi(g_N)| \times_{i=1}^n L_0^1(X_i, \mu_i)$ is a direct monomorphism.

Since $\Psi|M_N$ is a topological isomorphism from M_N to $\Psi(M_N)$, Ψ is C^∞ and for each $(g_N) \in M_N$, $D\Psi(g_N)| \times_{i=1}^n L_0^1(X_i, \mu_i)$ is a direct monomorphism, then Ψ is a regular immersion. □

As a consequence of the previous lemma we have

Proposition 4.4. *4.4. $\Psi(M_N)$ is a submanifold of $L^1(X_N, \mu_N)$ and $\Psi|M_N : M_N \to \Psi(M_N)$ is a C^∞-isomorphism.*

In the sequel we will think about $\Psi|M_N$ as a function from M_N onto $\Psi(M_N)$ and we will consider in $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N))$ the structure of normed vectorial space that makes the map

$$T : \mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) \to \mathcal{R}$$
$$f \to f - \Psi(g_N)$$

a topological isomorphism. With this structure, $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N))$ is a Banach space.

Now we are in condition to state and prove the main result of this section.

Theorem 4.5. *4.5.* $(\Psi(M_N), \mathcal{M}, \pi_{h_N}|\mathcal{M})$ *is a vector bundle with type fiber* $\mathcal{R}$.

Proof. Since $s_N : L^1(X_N, \mu_N) \to \times_{i=1}^n L^1(X_i, \mu_i)$ is C^∞, then $s_N|\mathcal{M} : \mathcal{M} \to M_N$ is C^∞. We also have that $\Psi : M_N \to \Psi(M_N)$ is C^∞ and $\pi_{h_N}|\mathcal{M} = \Psi p_{h_N} s_N|\mathcal{M} = \Psi s_N|\mathcal{M}$. Then, $\pi_{h_N}|\mathcal{M} : \mathcal{M} \to \Psi(M_N)$ is C^∞, because is the composition of C^∞ functions.

Now define

$$\varphi_{h_N} : (\pi_{h_N}|\mathcal{M})^{-1}(\Psi(M_N)) \to \Psi(M_N) \times \mathcal{R}$$
$$\Psi(g_N) + r \to (\Psi(g_N), r).$$

Clearly, φ_{h_N} is a C^∞-isomorphism and

$$p_{\Psi(M_N)} \varphi_{h_N}(\Psi(g_N) + r) = \Psi(g_N)$$
$$= (\pi_{h_N}|M)(\Psi(g_N))$$

where $p_{\Psi(M_N)} : \Psi(M_N) \times \mathcal{R} \to \Psi(M_N)$ is the projection. Taking into in account the structure of normed vectorial space defined in $\mathcal{M} \cap \pi_{h_N}^{-1}(\Psi(g_N)) = (\pi_{h_N}|\mathcal{M})^{-1}(\Psi(g_N))$, this set is a Banach space and the map

$$(\pi_{h_N}|\mathcal{M})^{-1}(\Psi(g_N)) \to$$
$$f \to f - \Psi(g_N)$$

is a topological isomorphism. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By the results of the previous section, if for each $i \in N$, $g_i \in \widetilde{X_i}$, then $U(g_N)$ consists of the nonnegative elements of the fiber $(\pi_{h_N}|\mathcal{M})^{-1}(\Psi(g_N))$. Also, any density function f in $L^1(X_N, \mu_N)$ with unidimensional marginals g_i, can be written as $f = \Psi(g_N) + r$ where r is in the type fiber $\mathcal{R}$.

5. Concluding remarks.

If the sets X_i are finite and we consider on them the cardinal measure, the above results reduce to those in Marchi (1972). As we have mentioned in the introduction, in the discrete case they motivate a great variety of new concepts of cooperative equilibrium in game theory. Now we could use the results obtained in this paper to extend all these concepts for continuous games. On the other hand, there are several properties of the derived fiber

bundle that can be studied. For example, problems concerning perturbation of the fibers and cross sections.

It is important to note that the set of density functions with given marginals appears in the study of the optimal transportation problem (see, e. g., Cafarelli, Feldman and McCann (2001), Cafarelli (2003)). More recently, it also appears in conexion with the application of game theory to physics, in particular, to quantum mechanics (see, e.g., Landsburg and references therein). We think that the results obtained here provide new geometrical tools to study all these matters.

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E. Marchi
Instituto de Matemática Aplicada San Luis
UNSL-CONICET
Ejército de los Andes 950,
5700 San Luis
Argentina
e-mail : emarchi@unsl.edu.ar

and

P. M. Morillas
Instituto de Matemática Aplicada San Luis
UNSL-CONICET
Ejército de los Andes 950
5700 San Luis
Argentina
e-mail : morillas@unsl.edu.ar