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OCCUPATION TIMES SEQUENCES AND MARTINGALES OF SIMPLE RANDOM WALKS ON THE REAL LINE

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Abstract

Given a simple random walk on the real line , we consider the sequences of occupation times on states and associate to them martingales defined by the moments of first order of this random walk. We deduce by this way recurrent relations for the expectations of the occupation times in states before a given time , and then remarkable identities for the expectations of the absolute values of the random walk.

Key words : *occupation times; simple random walks; predictible compensators; first passage times ; optional sampling theorem; first order absolute moments.*

On a certain space of probability a sequence of independent variables Y $= (Y_1, Y_2, \cdots, Y_n, \cdots)$ is considered from now on. , and the random walk on the real line $S = (S_n)_{n\in\mathbf{N}}$, with $S_n = Y_1 + Y_2 + \cdots + Y_n, S_0 = 0$. We shall deal here with simple random walks for which the common law of the sequence Y is of the form $p\partial_1 + q\partial_{-1} + r\partial_0$, where ∂_x is the Dirac measure in x, and $p + q + r = 1$. In other words , the support of the law is the set $\{-1, 0, 1\}$. It is denoted by μ the mean media and by σ^2 the variance of this law. In other works regarding random walks it has been thoroughly studied their Bernoulli case for simple random walks, that is when $p+q = 1$.

Given a set $A \in \mathbf{Z}$ and a random variable T with values in $\mathbf{N} \cup \{\infty\}$, it is defined the variable $N_T^A = \operatorname{card}\{k, 0 \leq k < T, S_k \in A\}$. It is called *the occupation time of the random walk in A before T*. The *sequence of occupation times in A* is $\left(N_n^A\right)_{n\in\mathbf{N}}$, this is that of occupation times before constant times. Subsequently we will be centered mainly in the occupation times sequence for sets A which are unitary.

Occupation times sequences have been widely studied within the classical theory for random walks (see [1] for example), related to so called Green functions, those of the form $x \to E\left(N_\infty^{A+x}\right)$. Even these times have been studied thoroughly, the general theory seems not have completely discussed the problem of the calculation of the expectations $E\left(N_T^x\right)$ for times T defined above.

Through this work we will prove that for simple random walks, occupation times sequences can be interpreted in terms of martingales ; in a more concrete way , regarding to the natural filtration of the random walk, these sequences are “almost ” the predictable compensators of sequences of the form $(|S_n - c|, n \geq 0)$, this is the absolute values of the translated random walks. This is basically the contents of theorem 1.3 of section 1.

We find in literature regarding random walks certain precedents about the result previously mentioned. In effect , given the symmetric and simple random walk S of variance 1 and transition kernel P, in the problem of the discrete Poisson equation over $\mathbf{Z}$, it is considered the equation $Pf - f = \partial_c$, with the condition for f to be positive over $\mathbf{Z}$. It is demonstrated in [1] that the solution is $f(x) = |x - c|$, which means that $\left(|S_n - c| - N_n^{\{c\}}\right)_{n\in\mathbf{N}}$ is a martingale of the natural filtration of S. Likewise, in E. Perkins works (see [6]), it is established a decomposition for the paths of a class of simple random walks which leads to one result of the same type.

Even though the result of section 1 is relatively easy to establish , it is strong enough as to deduce some new results over occupation times, properties of first passage times on boundaries, properties of the sequence of expectations $(E(|S_n - c|), n \geq 0)$, and others that appear in sections 2, 3 and 4. In all of them martingales related to the occupation times sequences are substantially capitalized on (theorem 1.5), and the possibility that these provide in order to apply the results of the theory of martingales in discrete time, among them the Doob optional sampling theorem.

It is considered the *natural filtration* of S, that is , the increasing sequence of algebras $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$, being F_n that generated by $Y_1, Y_2, \cdots, Y_n$. It will be denoted by F . As it is well known (see example [5]), all process X adapted to F admits a *predictable ompensator* with respect to F, and which we denote by X^* and it is such X_n^* is adapted to F_{n-1} for any n, and which achieves the condition that $X - X^*$ is a martingale of F. The predictible compensator is unique under the condition $X_0 = X_0^*$. It may be defined through its increments and its initial value $X_0 = X_0^*, X_n^* - X_{n-1}^* = E(X_n - X_{n-1}|F_{n-1})$.

Given the time T and an integer k we denote X_k^T the expectation $E\left(N_T^k\right)$. By this way a sequence X^T indexed by $\mathbf{Z}$ is then defined : $\mathrm{X}^T = \left(X_k^T\right)_{k\in\mathbf{Z}}$, with values in $\mathbf{R}^+ \cup \{+\infty\}$.

It is convenient to position oneself within the *canonical space* of the random walk, the space Ω of sequences $(x_n)_{n\in\mathbf{N}}$ with values in the set $\{-1, 0, 1\}$, provided with the product law and the product algebra. We define the *translation operator* $\theta : \Omega \to \Omega, (x_n) \to (x_{n+1})$.

We say that a real sequence $(z_k)_{k\in\mathbf{Z}}$ is *unimodal with maximum* in $k = k_0$, if the partial sequences $(z_k, k \leq k_0)$ and $(z_k, k \geq k_0)$ are respectively increasing and decreasing.

In the current work we will treat several stopping times T of F which are basically the constant times $T = n$, with n an integer, and those which are of the form a $T = \inf\{k : S_n \in A\}$, $A \subseteq \mathbf{Z}$, and which we denote by T^A. The stopping time T^A is the so called *first passage time through* A. In the current scenario of simple random walks it is enough to take into consideration those defined by sets A with one or two elements In the first case we speak of one level boundary while in the second of two level boundary . The time $T = \infty$ is interpreted as $T^{\oslash}$

When $A = \{a, b\}$ in $\mathbf{Z}, a < 0 < b$, it is known that T^A is integrable (see lemma 1.2 onwards for a direct proof). We denote $p_b^A = P\left(T^A = T^{\{b\}}\right), p_a^A = P\left(T^A = T^{\{a\}}\right)$. We have $p_b^A + p_a^A = 1$ (result of the integrability of T^A) and it is even possible to obtain explicit expressions :

$\mathrm{p}_b^A = \frac{a}{a-b}$, if $\mu = 0; p_b^A = \frac{\left(\frac{q}{p}\right)^a}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b}$, if $\mu \neq 0, p \neq 0$

(using for example the martingale method, thanks to the well known identities of Wald: $E\left(S_{T^{a,b}}\right) = \mu E\left(T^{a,b}\right), E\left(\left(\frac{q}{p}\right)^{S_{T^{a,b}}}\right) = 1$). (See for example [4], chapter 9).

The following limits : $\lim_{a \to -\infty} p_b^A, \lim_{b \to \infty} p_a^A$, exist, which we denote by p_b, p_a respectively and whose values are deducted easily from the relations above:

$$\begin{aligned} p_c &= 1 \quad \text{if } \operatorname{sign}(c) = \operatorname{sign}(\mu), \\ p_c &= \min\left(\left(\frac{q}{p}\right)^{|c|}, \left(\frac{p}{q}\right)^{|c|}\right) \text{ if } \operatorname{sign}(c) \neq \operatorname{sign}(\mu) \end{aligned}$$

In case the time T is of the form $T = T^{\{c\}}$, we have $T = \lim_{a \to -\infty} T^{\{a,c\}}$, if $c > 0$, $T = \lim_{a \to \infty} T^{\{a,c\}}$, if $c < 0$. In both cases the limits above are increasing. In fact , in case $c > 0$, on the event $\{T < \infty\}$ the variable T must be equal to $T^{\{a,c\}}$ for a large enough , and then on this event the limit on top is achieved and it is also stationary. Over $\{T = \infty\}$ it is true that $T^{\{a,c\}} = T^a$ for all $a < 0$, and given that $T^{\{a\}} \geq |a|$ we conclude here.

1. Calculation of predictable compensators

Let S be a simple random walk S. Lets denote by $(S-c)^+, (S-c)^-$, $|S-c|$ respectively the sequences $\left((S_n - c)^-, n \geq 0\right), \left((S_n - c)^-, n \geq 0\right)$, $(|S_n - c|, n \geq 0)$. Regarding the filtration defined in the introduction we look for the predictable compensators of these three sequences , for which is imperative a decomposition result for processes $\left((S_n - c)^-, n \geq 0\right)$, $\left((S_n - c)^-, n \geq 0\right)$, $(|S_n - c|, n \geq 0)$, as it is described next. Firstly an elementary lemma which proof we omit:

Lemma 1.1. *For $x \in \mathbf{Z}, y \in \{-1, 0, 1\}$, we have $(y+x)^+ - x^+ = y^+$ if $x = 0$, y if $x > 0$, 0 if $x < 0$.*

Theorem 1.1. *For all $c \in \mathbf{Z}$ we have:*

Theorem 1.2.

$$\begin{aligned}
(S_n - c)^+ &= c^- + \sum_{k=0}^{n} Y_k^+ 1_{\{S_{k-1}=c\}} + \sum_{k=0}^{n} Y_k 1_{\{S_{k-1}>c\}} \\
(S_n - c)^- &= c^+ + \sum_{k=0}^{n} Y_k^- 1_{\{S_{k-1}=c\}} - \sum_{k=0}^{n} Y_k 1_{\{S_{k-1}<c\}} \\
|S_n - c| &= |c| + \sum_{k=0}^{n} |Y_k| \, 1_{\{S_{k-1}=c\}} + \sum_{k=0}^{n} \left(Y_k 1_{\{S_{k-1}>c\}} - Y_k 1_{\{S_{k-1}<c\}} \right)
\end{aligned}$$

Proof

Lets observe that the third identity is a direct result of the two first. In the other hand as $(S-c)^- = (-S+c)^+$, it is enough to prove the result for $(S-c)^+$. Since $(S_0 - c)^+ = (-c)^+ = c^-$, the identity is equivalent to the identity of the increments between both members, this is :

$(S_{n+1} - c)^+ - (S_n - c)^+ = Y_{n+1}^+ 1_{\{S_n = c\}} + Y_{n+1} 1_{\{S_n > c\}}$, for all $n \geq 0$.

But this identity is a simple consequence of 1.1, setting $x = S_n - c$ and $y = Y_{n+1}$. $\square$

Note : Lets observe these identities just presume that variables Y_n take their values in $\{-1, 0, 1\}$, without restrictions over the law of sequences.

We find some antecedents of these formulae in the work of E. Perkins (see [6]). In effect, except one multiplicative constant on the Y_n, these formulae are found in the proof of a theorem for the case of simple random walks, even they are not enunciated explicitly. These are similar to Tanaka formulae for processes in continuous time in which furthermore local times processes intervene .

The decomposition formulae of previous theorem allow to determine the predictable compensator of sequences $(S-c)^+$, $(S-c)^-$, $|S-c|$, which we denote by $\left((S-c)^+\right)^*$, $\left((S-c)^-\right)^*$, $(|S-c|)^*$ respectively.

Theorem 1.3. *We have*
$\left((S-c)^+\right)_0^* = c^-$, $\left((S-c)^-\right)_0^* = c^+$, $(|S-c|)_0^* = |c|$
and for $n \geq 1$

Theorem 1.4.

$$\left((S-c)^+\right)_n^* = c^- + \mu \sum_{k>c} N_n^k + p N_n^c$$

$$\begin{aligned}\left((S-c)^-\right)^*_n &= c^+ - \mu\sum_{k<c} N^k_n + qN^c_n \\ \left(|S-c|\right)^*_n &= |c| + \mu\left(\sum_{k>c} N^k_n - \sum_{k<c} N^k_n\right) + (p+q)\,N^c_n\end{aligned}$$

Proof

The predictable compensator of the sum of two sequences being the sum of the respective predictable compensators, the results regarding $(|S-c|)^*$ are deduced directly from those of $\left((S-c)^+\right)^*, \left((S-c)^-\right)^*$. Therefore, as $\left((S-c)^-\right)^* = \left((-S+c)^+\right)^*$, it is enough to prove the result for $\left((S-c)^+\right)^*$. Given that $\left((S-c)^+\right)^*_0 = (S_0-c)^+ = (-c)^+ = c^-$, the formula for $\left((S-c)^+\right)^*_n, n \geq 1$,is equivalent to the following formula regarding the increments :

$\left((S-c)^+\right)^*_{n+1} - \left((S-c)^+\right)^*_n = \mu 1_{\{S_n>c\}} + p1_{\{S_n=c\}}, n \geq 0$

However , it is known that

$\left((S-c)^-\right)^*_{n+1} - \left((S-c)^+\right)^*_n = E\left((S_{n+1}-c)^+ - (S_n-c)^+ |_{F_n}\right)$, and that in the other hand , by virtue of theorem 1.1:

$(S_{n+1}-c)^+ - (S_n-c)^+ = Y^+_{n+1}1_{\{S_n=c\}} + Y_{n+1}1_{\{S_n>c\}}$,

so that

$$\begin{aligned} & E\left((S_{n+1}-c)^+ - (S-c)^+ |_{F_n}\right) \\ = \; & E\left(Y^+_{n+1}1_{\{S_n=c\}}|_{F_n}\right) + E\left(Y_{n+1}1_{\{S_n>c\}}|_{F_n}\right) \\ = \; & 1_{\{S_n=c\}}E\left(Y^+_{n+1}|_{F_n}\right) + 1_{\{S_n>c\}}E\left(Y_{n+1}|_{F_n}\right) \\ = \; & 1_{\{S_n=c\}}E\left(Y^+_{n+1}\right) + 1_{\{S_n>c\}}E\left(Y_{n+1}\right) \\ = \; & \mu 1_{\{S_n>c\}} + p1_{\{S_n=c\}}\end{aligned}$$

as it was wished for. □

We deduct a condition of uniform integrability :

Theorem 1.5. *If T is an integrable stopping time , the martingales stopped in T :*

$\left(S^T-c\right)^+ - \left(\left(S^T-c\right)^+\right)^*, \left(S^T-c\right)^- - \left(\left(S^T-c\right)^-\right)^*,$

$\left|S^T-c\right| - \left(\left|S^T-c\right|\right)^*$

are uniformly integrable .
In particular the following formulae are achieved :

$$
\begin{aligned}
E\left((S_T - c)^+\right) &= c^- + \mu E\left(\sum_{k>c} N_T^k\right) + pE\left(N_T^c\right)\\
E\left((S_T - c)^-\right) &= c^+ - \mu E\left(\sum_{k<c} N_T^k\right) + qE\left(N_T^c\right)\\
E\left(|S_hT - c|\right) &= |c| + \mu\left(E\left(\sum_{k>c} N_T^k\right) - E\left(\sum_{k<c} N_T^k\right)\right) +\\
(p+q)\,E\left(N_T^c\right)
\end{aligned}
$$

Proof

It is enough to demonstrate the martingale case $\left(S^T - c\right)^+ - \left(\left(S^T - c\right)^+\right)^*$. In the one hand the sequence $\left(S^T - c\right)^+$ is uniformly bounded by the variable $(T-c)^+$, whereas in virtue of 1.3, $\left(\left(S^T - c\right)^+\right)^*$ is smaller than $c^- + |\mu| \sum_{k>c} N_T^k + pN_T^c \leq c^- + T$. Then , both terms of this martingale are uniformly integrable thanks to the assumptions and we conclude.

The last identities are derived from the former and from the optional sampling theorem of Doob. □

We expose as an application of the former seen a proof of a known result of the theory of random walks . It is given a set $A = \{a, b\}$ in $\mathbf{Z}, a < 0 < b$, and we consider the stopping time T^A. The next result is a weak version of the so called Stein lemma (see [7], chap. 2) according to which T^A has moments of all orders.

Lemma 1.2. T^A *is integrable.*

Proof

We set $T = T^A$. The time T being the sum of variables N_T^x, x an integer of $]a,b[$, it is enough to demonstrate that for each of these integers the expectation $E(N_T^x)$ is finite.

Lets have then an integer c in $]a,b[$. It is attempted to establish that the numerical sequence $\left(E\left(N_{\min(T,n)}^c\right), n \geq 0\right)$ is bounded, which is enough as $N_T^c = \lim_{n\to\infty} N_{\min(T,n)}^c$ (including the eventual case of infinite values for T), and in virtue of the monotone convergence theorem the value $E(N_T^c)$ is finite.

If we apply theorem 1.5 to the time $\min(T,n)$ we obtain:

$$E\left(\left(S_{\min(T,n)} - c\right)^+\right)$$
$$= \mu E\left(\sum_{k>c} N_{\min(T,n)}^k\right) + pE\left(N_{\min(T,n)}^c\right), \tag{1.1}$$
$$\text{if } 0 \leq c < b$$

$$E\left(\left(S_{\min(T,n)} - c\right)^-\right)$$
$$= -\mu E\left(\sum_{k<c} N_{\min(T,n)}^k\right) + pE\left(N_{\min(T,n)}^c\right), \tag{1.2}$$
$$\text{if } a < c \leq 0$$

Lets consider several cases :

a) **Case** $\mu = 0$. Given that $\sigma^2 \neq 0$, we must have $p \neq 0, q \neq 0$. The former relations are written as follows :

$$\begin{aligned} E\left(\left(S_{\min(T,n)} - c\right)^+\right) &= pE\left(N_{\min(T,n)}^c\right), \text{ if } \quad 0 \leq c < b \\ E\left(\left(S_{\min(T,n)} - c\right)^-\right) &= qE\left(N_{\min(T,n)}^c\right), \text{ if } \quad a < c \leq 0 \end{aligned}$$

The random sequences $\left(\left(S_{\min(T,n)}-c\right)^{+}, n\geq 0\right)$, $\left(\left(S_{\min(T,n)}-c\right)^{-}\right.$, $\left. n\geq 0\right)$ being bounded , we obtain from the relations above that the numerical sequences $\left(E\left(N^{c}_{\min(T,n)}\right), n\geq 0\right)$ are bounded above .

b) **Case** $\mu\neq o, p\neq 0, q\neq 0$. Lets ponder the case $0\leq c<b$, reasoning by induction on the integer $b-c$. For $c=b-1$ the relation (1.1) is written $E\left(\left(S_{\min(T,n)}-c\right)^{+}\right)=pE\left(N^{c}_{\min(T,n)}\right)$, and as the left member is a bounded sequence the current case is concluded. We suppose the result is right for $c=b-1, b-2,\cdots,b-m$, with $b-m>0$. For $c=b-m-1$, lets observe that being the quantity $E\left(\sum\limits_{k>c}N^{k}_{\min(T,n)}\right)$ the sum of the quantities $E\left(N^{k}_{\min(T,n)}\right)$ with $c<k<b$, by induction assumption is then the term of a bounded sequence. Being the left member the term of a bounded sequence we deduce then that the sequence which general term is $E\left(N^{c}_{\min(T,n)}\right)$ is bounded above.

The case $a<c\leq 0$ is established similarly , reasoning by induction on the integer $c-a$ using relation (1.2).

c) **If** $\mu\neq 0, p=0$ **or** $q=0$. It is enough to ponder a subcase, for instance $\mu\neq 0, p=0$, that is $q=1, \mu=-1$. The relation (1.2) is written: $\mathrm{E}\left(\left(S_{\min(T,n)}-c\right)^{-}\right)=E\left(\sum_{k<c}N^{k}_{\min(T,n)}\right)+E\left(N^{c}_{\min(T,n)}\right)$, if $a<c\leq 0$

Using this relation and reasoning inductively on the integer $c-a$ as in b), we conclude for $a<c\leq 0$. For $0<c<b$ relation (1.1) is not useful, but it is easy to see that the variable N^{c}_{T} is null as the walk takes only values in $\mathbf{Z}^{-}$. □

2. Study of the sequence X^T (general case)

Given a time T it is considered the sequence $X^{T}=\left(X^{T}_{k}\right)_{k\in\mathbf{Z}}$, the sequence of expectations of the occupation times before T defined in the introduction.

Next recurrence relations are provided for these sequences under integrability conditions:

Theorem 2.1. *Let T be an integrable stopping time . Then :*
if $k\geq 0$: $pX^{T}_{k}-qX^{T}_{k+1}=P\left(S_{T}>k\right)$, if $k<0$: $-pX^{T}_{k}+qX^{T}_{k+1}=P\left(S_{T}\leq k\right)$

Proof

Lets observe that from lemma 1.1 one obtains : $(S_T - (k+1))^+ - (S_T - k)^+ = -1_{\{S_T - k > 0\}}$. In this relation the random variables appearing are all integrable, so that taking expectations on both sides it is obtained :

$$E\left(S_T - (k+1)^+\right) - E\left(S_T - k\right)^+ = -P_{\{S_T - k > 0\}} \tag{2.1}$$

In the other hand , in virtue of the identities of theorem 1.5, if $k \geq 0$:

$$\begin{aligned}
&E\left(S_T - (k+1)\right)^+ - E\left(S_T - k\right)^+ \\
&= \mu\left(E\left(N^{(k+1)+}_T\right) - E\left(N^{(k)+}_T\right)\right) + p\left(E\left(N_T^{(k+1)}\right) - E\left(N_T^{(k)}\right)\right) \\
&= -\mu E\left(N_T^{(k+1)}\right) + p\left(E\left(N_T^{(k+1)}\right) - E\left(N_T^{(k)}\right)\right) \\
&= qE\left(N_T^{(k+1)}\right) - pE\left(N_T^{(k)}\right) \\
&= -pX_k^T + qX_{k+1}^T,
\end{aligned}$$

and comparing with (2.1) we obtain the first identity.

The proof of the second identity , for $k < 0$, is analogous , pondering the difference $(S_T - (k+1))^- - (S_T - k)^-$, and rewriting (2.1) in terms of the negative part and using again theorem 1.5. □

By solving X_k^T in the above relations recurrent sequences are generated in the usual sense from a given value $X_{k_0}^T$, that is the sequences $\left(X_k^T, k \leq k_0\right), \left(X_k^T, k \geq k_0\right)$. The study of these sequences is relatively simple and it is similar to certain classical linear recurrences of first order. Next we will study in a detailed form this sequences for particular classes of times introduced in the preface.

Before we proceed some consequences of the previous theorem will be introduced for times not necessarily integrable:

Theorem 2.2. *Given an arbitrary stopping time T , the following is achieved : either the values of the sequence X^T are all real numbers (finite values) or all are infinite. In the case of finiteness the sequence is unimodal with a maximum in $k = 0$.*

Proof

Let T be a stopping time (with values eventually infinite). The relations of theorem 2.1 are achieved for each of the times $\min(T, n)$, n an integer, and given that the right member of these relations are probabilities, it is deduced that for any k in $\mathbf{Z}$ the sequence $\left(pX_k^{\min(T,n)} - qX_{k+1}^{\min(T,n)}\right)_{n\in\mathbf{N}}$ is bounded . From this and from $X_k^T = \lim_{n\to\infty} X_k^{\min(T,n)}$, follows the first assertion.

For the second assertion we discuss by the sign of μ. In the case of $\mu = 0$, that is $p = q$, relations of theorem 2.1 imply that :

$$\begin{aligned} \text{if} \quad k &\geq 0 : X_k^T = X_{k+1}^T + \frac{P(S_T > k)}{q} \geq X_{k+1}^T \\ \text{if} \quad k &< 0 : X_{k+1}^T = X_k^T + \frac{P(S_T \leq k)}{q} \geq X_k^T \end{aligned}$$

And this is the property of unimodality with a maximum value in $k = 0$.

If $p = 0$, the sequence X^T is null in $\mathbf{Z}^+$, and if $k < 0$ we have $X_{k+1}^T = P(S_T \leq k)$, a quantity which increases with k. The case $q = 0$ is similar.

Lets suppose now $p \neq 0, q \neq 0, \mu > 0$. We need the following auxiliary lemma :

Lemma 2.1. *Let be a bounded sequence $Z = (z_k)_{k\in\mathbf{N}}$ that satisfies :*

$$pz_k - qz_{k+1} = a_k, \tilde{A}\tilde{A}$$

with (a_k) decreasing and $p > q > 0$. Then Z es decreasing.

Proof of 2.1

Lets suppose there is $k' \geq 0$ such that $z_{k'+1} - z_{k'} > 0$, and lets call c this quantity. Since a_k decreases along k, we obtain that $pz_k - qz_{k+1} \leq pz_{k+1} - qz_{k+2}$, this is :

$$\frac{p}{q}(z_k - z_{k+1}) \leq z_{k+1} - z_{k+2},$$

and therefore for $k \geq k'$:
$z_k - z_{k+1} \geq \left(\frac{p}{q}\right)^{k-k'} (z_{k'} - z_{k'+1}) = \left(\frac{p}{q}\right)^{k-k'} c.$

But then we would have that $z_k - z_{k+1}$ diverges to the infinite $(when\ k \to \infty)$, which contradicts the fact that Z is bounded. Then the assumption that there exists $k' \geq 0$ such that $z_{k'+1} - z_{k'} > 0$ is false, and we conclude . □

(proof of 2.2, continuation)

If $k < 0$, the relation $-pX_k^T + qX_{k+1}^T = P(S_T \leq k)$ implies that $X_{k+1}^T = \frac{p}{q}X_k^T + \frac{P(S_T \leq k)}{q} \geq X_k^T$, as $p > q$.Then $\left(X_k^T, k \leq 0\right)$ is increasing. In the other hand the sequence $\left(X_k^T, k \geq 0\right)$ satisfies in this case conditions of 2.1 above and from this result we conclude that $\left(X_k^T, k \geq 0\right)$ is decreasing , and this leads to the final conclusion.

The case $\mu < 0, p \neq 0, q \neq 0$, is discussed similarly. □

Example 2.1. Recurrent relations for constant times and for first passage times through two level boundaries

Constant stopping times and those of the form T^a, with $A = \{a, b\}$ in $\mathbf{Z}, a < 0 < b$, being integrable (see lemma 1.2), results from this section may be applied directly. Lets calculate the quantities $P(S_T > k)$, $P(S_T \leq k)$ which appear in theorem 2.2:

Example 2.1 $: T$ **is the constant n**

These quantities are written as $P(S_n > k), P(S_n \leq k)$. Given that $|S_n|$ is bounded by n, we will have $P(S_n > k) = P(S_n \leq k) = 0 \ \ if \ \ |k| > n$. Then we have:

$$pX_k^T - qX_{k+1}^T = 0, \text{ if } \ k \geq n+1 \ \text{ and } \ -pX_k^T + qX_{k+1}^T = 0, \ \text{ if } \ k < -n$$

relations which are congruent with the fact that $X_k^T = 0$ if $|k| > n$. This property plus the relations of theorem 2.1 and the knowledge of values $P(S_n > k)$ allow to calculate recurrently the sequences $\left(X_k^T, -n \leq k \leq 0\right), \left(X_k^T, n \geq k \geq 0\right)$. According to theorem 2.2 the sequence X^T is unimodal with maximum in $k = 0$ (maximality in $k = 0$, but not unimodality, is proved in [1], chap. 1, using other methods).

Example 2.2 : $T = T^A$

$P(S_T > k) = 0$ if $k \geq b$, $and = p_b^A$ if $0 \leq k < b$; in the other hand $P(S_n \leq k) = 0$ if $k < a$,and $= p_a^A$ if $a \leq k \leq 0$. Relations of theorem 2.1 are written in the following ways :

$$\begin{cases} pX_{k-1}^T - qX_k^T &= p_b^A \text{ if } 1 \leq k \leq b \\ pX_{k-1}^T - qX_k^T &= -p_a^A \text{ if } 0 \geq k \geq a+1 \\ X_k^T &= 0 \text{ if } k \notin \,]a, b[\end{cases}$$

From these identities some properties may be deducted and easily verifiable:

a) If $p = 0$ (resp. $q = 0$) then X_k^T is null if $k \geq 1$, and equals $\frac{1}{q}$ if $a < k \leq 0$ (resp. is null if $k < 0$, and equals $\frac{1}{p}$ if $b > k \geq 0$).

b) For a centered walk the sequence X^T is arithmetical in $\{a+1, \cdots, 0\}$ (resp. in $\{0, \cdots, b-1\}$) with rate $\frac{2b}{(b-a)\sigma^2}$ $\left(\text{resp.} -\frac{2a}{(a-b)\sigma^2}\right)$.

c) X^T is unimodal with maximum in $k = 0$.

d) Supposing $p \neq 0, q \neq 0$ and $\mu \neq 0$ the sequence X^T is bounded above by $\frac{p_c^A}{|\mu|}$, where $c = b$ if $\mu < 0$.

Property d) requires a formal proof through the following lemma (whose proof we omit) : let $Z = (z_k, k \in \mathbf{Z})$, be a real sequence such that for $k \in \mathbf{Z}$ the relation

$z_{k+1} = A + Bz_k, z_0 > 0$, with $A, B > 0$. holds . Then :

Lemma 2.2. *Z is strictly increasing and diverges to $+\infty$ if $B > 1$ and $z_0 > \frac{A}{1-B}$ or if $B = 1$ and $A > 0$. If $0 < B < 1$ it is convergent to $\frac{A}{1-B}$, increasingly if $z_0 \leq \frac{A}{1-B}$ and decreasingly if $z_0 > \frac{A}{1-B}$.*

Proof of d)

We treat just the case $\mu > 0$, the case $\mu < 0$ being analogous. This means that $\frac{p}{q} > 1$ and from recurrent relations (2) and the first part of lemma 2.2 applied to the sequence $z_k = X_k^T$ it is deduced that X^T is strictly increasing in $\{a+1, \cdots, 0\}$. Within the interval $\{0, \cdots, b-1\}$, along with notations of 2.2 and defining $z_k = X_{b-k}^T$, we have $A = \frac{p_b^A}{p}, B = \frac{q}{p}$, then A, B are both strictly positive if $B < 1$. The value X_{b-1}^T is smaller than $\frac{A}{1-B} = \frac{p_b^A}{p-q}$ as $X_{b-1}^T = \frac{p_b^A}{p}$. We deduce from the lemma that $\frac{p_b^A}{p-q}$ is an upper bound of the sequence in $\{0, \cdots, b-1\}$. Being X^T unimodal with maximum in $k = 0$ (c) above) the former value $\frac{p_b^A}{p-q}$ is then an upper bound for X^T in all $\mathbf{Z}$.

3. Study of X^T, for first passage time through boundaries

In this section we will extend the study of the example given in the previous section regarding the expectations X_k^T, for $T = T^A$ and sets A which are unitary in $\mathbf{Z}$ or those which contain 0, which do not necessarily have the property of integrability. Results of section 2 are not directly applicable, but from what it was established in the example, it is possible to study the possible relations for X^{TA} by taking limits in certain cases, or through markovian properties in other cases .

The study is divided into cases , depending if 0 belongs or not to A.

3.1. Case $\mathbf{T} = T^{\{c\}}, with\ c \neq 0$

Consider the sequence $X^T = \left(X_k^T\right)_{k\in\mathbf{Z}}$, defined by the time $T = T^{\{c\}}, c \neq 0$. In virtue of what was exposed in the introduction , the variable $N^k_{T\{a,c\}}$ is increasing in $|a|$ and so from the monotone convergence theorem , for $T = T^{\{c\}}$ we obtain : $\mathrm{E}\left(N_T^k\right) = \lim_{a\to-\infty} E\left(N^k_{T\{a,c\}}\right)$ if $c > 0$, $E\left(N_T^k\right) = \lim_{a\to\infty} E\left(N^k_{T\{c,a\}}\right)$ if $c < 0$ which along with notations previously introduced are written :

$$X_k^T = \lim_{a\to-\infty} X_k^{T\{a,c\}} \quad \text{if} \quad c > 0, \tag{3.1}$$

$$X_k^T = \lim_{a\to\infty} X_k^{T\{c,a\}} \quad \text{if} \quad c < 0 \tag{3.2}$$

In (3.1) and (3.2) the limit values are finite. In effect , assuming $c > 0$, the relation $X_k^T = 0$ holds if $k \geq c$, and it is concluded through theorem 2.2.

We study the relations (2) using (3.1) and (3.2) as a limit type definition of X^T and the definitions of probabilities p_c as limits (see introduction). Suppose $c > 0$. In relations (2) it exists the limit of the right member ($a \to \infty$, taking $b = c$) and equals p_c if $\mathrm{sign}(k) = \mathrm{sign}(c)$, or $p_c - 1$ if $\mathrm{sign}(k) = -\mathrm{sign}(c)$. Since , as it was signaled above, the limit values of $X_k^{T\{a,c\}}$ ($a \to -\infty$, taking $b = c$) are all finite, we have $\underset{a\to-\infty}{group} \lim \left(pX_{k-1}^{T\{a,c\}} - qX_k^{T\{a,s\}}\right) = pX_{k-1}^T - qX_k^T$. An analogous discussion for $c < 0$ leads then to the following result :

$$\left\{\begin{array}{l} \text{the sequence } X^T \text{ consists only of reals which satisfy :} \\ pX_{k-1}^T - qX_k^T = \mathrm{sign}(c)\, p_c \quad \text{if } \mathrm{sign}(k) = \mathrm{sign}(c)\,, \\ pX_{k-1}^T - qX_k^T = -\,\mathrm{sign}(c)\,(1 - p_c) \quad \text{if } \mathrm{sign}(k) = -\,\mathrm{sign}(c)\,, \\ X_k^T = 0 \quad \text{if} \quad k - c = \mathrm{sign}(c) \end{array}\right. \tag{3.3}$$

Note : The fact that sequence X^T consists only of finite values is proved in [1] (chap. 3, section 10) in the general case of walks over $\mathbf{Z}$ and using different methods.

In some particular cases these relations allow to describe the sequence X^T fully (see previous example for a similar discussion):

a) If $p = 0$ (resp. $q = 0$) X_k^T is null if $k \geq 1$, equals $\frac{1}{q}$ if $k \leq 0$ and $k - c \neq$ sign(c) $\Big($resp. is null if $k < 0$, equals $\frac{1}{p}$ if $k \geq 0$ and $k - c \neq$ sign $(c)\Big)$.

b) For a centered random walk , X^T is arithmetical within the interval with extremes $0, c$, with rate $-$ sign$(c)\, \frac{2}{\sigma^2}$, and constant in $\{k : \text{sign}\,(k) = -\text{sign}\,(c)\}$

c) X^T is unimodal with maximum in X_0^T.

d) When $\mu \neq 0$, X^T is bounded above by :

$$\frac{1}{|\mu|}, \text{ if sign}\,(c) = sign\,(\mu)\,, \quad \frac{1 - p_c}{|\mu|} \quad \text{if sign}\,(c) \neq sign\,(\mu)$$

e) Under the condition $p \neq 0, q \neq 0, \mu \neq 0$, it holds that, for sign$(c) = -$ sign(k) :

$$\text{If} \quad \text{sign}\,(c) = \text{sign}\,(\mu) : X_k^T = \min\left(\left(\frac{q}{p}\right)^{|k|}, \left(\frac{p}{q}\right)^{|k|}\right) X_0^T$$

$$\text{If} \quad \text{sign}\,(c) \neq \text{sign}\,(\mu) : X_k^T = \frac{1 - p_c}{|\mu|}$$

Regarding d) it is enough to take the limit in d) from the example in 2, and taking into account the possible values of p_c given in the introduction. Property e) must be justified with more detail : we analyze just the case $c > 0$, as case $c < 0$ is analogous. When $\mu > 0$ it holds that $p_c = 1$, and the relations (3.3) imply that $\frac{X_k^T}{X_{k+1}^T} = \frac{q}{p}$ for $k \leq -1$, from which the relation $X_k^T = \left(\frac{q}{p}\right)^{|k|} X_0^T$ holds for $k \leq 0$. When $\mu < 0$, from the same relations (3.3) we find that $X_k^T = A + BX_{k+1}^T$ if $k \leq -1$, with $A = -\frac{1-p_c}{p}, p \neq 0$, and $B = \frac{q}{p}$, with $B > 1$ by hypothesis. According to 2.2 this sequence (indexed in decreasing order of k) diverges to $+\infty$ if $X_0^T \neq \frac{1-p_c}{|\mu|}$, which cannot occur because it is bounded by X_0^T in virtue of c). Then $X_0^T = \frac{1-p_c}{|\mu|}$ and in this case X_k^T is constant in $\mathbf{Z}^-$ and equals X_0^T.

We may establish now a classical result on integrability for first passage times through boundaries of one level(see [7], chap. 2, for a general discussion):

Proposition 3.1. *If $\mu \neq 0$ the time T is integrable if and only if sign$(c) =$ sign(μ)*

Proof : The only case analyzed is $c > 0$, the case $c < 0$ being analogous. For $c > 0$ the expectation of T is expressed as follows :

$$E(T) = \sum_{k \in \mathbf{Z}; k < c} X_k^T$$

this is, the sum of series of positive, non null terms. The order of summation being irrelevant , we may define the series in the increasing order of k.Lets assume $\mu > 0$. If $p = 1$ the result is immediate from a). If $p \neq 0, q \neq 0$, from e) above we know that $X_k^T = \left(\frac{q}{p}\right)^{|k|} X_0^T$ for $k \leq 0$. As $\frac{q}{p} < 1$, the general term of the series of $E(T)$ is convergent.

Lets assume $\mu < 0$. Using a) when $p = 0$ and e) when $p \neq 0, q \neq 0$ we obtain that the general term of the series converges to a non zero real , then it is divergent . $\square$

In the corollary the case $p = q.$ is not regarded. In this case it can be shown that T is not integrable but it is almost surely finite ([7], chap. 2)

3.2. Case of times T^A, with $0 \in A$

As to end the discussion in this section first passage times through boundaries containing zero are considered . This comprises the cases of of times of the type $T^{\{0,b\}}$ (with $0 < b$) ,$T^{\{a,0\}}$ (with $a < 0$) and $T^{\{0\}}$. If T designs one of this cases we will demonstrate that the sequence X^T is deducted easily from 3.1. In effect , it is just a matter to apply the markovian character of the process .

Lets consider for this the canonical space of the random walk (see definitions and notations) and the case $X^{\{0,b\}}$. Within this space the stopping times of crossing boundaries may be decomposed the following way from θ : $T^{\{a,b\}} = 1 + T^{\{a-X_1,b-X_1\}} \circ \theta, T^{\{c\}} = 1 + T^{\{c-X_1\}} \circ \theta$ which implies that $T^{\{0,b\}}$ equals to : 1 if $X_1 = 0$, $1 + T^{\{-1,b-1\}} \circ \theta$ if $X_1 = 1$, $1 + T^{\{1,b+1\}} \circ \theta$ if $X_1 = -1$

Then given , given $k \in \mathbf{Z}$ and n an integer, the events

$$\left\{S_n = k, n < T^{\{0,b\}}\right\} \quad \text{and} \quad \left\{S_{n-1} \circ \theta = k - X_1, n - 1 < T^{\{-X_1,b-X_1\}}\right\}$$

coincide. This means in particular that $N^k_{T^{\{0,b\}}} = N^{k-X_1}_{T^{\{-X_1,b-X_1\}} \circ \theta} \circ \theta$ holds if $X_1 \neq 0$. As $H \circ \theta$ is independent from X_1 whatever is the variable H over Ω (markovian property), we deduce from the last relations the following formula : $\mathrm{E}\Big(N^k_{T^{\{0,b\}}}\Big) = pE\left(N^{k-1}_{T^{\{-1,b-1\}}}\right) + qE\left(N^{k+1}_{T^{\{1,b-1\}}}\right)$ which in the adopted notation is written as follows : $\mathrm{X}^{\{0,b\}}_k = pX^{\{-1,b-1\}}_{k-1} + qX^{\{1,b+1\}}_{k+1}$

But as $X^{\{a,b\}}_k = 0$ if $k \notin]a,b[$ (see (2)), the value of $X^{\{a,b\}}_k$ is : $pX^{\{-1,b-1\}}_{k-1}$ if $0 < k < b$, $qX^{\{1\}}_{k+1}$ if $k < 0$, and 0 if $k > b$.

An analogous discussion for $X^{\{a,0\}}$ and the fact that $X^{\{0\}}_k$ is the limit value of $X^{\{0,b\}}_k$ when $b \to \infty$, lead to the following relations :

$$(3.4)\begin{cases} \text{The value of } X^{\{0,b\}}_k \text{ is :} \\ pX^{\{-1,b-1\}}_{k-1} \text{ if } 0<k<b, \quad qX^{\{1\}}_{k+1} \text{ if } k<0, \quad 0 \quad \text{if } k>b \\ \text{and that of } X^{\{a,0\}}_k \text{ is :} \\ qX^{\{1,a+1\}}_{k-1} \text{ if } a<k<0, \quad pX^{\{-1\}}_{k-1} \text{ if } k>0, \quad 0 \quad \text{if } k<a \end{cases}$$

$$(3.5)\begin{cases} \text{The value of } X^{\{0\}}_k \text{ is :} \\ pX^{\{-1\}}_{k-1} \text{ if } k>0, \quad qX^{\{1\}}_{k+1} \text{ if } k<0. \end{cases}$$

Thanks to relations above along with those of (2) and (3.3) is possible to determine the values of $X^{\{0,b\}}, X^{\{a,0\}}$ and of $X^{\{0\}}$. In particular , from (3.4) and (3.5) is easy to answer that question of what is the most visited state before a given time :

a) For $T = T^{\{0,b\}}$ or $T^{\{a,0\}}$ the sequence X^T is unimodal with maximum value 1 , attained in : k=1 if $\mu > 0$, $k = -1$ if $\mu < 0$, $k = 1$ and $k = -1$ if $\mu = 0$

b) The maximum value of $X^{\{0\}}$ is 1 in all cases and it is attained in all $\mathbf{Z}^*$ if $\mu = 0$, in $k = 1$ (only) if $\mu > 0$, in $k = -1$ (only) if $\mu < 0$

(**Note:** The result b), for the case $\mu = 0$ is shown in [1], chap. 3, section 10, for walks in $\mathbf{Z}^d$. the techniques applied are of very different nature if compared to ours **)**

Proof of a)

It is enough to consider the case $T = T^{\{0,b\}}$. We omit the cases $p = 0, q = 0$ which are immediate. The property of unimodality of X^T is a result of (3.4) and the properties of unimodality of $X^{\{-1,b-1\}}$ in $]-1, b-1[$ and of $X^{\{1\}}$ in $\mathbf{Z}$. The maximum values of X^T are attained on in $k = 1$ or -1. To determine which of this two values is bigger we turn to relations (2) and (3.3) to obtain : $X_0^{\{1\}} = \frac{p_1}{p}$, $X_0^{\{-1,b-1\}} = \frac{p_{-1}}{q}$. Then it is deducted : $X_1^T = p\left(\frac{p_{-1}}{q}\right)$, $X_{-1}^T = q\left(\frac{p_1}{p}\right)$. From these identities a nd values of p_{-1} and p_1 : $p_{-1} = p_1 = 1$ if $\mu = 0$; $p_{-1} = \frac{q}{p}$, $p_1 = 1$ if $\mu > 0$; $p_{-1} = 1, p_1 = \frac{p}{q}$ if $\mu < 0$ it is obtained that 1 is the maximum value of X^T in all cases, likewise those points in which it is reached . □

Proof of b)

Given that sequences $X^{\{-1\}}$ and $X^{\{1\}}$ are unimodal with a maximum in $k = 0$ (see c) of 3.1), from b) above we have that $X^{\{0\}}$ is also unimodal, which maximum is reached at least in $k = 1$ or -1. Using relations (3.3), it is calculated $X_1^{\{0\}}, X_{-1}^{\{0\}}$ in different cases:

Case $\mu = 0$. In this case lets remember the fact that $X^{\{c\}}$ is constant within the subinterval of integers of opposite sign to c. Then we have :

$$\text{if} \quad k > 0 : X_k^{\{0\}} = pX_{k-1}^{\{-1\}} = p\frac{1}{p} = 1$$
$$\text{if} \quad k < 0 : X_k^{\{0\}} = qX_{k+1}^{\{1\}} = q\frac{1}{q} = 1$$

Case $\mu > 0$. We have : $X_1^{\{0\}} = pX_0^{\{-1\}} = p\left(\frac{p_{-1}}{q}\right) = p\frac{q}{p}\frac{1}{q} = 1$, and in the other hand : $X_{-1}^{\{0\}} = qX_0^{\{1\}} = q\left(\frac{p_1}{q}\right) = \frac{q}{p} < 1$.

Case $\mu < 0$. Analogous to the above mentioned. □

3.3. Case $T = \infty$

Given that : $X_k^\infty = \lim_{a\to-\infty} X_k^{T^{\{a\}}} = \lim_{a\to\infty} X_k^{T^{\{a\}}} = \lim_{n\to\infty} X_k^n$ it is quite simple to deduct relations for the case $T = \infty$ from the relations (3.3). In effect :

- If $p = 0$ (resp. $q = 0$), in virtue of a) and 3.1 we obtain that X_k^∞ is null if i $k \geq 1$, equal to $\frac{1}{q}$ if $k \leq 0$ (resp. is null if $k < 0$, equal to $\frac{1}{p}$ if $k \geq 0$).
- If $p \neq 0, q \neq 0, \mu \neq 0$, choose c real so that $\text{sign}(c) \neq \text{sign}(\mu)$, and let $T = T^{\{c\}}$. From e) in section 3.1 we know that $X_k^T = \frac{1-p_c}{|\mu|}$ if $\text{sign}(\mu) = \text{sign}(k)$. Given that $p_c \to 0$ if $c \to -\text{sign}(\mu)\infty$, it is deducted that X_k^∞ has a value $\frac{1}{|\mu|}$ if $\text{sign}(\mu) = \text{sign}(k)$. Taking now

c so that $\text{sign}(c) = \text{sign}(\mu)$ and $T = T^{\{c\}}$, it is known again by e) that $X_k^T = \min\left(\left(\frac{q}{p}\right)^{|k|}, \left(\frac{p}{q}\right)^{|k|}\right) X_0^T$ if $\text{sign}(k) = -\,\text{sign}(\mu)$, and as proven above $X_0^\infty = \frac{1}{|\mu|}$, by taking the limit when $c \to \text{sign}(\mu)_\infty$ it is obtained : $X_k^\infty = \min\left(\left(\frac{q}{p}\right)^{|k|}, \left(\frac{p}{q}\right)^{|k|}\right) \frac{1}{|\mu|}$.

- In case $\mu = 0$, be $c > 0$ and $T = T^{\{c\}}$. Applying b) of 3.1 it is obtained the relation $X_k^T = (c-k)\frac{2}{\sigma^2}$, for $0 \le k \le c$ and $X_k^T = c\frac{2}{\sigma^2}$ for $k \le 0$. By taking the limit as $c \to -\,\text{sign}(\mu)^\infty$ in these relations it is deducted that $X_k^\infty = \infty$.

Summarizing :

$$(3.6)\begin{cases} X_k^\infty = \infty \quad \text{if} \quad \mu = 0, \\ X_k^\infty = \min\left(\left(\frac{q}{p}\right)^{|k|}, \left(\frac{p}{q}\right)^{|k|}\right) \frac{1}{|\mu|} \quad \text{if } \text{sign}\,(k) = -\,\text{sign}\,(\mu) \neq 0 \\ X_k^\infty = \frac{1}{|\mu|} \quad \text{if } \text{sign}\,(k) = \text{sign}\,(\mu) \neq o. \end{cases}$$

(**Note:** relation (3.6) for $\mu = 0$ is established in [1] in a general case (P3, chap. 1)**.** For $\mu \neq 0$, the result above is resolved in E2, [1], chap. 1, for the Bernoulli case, $r = 0$. It is to be observed that $\mu \neq 0$, the author resorts to a relation that in our notations would be written as follows : $X_k^\infty - \delta(k, 0) = pX_{k-1}^\infty + qX_{k+1}^\infty$. This relation also may be obtained from (3.3), which is easy to verify)

In the other hand , as $X_k^\infty = \lim\limits_{n\to\infty} X_k^{(n)}$, whereas $X_k^{(n)} = X_k^T$, with $T = n$, following the same set of reasoning applied to establish (3.6) it may be obtained the following :

Corollary 3.1. $\lim\limits_{n\to\infty} \frac{X_k^{(n)}}{X_0^{(n)}}$ *exists and is less than 1. It equals 1 if* $\mu = 0$.

The result for the general case of this corollary is proved in [1], (P5 of section 1 and P6 of section 2, chap. 1).

4. Some other remarkable relations for the expectations of the random walk

The relations of theorem 1.5 lead to remarkable identities and inequalities related to the expectations $(E(W_n))_{n\in\mathbf{N}}$, $\left(E\left(W_{\min(T,n)}\right)\right)_{n\in\mathbf{N}}$, $E(W_T)$, being $(W_n)_{n\in\mathbf{N}}$ one of the process $(S-c)^+, (S-c)^-, |S-c|$ and for certain stopping times T.

In the first place it is easy to see that the identity of Wald : $E(S_T) = \mu E(T)$ for T integrable, is one of the result. In effect, taking $c=0$ and subtracting the expressions $E\left(S_T^+\right)$ and $E\left(S_T^-\right)$ it is obtained $E(S_T) = E\left(S_T^+\right) - E\left(S_T^-\right) = \mu\left(\sum_{c\in\mathbf{Z}^*} E(N_T^c)\right) + \mu E(N_T^0) = \mu\left(\sum_{c\in\mathbf{Z}} E(N_T^c)\right) = \mu E(T)$.

The most interesting case is that of centered random walks , $\mu = 0$, for which theorem 1.5 is simplified considerably . In effect, in this case $p = q = \frac{\sigma^2}{2}$, and being T an integrable time , the relations of theorem 1.5 are written as :

$$
\left\{\begin{array}{c} E\left((S_T-c)^+\right) = c^- + \frac{\sigma^2}{2} E(N_T^c) \\ E\left((S_T-c)^-\right) = c^+ + \frac{\sigma^2}{2} E(N_T^c) \\ E(|S_T-c|) = |c| + \sigma^2 E(N_T^c) \end{array}\right. \tag{4.1}
$$

c being any integer.

We consider the $c=0$ in (4.1) and the times T which are constant . The identities of this theorem in the case $\mu = 0$ are written for each integer n as follows :

$$
E(S_n^+) = \frac{\sigma^2}{2} E(N_n^0), \quad E(S_n^-) = \frac{\sigma^2}{2} E(N_n^0), \quad E(|S_n|) = \sigma^2 E(N_n^0) \tag{4.2}
$$

and introducing the probabilities $p_n = P(S_n = 0)$, (in which $p_0 = 1$), we may additionally write $E(N_n^0) = \sum_{k=0}^{n-1} p_k$. It should be recalled that in this case $p_n = \frac{n!}{\left(\frac{n}{2}\right)!^2 2^n}$, and that $(n-1)p_{n-2} = np_n$.

Theorem 4.1. *For a centered random walk with* $\sigma^2 = 1$, *for each even integer* n *the following is achieved :* $E\left(\frac{|S_n|}{n}\right) = p_n, \quad p_n = \frac{\sum_{k=0}^{n-1} p_k}{n}$

(**Note:** it is known already that for n odd, p_n is null).

Proof

Lets observe that the second identity is the result of the first one as it is enough to use (4.2) and do a substitution . For the same identity we use an inductive reasoning on n. For $n = 2$, we have $E(|S_2|) = 2(1 - p_2)$, and as $p_2 = \frac{1}{2}$ it is concluded.

Lets suppose that the result is true for the even number n, and lets consider the next even number $n + 2$.

In virtue of (4.2) we have : $E(|S_{n+2}|) = E(N^0_{n+2}) = E(N^0_n) + p_n = E(|S_n|) + p_n$, and by induction hypothesis this equals $np_n + p_n = (n+1)p_n$, and it is enough to use the above identity to conclude that : $E(|S_{n+2}|) = (n+2)p_{n+2}$. □

Given that under the assumption of theorem 4.1 the law of variable S_n is known, the relations of this theorem areas actually combinatorial identities. For instance , the identity is obtained :

$$\frac{1}{2^{n-2}} \sum_{k=0}^{\frac{n}{2}} \left(\frac{n}{2} - k\right) \binom{n}{k} = np_n$$

This appears in [3] as an exercise in chap. 5 and it is verified by resorting to the exhibited theory in this book (the theory of hypergeometrical sums). It i s interesting to note that it may be established (in the same [3]) that for the question of finding closed formulas for the quantity $E(|S_n|)$ and p_n for centered random walks without condition $\sigma^2 = 1$, there is no solution.

We may establish an asymptotic result regarding the growth of expectations :

Theorem 4.2. *For simple centered random walks, the relation* $E(|S_n|) \sim \sigma\sqrt{\frac{2n}{\pi}}$, *holds when* $n \to \infty$.

(**Note:** In view of symmetry of the law , the result is equivalent to $E(S_n^+) \sim \sigma\sqrt{\frac{n}{2\pi}}$, or to $E(S_n^-) \sim \sigma\sqrt{\frac{n}{2\pi}}$).

Proof

According to the above relation (4.2) it is enough to verify that $E(N^0_n) \sim \frac{1}{\sigma}\sqrt{\frac{2n}{\pi}}$. But in the other hand, in virtue of the local central limit theorem, when $n \to \infty, p_n \sim \Phi(0)\frac{1}{\sigma\sqrt{n}}$, where $\Phi(x) = \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}}$, this is $p_n \sim \frac{1}{\sigma\sqrt{2\pi n}}$. From this and from (4.2) it is deducted, taking into consideration the divergence of the series $\sum_{k\geq 1} \frac{1}{\sqrt{k}}$, that $E(N^0_n) \sim \frac{1}{\sigma\sqrt{2\pi}} \int_2^n x^{-\frac{1}{2}} dx \sim \frac{1}{\sigma}\sqrt{\frac{2n}{\pi}}$, as it was looked for. □

(In [1] it is demonstrated that this result holds in the general case of recurrent random walks (P3 , section 18, chap. IV). Contrary to the proof of this author , we use the local version of the central limit theorem instead of the global one, which looks more natural within the context of simple random walks. The proof is thus simplified considerably).

The study of " tail " probabilities for certain times T, this is that of probabilities of the form $P(T > K)$, for large K , may be done through the above identities. Lets consider the case $T = T^0$, for a centered random walk , using the identity $E(|S_T|) = \sigma^2 E(N_T^0)$. As in this case T is not integrable we apply the identity to $T' = \min(T, n)$, for an integer n , and we obtain $E(|S_T|) = \sigma^2 E(N_T^0)$. But as $E(|S_{T'}|) = E(|S_T|; T > n)$, and $E(N_{T'}^0) = 1$, we obtain $E(|S_n|; T > n) = \sigma^2$. Given that $E(|S_n|; T > n) \leq nP(T > n)$, from the above relation we obtain :

Corollary 4.1. *For a centered simple random walk, for all integer n :* $P(T > n) \geq \frac{\sigma^2}{n}$

Note : It is known , as explained in [1], that $P(T > n)$ is equivalent asymptotically to $\frac{1}{\sqrt{n}}$, which is congruent with the above result.

References

[1] F. Spitzer; **Principles of Random Walk,** Springer Verlag, New York, edición de (1976).

[2] H. Wilf; **Generatingfunctionology,** Academic Press, San Diego, (1994).

[3] M. Petrovsek, H. Wilf, D. Zeilberger; **A=B,** AK Peters, Wellesley, Massachusetts, (1996).

[4] P. Hoel, S. Port, Ch. Stone; **Introduction to probabily theory,** Houghton Mifflin Company, Boston, (1971).

[5] J. Neveu; **Martingales à temps discret,** Masson, Paris, (1972).

[6] E. Perkins; *A global continous characterizaction of brownian local time*, Annals of Probability 9, pp. 800 – 817, (1981).

[7] D. Dacunha-Castelle, M. Duflo; **Probabilités et Statistiques,** tomo 1, Masson, Paris, (1982).

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