Proyecciones
Vol. 24, N$^o$ 3, pp. 229-237, December 2005.
Universidad Católica del Norte
Antofagasta - Chile

# FIXED POINTS OF A FAMILY OF EXPONENTIAL MAPS

*ERIC BLABAC* *

*and*

*JUSTIN PETERS*

*Iowa State University, U. S. A.*

*Received : September 2005. Accepted : November 2005*

**Abstract**

*We consider the family of functions* $f_\lambda(z) = \exp(i\lambda z)$, $\lambda$ *real. With the help of MATLAB computations, we show* $f_\lambda$ *has a unique attracting fixed point for several values of* $\lambda$. *We prove there is no attracting periodic orbit of period* $n \geq 2$.

**Keywords subjclass :** *37510*

*Eric Blabac is an MS student in the Mathematics Department at Iowa State University.

## 1. Introduction

In this note we show the existence and uniqueness of an attracting fixed point for the map $f_\lambda(z) = \exp(i\lambda z),\ z \in \mathbf{C}$, for certain (real) values of the parameter $\lambda$. The proofs depend on MATLAB calculations, and as such can be viewed as computer-assisted proofs. By contrast, the exponential map $z \mapsto \exp(z)$ admits no attracting fixed point. ([2], see Remark 12)

We began with a MATLAB program in which one inputs a value $\lambda$, a point $z_0$ a positive integer $N$, and a tolerance $\epsilon$. The program outputs $f^n(z_0)$, where $n$ is the least integer $k \leq N$ for which

$$|f^{k-1}(z_0) - f^k(z_0)| < \epsilon$$

if there is such an $n$. Here $f^n(z)$ is the $n-$fold iterate of $f$ at $z$: thus, $f^1(z) = f(z)$, and $f^n(z) = f(f^{n-1}(z))$ for $n > 1$.

We experimented with various values of $\lambda$ and the computations indicated the function $f_\lambda$ had a fixed point inside the circle $|z| \leq 1/|\lambda|$ for $|\lambda| \leq 1.96$ (approximately). However in order to prove the existence of a fixed point, one must use a theorem, which usually involves an invariant domain. Since it was not easy to find such a domain, we took another approach, using the Maximum Modulus Theorem. While the analysis is elementary, the computer-assisted proofs have yielded new results.

**Definition 1.** A point $z \in \mathbf{C}$ is called an **attracting fixed point** of a map $f$ if $z$ satisfies $f(z) = z$ and $|f'(z)| < 1$.

Suppose that $z_0$ is an attracting fixed point of the map $f(z)$. Then for $z$ sufficiently close to $z_0$, the iterates $z,\ f(z),\ f^2(z), \dots$ converge to $z_0$.

Let's begin by stating some facts. Let $f$ be the function $f_\lambda$.

*Fact 1.* If $z_0$ is an attracting fixed point of $f$, then $|z_0| < 1/|\lambda|$.

Conversely, any fixed point in the disk $|z| < 1/|\lambda|$ is attracting.

**Proof.** We have $|f'(z_0)| = |i\lambda f(z_0)| = |\lambda||z_0| < 1 \iff |z_0| < 1/|\lambda|$. □

*Fact 2.* For $z \in \mathbf{C},\ z = x + iy\ (x,\ y \text{ real})$, $|f'(z)| < 1 \iff (\operatorname{sgn} \lambda)y > \frac{1}{|\lambda|}\log(|\lambda|)$.

**Proof.** If $z = x + iy$,

$$|f'(z)| = |i\lambda \exp(i\lambda z)| = |\lambda|e^{-\lambda y},$$

The conclusion is an easy exercise. □

*Fact 3.* The region $\mathbf{R}_\lambda$ defined by the inequalities

$$|z| < \frac{1}{|\lambda|}, \quad \mathrm{sgn}(\lambda)y > \frac{1}{|\lambda|}\log(|\lambda|),\ z = x + iy$$

is empty if $|\lambda| \geq e$.

**Proof.** $\frac{1}{|\lambda|}\log(|\lambda|) < \mathrm{sgn}(\lambda)y \leq |z| < \frac{1}{|\lambda|}$, so $\log(|\lambda|) < 1$, or $|\lambda| < e$. □

Our first goal was to prove the existence of a fixed point inside the circle $|z| = 1/|\lambda|$. We did this using the Maximum Modulus Theorem, or rather a Corollary, called the Minimum Modulus Theorem. Our MATLAB calculations indicated the attracting fixed point should lie in the intersection of $|z| \leq 1/\lambda$ with the first quadrant (for $\lambda > 0$), denoted $\mathcal{Q}_\lambda$ (or simply $\mathcal{Q}$ if $\lambda$ is fixed), so it was in that region we applied the the Minimum Modulus Principle.

A more conventional approach to the existence of a fixed point using, say, the Brouwer Theorem, requires a having region which is mapped into itself by $f$. But neither $\mathcal{Q}_{\lambda\lambda}$ nor $\mathcal{R}_\lambda$ (see Fact 3) is invariant. Say $\lambda > 1$, $0 \in \mathcal{Q}_\lambda$ but $1 = f_\lambda(0) \notin \mathcal{Q}_{\lambda\lambda}$. Also $\frac{i}{\lambda} \in \mathcal{R}_\lambda$, but $f_\lambda(\frac{i}{\lambda}) = e^{-1} \notin \mathcal{R}_\lambda$ since $\mathcal{R}_\lambda$ is disjoint from the real axis if $\lambda > 1$. (But see Remark 9.)

**Remark 2.** It is enough to determine fixed points of $f_\lambda$ for $\lambda > 0$, for

$$f_{-\lambda}(\overline{z}) = \exp(-i\lambda\overline{z}) = \exp(\overline{(i\lambda z)}) = \overline{f_\lambda(z)},$$

the last equality resulting from the fact the power series has real coefficients. Also, it is an easy observation that $\overline{z_0}$ is an attracting fixed point for $f_{-\lambda}$ iff $z_0$ is an attracting fixed point for $f_\lambda$.

**Notation 1.** We use both ‘ $\exp(\cdot)$ ’ and ‘$e^{\cdot}$ ’ to denote the exponential function.

## 2. Existence and Uniqueness of the Fixed Point

**Theorem 3.** Minimum Modulus Principle ([3], p. 313)

Let $U$ be a bounded open set in $\mathbf{C}$, and let $f$ be analytic on $U$ and continuous on the closure $\bar{U}$. Assume that $f$ never vanishes on $\bar{U}$. Then the minimum value of $|f|$ on $\bar{U}$ occurs on the boundary, $\partial U$.

**Theorem 4.**

Let $\lambda \in \mathbf{R}$ be one of the values $0.1, 0.2, 0.3, \ldots, 1.8, 1.9, 1.95,$ or $1.96$. Let $\mathcal{Q}$ be the intersection of the first quadrant $\Re z \geq 0,\ \Im z \geq 0$, with the closed disk $|z| \leq 1/\lambda$. Then there exists a fixed point $z$ in the interior $\mathcal{Q}^o$ of $\mathcal{Q}$ for the map $f(z) = \exp(i\lambda z)$.

**Proof.** Our calculations using MATLAB show that

$$[\min\{z \in \mathcal{Q} : |z - f(z)|\} < 10^{-6}.$$

The idea of the proof is to show $|z - f(z)|$ is bounded below along the boundary of $\mathcal{Q}$ by some constant which is greater than $10^{-6}$. Theorem would then assert the existence of a fixed point for $f$ in $\mathcal{Q}$. Writing $z = x + iy$, for $x, y \in \mathbf{R}$ we have

$$\begin{aligned} |f(z) - z|^2 &= |exp(i\lambda z) - z|^2 \\ &= |e^{-\lambda y} e^{i\lambda x} - (x + iy)|^2 \\ &= |e^{-\lambda y}(\cos(\lambda x) + i\sin(\lambda x)) - (x + iy)|^2 \\ &= e^{-2\lambda y} + x^2 + y^2 - 2e^{-\lambda y}(x\cos(\lambda x) + y\sin(\lambda x)). \end{aligned}$$

Denote the right hand side of the above by $g(x, y)$. We now have to check the values of $g(x, y)$ along the boundaries of $\mathcal{Q}$. We will start with $y = 0$. Then for $0 \leq x \leq 1/\lambda$

$$g(x, 0) = x^2 + 1 - 2x\cos(\lambda x).$$

Thus

$$g(x, 0) = (x - 1)^2 + 2x(1 - \cos(\lambda x)).$$

Since $0 < \lambda < e$ we have $g(x, 0) \geq (1 - e^{-1})^2$ on $0 \leq x \leq e^{-1}$ and by $2e^{-1}(1 - \cos(\frac{\lambda}{e})) \geq 2e^{-1}(1 - \cos(1))$ on $e^{-1} \leq x \leq \lambda^{-1}$.

Next we check the boundary $x = 0,\ 0 \leq y \leq 1/\lambda$:

$$g(0, y) = y^2 + e^{-2\lambda y} \geq e^{-2\lambda y} \geq \frac{1}{e^2}.$$

Finally we need to check the boundary on the quarter circle; it is convenient to convert to polar coordinates

$$x = (1/\lambda)\cos\theta, \quad y = (1/\lambda)\sin\theta, \quad \text{for } 0 \leq \theta \leq \pi/2.$$

Then,

$$g\left(\tfrac{1}{\lambda}\cos\theta, \tfrac{1}{\lambda}\sin\theta\right) =$$
$$(\tfrac{1}{\lambda})^2 + e^{-2\sin\theta} - e^{-\sin\theta}\left(\tfrac{2}{\lambda}\cos\theta\,\cos(\cos\theta) - \tfrac{2}{\lambda}\sin\theta\,\sin(\cos\theta)\right)$$

and using the identity $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ we see that

$$\begin{aligned} g(\tfrac{1}{\lambda}\cos\theta, \tfrac{1}{\lambda}\sin\theta) &= (\tfrac{1}{\lambda})^2 + e^{-2\sin\theta} - e^{-\sin\theta}(\tfrac{2}{\lambda}\cos(\theta - \cos\theta)) \\ &= (\lambda^{-1} - e^{-\sin\theta})^2 + \tfrac{2}{\lambda}e^{-\sin\theta}(1 - \cos(\theta - \cos\theta)). \end{aligned}$$

Since both $(\lambda^{-1} - e^{-\sin\theta})^2$ and $\frac{2}{\lambda}e^{-\sin\theta}(1 - \cos(\theta - \cos\theta))$ are nonnegative, we must see where both terms are zero: the second expression is zero only when $\cos\theta = \theta$. Call this $\theta_0$; so $\theta_0$ is approximately .739085. Putting $\theta = \theta_0$ into the first term and setting it to zero yields $\lambda = \lambda_0 := e^{\sqrt{1-\theta_0^2}}$, or approximately 1.96131.

From this we can conclude: suppose $\lambda$ is in the interval $0 < \lambda \leq 1.96$. Then for $0 \leq \theta \leq .741$,

$$\lambda^{-1} - e^{-\sin\theta} \geq 1.96^{-1} - e^{-\sin .741} > 10^{-3}.$$

And for $.741 \leq \theta \leq \frac{\pi}{2}$,

$$\begin{aligned} &\tfrac{2}{\lambda}e^{-\sin\theta}(1 - \cos(\theta - \cos\theta)) \\ &\geq \tfrac{2}{1.96}e^{-\sin .741}(1 - \cos(.741 - \cos .741)) > (2.67)10^{-6} \end{aligned}$$

We conclude that $|g(x,y)|^{\frac{1}{2}} > 10^{-3}$ on the boundary of $\mathcal{Q}$. Since

$$\min_{z\in\mathcal{Q}}\{|z - f(z)|\} = \min_{(x,y)\in\mathcal{Q}}\{g(x,y)^{\frac{1}{2}}\} < 10^{-6}$$

for any $\lambda$ in the Table, it follows from the Minimun Modulus Principle that the function $f(z) = \exp(i\lambda z)$ has a fixed point. □

**Corollary 5.** From Fact 1, any fixed point for $f$ inside the circle $|z| < \frac{1}{\lambda}$ is attracting, so that Theorem 4 establishes the existence of an attracting fixed point in $\mathcal{Q}^\circ$. By Fact 2, the fixed point lies in the intersection $\mathcal{Q}^\circ \cap \mathbf{R}$.

Below are the approximate fixed points for different values of $\lambda$. For each $\lambda$ the program terminated when $|f(z) - z| < 10^{-8}$.

Table of Fixed Points

| $\lambda$ | Fixed Point | $\lambda$ | Fixed point |
|---|---|---|---|
| .1 | $0.9854986 + 0.0974364i$ | 1.1 | $0.5463106 + 0.3745068i$ |
| .2 | $0.9470891 + 0.1815722i$ | 1.2 | $0.5191342 + 0.3729401i$ |
| .3 | $0.8954147 + 0.2464877i$ | 1.3 | $0.4945130 + 0.3703950i$ |
| .4 | $0.8396697 + 0.2931240i$ | 1.4 | $0.4721254 + 0.3671575i$ |
| .5 | $0.7852571 + 0.3251993i$ | 1.5 | $0.4516947 + 0.3634355i$ |
| .6 | $0.7346292 + 0.3465471i$ | 1.6 | $0.4329840 + 0.3593815i$ |
| .7 | $0.6885773 + 0.3602367i$ | 1.7 | $0.4157904 + 0.3551080i$ |
| .8 | $0.6470987 + 0.3685124i$ | 1.8 | $0.3999401 + 0.3506985i$ |
| .9 | $0.6098583 + 0.3729619i$ | 1.9 | $0.3852840 + 0.3462151i$ |
| 1.0 | $0.5764127 + 0.3746990i$ | 1.95 | $0.3783627 + 0.3439606i$ |
| | | 1.96 | $0.3770090 + 0.3435093i$ |

One might infer from our results that $f_\lambda$ has a attracting fixed point for all values of $\lambda,\ 0 < \lambda < \lambda_0$. But our method of proof can only be applied to finitely many $\lambda$.

**Remark 6.** For $\theta_0,\ \lambda_0$ as in the proof of the theorem, the proof shows that the function $f_0(z) = \exp(i\lambda_0 z)$ has a fixed point at $z = z_0 := \lambda_0^{-1} e^{i\theta_0}$. Since $z_0$ lies on the circle $|z| = 1/\lambda_0$, it follows $|f_0'(z_0)| = 1$. Such a point is called a nonhyperbolic, or neutral fixed point.

**Remark 7.** It is possible to express the fixed point $z$ of $f_\lambda$ as an analytic function of $\lambda$. Solving $z = f_\lambda(z)$ for $\lambda$ yields $\lambda = -i\ \log(z)/z$. The inverse function is given by $z = g(\lambda) := iW(-i\lambda)/\lambda$ where $W$ is the Lambert $W$-function, or the principal branch of the inverse of $w \to we^w$. Since the values of $W$ are not easy to calculate, this does not simplify the question of deciding when the fixed point $z = g(\lambda)$ is attracting, i.e., when it satisfies $|g(\lambda)| < \frac{1}{|\lambda|}$. (Cf [1].)

Our MATLAB computation indicates the fixed point is unique. That is indeed the case, as we now prove.

**Theorem 8.** Let $\lambda$ be one of the values in the Table. Then the map $f(z) = \exp(i\lambda z)$ has a unique attracting fixed point.

**Proof.**

Let $\mathbf{R}$ be the region in $\mathcal{C}$ determined by the two inequalities $x^2 + y^2 < 1/\lambda^2$ and $y > \frac{1}{\lambda}\log(\lambda)$. The region $\mathbf{R}$ is convex, and by Facts 1, 2, and 3 it is nonempty, and contains all attracting fixed points of the map $f$.

Suppose now that $z_1,\ z_2$ are two distinct fixed points of $f$. Then $z_1$, $z_2$ lie in $\mathbf{R}$, and if $\mathcal{C}$ is a contour joining $z_1$and $z_2$,

$$\begin{aligned}|z_1 - z_2| = |f(z_1) - f(z_2)| &= |\textstyle\int_{\mathcal{C}} f\prime(z)\, dx| f\prime(z)\, dx| \\ &\leq \textstyle\int_{\mathcal{C}} |f\prime(z)|\, d|z| \leq \max_{z\ \in\mathcal{C}}\{|f\prime(z)|\} length(\mathcal{C}).\end{aligned}$$

If $\mathcal{C}$ is the straight line contour joining $z_1$and $z_2$, then $\mathcal{C} \subset \mathbf{R}$ so that $|f\prime(z)| < 1$ for $z \in \mathcal{C}$, and since $\mathcal{C}$ is compact, $\max_{z\in\mathcal{C}}\{|f\prime(z)|\} < 1$.Since length($\mathcal{C}$) $= |z_1 - z_2|$ the calculation above implies $|z_1 - z_2| < |z_1 - z_2|$, which is absurd. Thus the fixed point is unique. □

**Remark 9.** An alternative, more conventional approach to the existence of a fixed point may be possible using a standard fixed point theorem, such as the Brouwer Theorem. Assume that $z_0$satisfies $|f(z_0) - z_0| < 10^{-6}$, and $|f'(z_0)| < 1$.Let $\epsilon = 1 - |f'(z_0)|$.There is a $\delta > 0$such that $|f'(z)| < 1 - \epsilon/2$ for $|z - z_0| < \delta$. So for $|z - z_0| \leq \delta$,

$$\begin{aligned}|f(z) - z_0| &\leq |f(z) - f(z_0)| + |f(z_0) - z_0| \\ &< \max\{|f'(z)| : z \in \mathcal{C}\}\,\delta + 10^{-6} \\ &< (1 - \epsilon/2)\delta + 10^{-6} \\ &< \delta\end{aligned}$$

(where $\mathcal{C}$ is the line segment joining $z$ and $z_0$) is valid as long as $10^{-6} < \frac{\epsilon}{2}\delta$. A tolerance finer than $10^{-6}$ may be required. We have not carried out these calculations. However, we do not see how the critical value $\lambda_0$ could be obtained through this approach.

## 3. Attracting Orbits

If $z_0,\ z_1, \ldots, z_{n-1}$ is a set of points satisfying

**Remark 1.**

$$z_1 = f(z_0),\ z_2 = f(z_1), \ldots, z_0 = f(z_{n-1})$$

then $z_0,\ z_1, \ldots, z_{n-1}$ is called an *orbit of period* $n$.

**Definition 10.** Let $z_0,\ z_1, \ldots, z_{n-1}$ be an orbit of period $n$. It is said to be an *attracting* orbit if $|f'(z_k)| < 1,\ 0 \leq k \leq n-1$.

Let $f(z) = \exp(i\lambda z),\ \lambda > 0$, and $z_0,\ z_1, \ldots, z_{n-1}$ a period $n$ orbit, and assume the orbit is attracting. Observe

$$|f'(z_0)| = |i\lambda\exp(i\lambda z_0)| = \lambda|z_1| < 1.$$

Similarly, $z_2, \ldots, z_{n-1},\ z_0$ lie in the circle $|z| < 1/\lambda$.

Furthermore, it follows from Fact 2 that $z_k$ satisfy $y_k > \frac{1}{\lambda}\log(\lambda)$, where $z_k = x_k + iy_k$. Thus, $z_0,\ z_1, \ldots, z_{n-1}$ lie in the region $\mathbf{R}$ in the complex plane determined by the two inequalities $x^2 + y^2 < 1/\lambda^2$ and $y > \frac{1}{\lambda}\log(\lambda)$.

**Theorem 11.** Let $0 \neq \lambda \in \mathbf{R}$. Then the map $f(z) = \exp(i\lambda z)$ does not have any attracting periodic orbit of period $n$, for $n \geq 2$.

**Proof.** As noted above (cf Remark 2) it is enough to prove the assertion for $\lambda > 0$. The proof is in the spirit of the uniqueness proof (Theorem 8). $\square$

Note our definition of attracting orbit is stronger than the standard definition ([2]): $|(f^n)'(z_0)| < 1$, or equivalently that $|f'(z_0)\, f'(z_1) \cdots f'(z_{n-1})| < 1$.

**Remark 12.** Recall that the Julia set of a map $f$ is the closure of the repelling periodic points. It's interesting to note the difference between the maps $f_\lambda$ and the exponential map, $z \to e^z$. For the exponential map, it is shown in [2] that the Julia set is all of $\mathbf{C}$. But for the maps $f_\lambda$ (at least for the values of $\lambda$ in the table), the Julia set is a proper subset: indeed, there is an open neighborhood $U$ of the attracting fixed point, which is invariant under $f_\lambda$, not containing any other periodic points. Thus, the Julia set of $f_\lambda$ is a proper subset of $\mathbf{C}$.

Of course each $f_\lambda$ has a Julia set which is unbounded, and hence $f_\lambda$ has infinitely many repelling periodic points.

## References


[1] Borwein, Jonathan M, and Corless, Robert M., *Emerging Tools for Experimental Mathematics*, Amer. Math. Monthly **106**, No. 10, pp. 899–909, (1999).

[2] Devaney, Robert L., *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, (1989).

[3] Rubenfeld, Lester A., *A First Course in Applied Complex Variables*, John Wiley & Sons, (1985).

**Eric M. Blabac**
Department of Mathematics
Iowa State University
Ames, IA 50011
U. S. A.
e-mail : ebikeman@iastate.edu

and

**Justin R. Peters**
Department of Mathematics
Iowa State University
Ames, IA 50011
e-mail : peters@iastate.edu