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A NEW FORM OF FUZZY β -COMPACTNESS *

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Abstract

A new form of β -compactness is introduced in L -topological spaces by means of β -open L -sets and their inequality where L is a complete de Morgan algebra. This new form doesn't rely on the structure of basis lattice L . It can also be characterized by means of β -closed L -sets and their inequality. When L is a completely distributive de Morgan algebra, its many characterizations are presented. Meanwhile countable β -compactness and the β -Lindelöf property are also researched.

Key Words and Phrases: *L -topology, compactness, β -compactness, countable β -compactness, the β -Lindelöf property*

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1. Introduction

As we know, stronger and weaker forms of compactness occupy very important places in general topology. In [5], Abd El-Monsef et al introduced the concepts of β -open sets and β -continuous functions in general topology, and Fath Alla in [1] introduced these concepts in $[0,1]$ -topological spaces. In [2], G. Balasubramanian generalized the concept of β -compactness [6] to $[0,1]$ -topological spaces along the line of Chang's compactness which is not a good extension.

In [12, 13], a new definition of fuzzy compactness is presented in L -topological spaces by means of open L -sets and their inequality where L is a complete de Morgan algebra. This new definition doesn't depend on the structure of L . When L is completely distributive, it is equivalent to the notion of fuzzy compactness in [9, 10, 15]. In this paper, following the lines of [12, 13], we shall introduce a new form of β -compactness in L -topological spaces by means of β -open L -sets and their inequality. This new form of β -compactness has many characterizations if L is completely distributive.

2. Preliminaries

Throughout this paper $(L, \vee, \wedge, ')$ is a complete de Morgan algebra, X a nonempty set. L^X is the set of all L -fuzzy sets (or L -sets for short) on X . The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$.

An element a in L is called prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in L is called co-prime element if a' is a prime element [7]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $M(L)$.

The binary relation \prec in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [4]. In a completely distributive de Morgan algebra L , each element b is a sup of $\{a \in L \mid a \prec b\}$. $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of b in the sense of [9, 15], in symbol $\beta(b)$. Moreover for $b \in L$, define $\beta^*(b) = \beta(b) \cap M(L)$, $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations in [14].

$$A_{[a]} = \{x \in X \mid A(x) \geq a\}, \quad A^{(a)} = \{x \in X \mid A(x) \not\geq a\}.$$

An L -topological space (or L -space for short) is a pair (X, \mathcal{T}) , where \mathcal{T} is a subfamily of L^X which contains \mathcal{X}_\emptyset , \mathcal{X}_X and is closed for any suprema

and finite infima. \mathcal{T} is called an L -topology on X . Members of \mathcal{T} are called open L -sets and their complements are called closed L -sets. We often don't differ a crisp subset A of X and its character function \mathcal{X}_A .

Definition 2.1 ([9, 15]) : An L -space (X, \mathcal{T}) is called weak induced if $\forall a \in L, \forall A \in \mathcal{T}$, it follows that $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in \mathcal{T} .

Definition 2.2 ([9, 15]) : For a topological space (X, τ) , let $\omega_L(\tau)$ denote the family of all the lower semi-continuous maps from (X, τ) to L , i.e., $\omega_L(\tau) = \{A \in L^X \mid A^{(a)} \in \tau, a \in L\}$. Then $\omega_L(\tau)$ is an L -topology on X , in this case, $(X, \omega_L(\tau))$ is called topologically generated by (X, τ) . A topologically generated L -space is also called an induced L -space.

It is obvious that $(X, \omega_L(\tau))$ is weak induced.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamily of Φ . $2^{[\Phi]}$ denotes the set of all countable subfamily of Φ .

Definition 2.3 ([12, 13]) : Let (X, \mathcal{T}) be an L -space, $G \in L^X$ is called (countably) compact if for every (countably) family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Definition 2.4 [13] : Let (X, \mathcal{T}) be an L -space, $G \in L^X$ is said to have the Lindelöf property if for every family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Lemma 2.5 [13] : Let L be a complete Heyting algebra, $f : X \rightarrow Y$ be a map, $f_L^\rightarrow : L^X \rightarrow L^Y$ is the extension of f , then for any family $\mathcal{P} \subseteq L^Y$, we have:

$$\bigvee_{y \in Y} \left(f_L^\rightarrow(G)(y) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^\leftarrow(B)(x) \right).$$

The following definition is presented in $[0,1]$ -topological spaces. Analogously we can generalize it to L -fuzzy setting as follows:

Definition 2.6 [1] : An L -set A in an L -space (X, δ) is said to be β -open if $A \leq cl(int(cl(A)))$.

Definition 2.7. A map $f : (X, \delta) \rightarrow (Y, \mu)$ is said to be fuzzy β -continuous [1] (resp. $M\beta$ -continuous [8] if the inverse image $f_L^-(B)$ of every open (resp. β -open) L -set B in Y is β -open in X .

3. Definition and characterizations of β -compactness

Definition 3.1 : Let (X, \mathcal{T}) be an L -space. $G \in L^X$ is called (countably) β -compact if for every (countable) family \mathcal{U} of β -open L -sets, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Definition 3.2 : Let (X, \mathcal{T}) be an L -space. $G \in L^X$ is said to have the β -Lindelöf property (or be an β -Lindelöf L -set) if for every family \mathcal{U} of β -open L -sets, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Obviously we have the following theorem.

Theorem 3.3 : β -compactness implies countably β -compactness and the β -Lindelöf property. Moreover an L -set having the β -Lindelöf property is β -compact if and only if it is countably β -compact.

Since an open L -set must be β -open, we have the following theorem.

Theorem 3.4 : β -compactness implies compactness, countably β -compactness implies countably compactness, and the β -Lindelöf property implies the Lindelöf property.

From Definition 3.1 and Definition 3.2 we can obtain the following two theorems by simply using complement.

Theorem 3.5 : Let (X, \mathcal{T}) be an L -space. $G \in L^X$ is (countably) β -compact if and only if for every (countable) family \mathcal{B} of β -closed L -sets, it follows that

$$\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{B}} B(x) \right) \geq \bigwedge_{\mathcal{F} \in 2^{[\mathcal{B}]}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right).$$

Theorem 3.6 : Let (X, \mathcal{T}) be an L -space. $G \in L^X$ has the β -Lindelöf property if and only if for every family \mathcal{B} of β -closed L -sets, it follows that

$$\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{B}} B(x) \right) \geq \bigwedge_{\mathcal{F} \in 2^{[\mathcal{B}]}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right).$$

In order to present characterizations of β -compactness, countable β -compactness and the β -Lindelöf property, we generalize the notions of a -shading and a -R-neighborhood family in [12, 13] as follows:

Definition 3.7 : Let (X, \mathcal{T}) be an L -space, $a \in L \setminus \{1\}$ and $G \in L^X$. A family $\mathcal{A} \subseteq L^X$ is said to be

- (1) An a -shading of G if for any $x \in X$, $\left(G'(x) \vee \bigvee_{A \in \mathcal{A}} A(x) \right) \not\leq a$.
- (2) A strong a -shading of G if $\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{A}} A(x) \right) \not\leq a$.
- (3) An a -remote family of G if for any $x \in X$, $\left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq a$.
- (4) A strong a -remote family of G if $\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq a$.

It is obvious that a strong a -shading of G is an a -shading of G , a strong a -remote family of G is an a -remote family of G , and \mathcal{P} is a strong a -remote family of G if and only if \mathcal{P}' is a strong a' -shading of G .

Definition 3.8 : Let $a \in L \setminus \{0\}$ and $G \in L^X$. A subfamily \mathcal{A} of L^X is said to have weak a -nonempty intersection in G if

$$\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{A \in \mathcal{A}} A(x) \right) \geq a.$$
 \mathcal{A} is said to have the finite (countable) weak a -intersection property in G if every finite (countable) subfamily \mathcal{F} of \mathcal{A} has weak a -nonempty intersection in G .

From Definition 3.1, Definition 3.2, Theorem 3.5 and Theorem 3.6 we immediately obtain the following two results.

Theorem 3.9 : Let (X, \mathcal{T}) be an L -space and $G \in L^X$. Then the following conditions are equivalent:

- (1) G is (countably) β -compact.
- (2) For any $a \in L \setminus \{1\}$, each (countable) β -open strong a -shading \mathcal{U} of G has a finite subfamily which is a strong a -shading of G .
- (3) For any $a \in L \setminus \{0\}$, each (countable) β -closed strong a -remote family \mathcal{P} of G has a finite subfamily which is a strong a -remote family of G .
- (4) For any $a \in L \setminus \{0\}$, each (countable) family of β -closed L -sets which has the finite weak a -intersection property in G has weak a -nonempty intersection in G .

Theorem 3.10 : Let (X, \mathcal{T}) be an L -space and $G \in L^X$. Then the following conditions are equivalent:

- (1) G has the β -Lindelöf property.
- (2) For any $a \in L \setminus \{1\}$, each β -open strong a -shading \mathcal{U} of G has a countable subfamily which is a strong a -shading of G .
- (3) For any $a \in L \setminus \{0\}$, each β -closed strong a -remote family \mathcal{P} of G has a countable subfamily which is a strong a -remote family of G .
- (4) For any $a \in L \setminus \{0\}$, each family of β -closed L -sets which has the countable weak a -intersection property in G has weak a -nonempty intersection in G .

4. Properties of (countable) β -compactness

Theorem 4.1 : Let L be a complete Heyting algebra. If both G and H are (countably) β -compact, then $G \vee H$ is (countably) β -compact.

Proof. For any (countable) family \mathcal{P} of β -closed L -sets, by Theorem 3.5 we have that

$$\begin{aligned}
 & \bigvee_{x \in X} \left((G \vee H)(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \\
 = & \left\{ \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \right\} \vee \left\{ \bigvee_{x \in X} \left(H(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \right\} \\
 \geq & \left\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \vee \left\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left(H(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \\
 = & \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left((G \vee H)(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right).
 \end{aligned}$$

This shows that $G \vee H$ is (countably) β -compact.

Analogously we have the following result.

Theorem 4.2 : Let L be a complete Heyting algebra. If both G and H have the β -Lindelöf property, then $G \vee H$ has the β -Lindelöf property.

Theorem 4.3 : If G is (countably) β -compact and H is β -closed , then $G \wedge H$ is (countably) β -compact.

Proof : For any (countable) family \mathcal{P} of β -closed L -sets, by Theorem 3.5 we have that

$$\begin{aligned}
 & \bigvee_{x \in X} \left((G \wedge H)(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \\
 = & \bigvee_{x \in X} \left((G \wedge H)(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \\
 \geq & \bigwedge_{\mathcal{F} \in 2(\mathcal{P} \cup \{H\})} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \\
 = & \left\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \wedge \\
 & \left\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left(G(x) \wedge H(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \\
 & \left\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left(G(x) \wedge H(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\}
 \end{aligned}$$

$$\bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigvee_{x \in X} \left((G \wedge H)(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right).$$

This shows that $G \wedge H$ is (countably) β -compact.

Analogously we have the following result.

Theorem 4.4 : If G has the β -Lindelöf property and H is β -closed , then $G \wedge H$ has the β -Lindelöf property.

Theorem 4.5 : Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ be $M\beta$ -continuous. If G is an β -compact (respectively a countably β -compact, an β -Lindelöf) L -set in (X, \mathcal{T}_1) , then so is $f_L^{\rightarrow}(G)$ in (Y, \mathcal{T}_2) .

Proof. We only prove that the theorem is true for β -compactness. Suppose that \mathcal{P} is a family of β -closed L -sets in (Y, \mathcal{T}_2) , by Lemma 2.5 and β -compactness of G we have that

$$\begin{aligned} & \bigvee_{y \in Y} \left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) \\ &= \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^{\leftarrow}(B)(x) \right) \\ &\geq \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} f_L^{\leftarrow}(B)(x) \right) \\ &= \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigvee_{y \in Y} \left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B \in \mathcal{F}} B(y) \right). \end{aligned}$$

Therefore $f_L^{\rightarrow}(G)$ is β -compact.

Analogously we have the following result.

Theorem 4.6 : Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ be β -continuous. If G is an β -compact (respectively a countably β -compact, an β -Lindelöf) L -set in (X, \mathcal{T}_1) , then $f_L^{\rightarrow}(G)$ is a compact (countably compact, Lindelöf) L -set in (Y, \mathcal{T}_2) .

Definition 4.7 : Let (X, \mathcal{T}_1) and (Y, \mathcal{T}_2) be two L -spaces. A map $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ is called strongly $M\beta$ -continuous if $f_L^{\leftarrow}(G)$ is open in (X, \mathcal{T}_1) for every β -open L -set G in (Y, \mathcal{T}_2) .

It is obvious that a strongly $M\beta$ -continuous map is $M\beta$ -continuous. Analogously we have the following result.

Theorem 4.8 : Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ be a strongly $M\beta$ -continuous map. If G is a compact (respectively countably compact, Lindelöf) L -set in (X, \mathcal{T}_1) , then $f_L^{\rightarrow}(G)$ is an β -compact (a countably β -compact, an β -Lindelöf) L -set in (Y, \mathcal{T}_2) .

5. Further characterizations of β -compactness and goodness

In this section, we assume that L is a completely distributive de Morgan algebra.

Now we generalize the notions of open β_a -cover and open Q_a -cover [13] as follows:

Definition 5.1 : Let (X, \mathcal{T}) be an L -space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a β_a -cover of G if for any $x \in X$, it follows that $a \in \beta \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right)$. \mathcal{U} is called a strong β_a -cover of G if $a \in \beta \left(\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \right)$.

Definition 5.2 : Let (X, \mathcal{T}) be an L -space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathbf{U} \subseteq L^X$ is called a Q_a -cover of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A \in \mathbf{U}} A(x) \geq a$.

It is obvious that a strong β_a -cover of G must be a β_a -cover of G and a β_a -cover of G must be a Q_a -cover of G .

Analogous to the proof of Theorem 2.9 in [13] we can obtain the following theorem.

Theorem 5.3 : Let (X, \mathcal{T}) be an L -space and $G \in L^X$. Then the following conditions are equivalent.

- (1) G is β -compact.
- (2) For any $a \in L \setminus \{0\}$, each β -closed strong a -remote family \mathcal{P} of G has a finite subfamily which is a strong a -remote family of G .
- (3) For any $a \in L \setminus \{0\}$, each β -closed strong a -remote family \mathcal{P} of G has a finite subfamily which is an a -remote family of G .
- (4) For any $a \in L \setminus \{0\}$ and any β -closed strong a -remote family \mathcal{P} of G , there exist a finite subfamily \mathcal{F} of \mathcal{P} and $b \in \beta(a)$ such that \mathcal{F} is a strong b -remote family of G .
- (5) For any $a \in L \setminus \{0\}$ and any β -closed strong a -remote family \mathcal{P} of G , there exist a finite subfamily \mathcal{F} of \mathcal{P} and $b \in \beta(a)$ such that \mathcal{F} is a b -remote family of G .
- (6) For any $a \in L \setminus \{1\}$, each β -open strong a -shading \mathcal{U} of G has a finite subfamily which is a strong a -shading of G .
- (7) For any $a \in L \setminus \{1\}$, each β -open strong a -shading \mathcal{U} of G has a finite subfamily which is an a -shading of G .
- (8) For any $a \in L \setminus \{1\}$ and any β -open strong a -shading \mathcal{U} of G , there exist a finite subfamily \mathcal{V} of \mathcal{U} and $b \in \alpha(a)$ such that \mathcal{V} is a strong b -shading of G .
- (9) For any $a \in L \setminus \{1\}$ and any β -open strong a -shading \mathcal{U} of G , there exist a finite subfamily \mathcal{V} of \mathcal{U} and $b \in \alpha(a)$ such that \mathcal{V} is a b -shading of G .
- (10) For any $a \in L \setminus \{0\}$, each β -open strong β_a -cover \mathcal{U} of G has a finite subfamily which is a strong β_a -cover of G .
- (11) For any $a \in L \setminus \{0\}$, each β -open strong β_a -cover \mathcal{U} of G has a finite subfamily which is a β_a -cover of G .
- (12) For any $a \in L \setminus \{0\}$ and any β -open strong β_a -cover \mathcal{U} of G , there exist a finite subfamily \mathcal{V} of \mathcal{U} and $b \in L$ with $a \in \beta(b)$ such that \mathcal{V} is a strong β_b -cover of G .
- (13) For any $a \in L \setminus \{0\}$ and any β -open strong β_a -cover \mathcal{U} of G , there exist a finite subfamily \mathcal{V} of \mathcal{U} and $b \in L$ with $a \in \beta(b)$ such that \mathcal{V} is a β_b -cover of G .
- (14) For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each β -open Q_a -cover of G has a finite subfamily which is a Q_b -cover of G .
- (15) For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each β -open Q_a -cover of G has a finite subfamily which is a β_b -cover of G .
- (16) For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each β -open Q_a -cover of G has a finite subfamily which is a strong β_b -cover of G .

Remark 5.4 : In Theorem 5.3, $a \in L \setminus \{0\}$ and $b \in \beta(a)$ can be replaced by $a \in M(L)$ and $b \in \beta^*(a)$ respectively, $a \in L \setminus \{1\}$ and $b \in \alpha(a)$ can be replaced by $a \in P(L)$ and $b \in \alpha^*(a)$

Now we consider the goodness of β -compactness.

For $a \in L$ and a crisp subset $D \subset X$, we define $a \wedge D$ and $a \vee D$ as follows:

$$(a \wedge D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \notin D. \end{cases} \quad (a \vee D)(x) = \begin{cases} 1, & x \in D; \\ a, & x \notin D. \end{cases}$$

Theorem 5.5 ([14]) :For an L -set $A \in L^X$, the following facts are true.

- (1) $A = \bigvee_{a \in L} (a \wedge A_{(a)}) = \bigvee_{a \in L} (a \wedge A_{[a]})$.
- (2) $A = \bigwedge_{a \in L} (a \vee A^{(a)}) = \bigwedge_{a \in L} (a \vee A^{[a]})$.

Theorem 5.6 ([15])Let $(X, \omega_L(\tau))$ be the L -space topologically generated by (X, τ) and $A \in L^X$. Then the following facts hold.

- (1) $cl(A) = \bigvee_{a \in L} (a \wedge (A_{(a)})^-) = \bigvee_{a \in L} (a \wedge (A_{[a]})^-)$;
- (2) $cl(A)_{(a)} \subset (A_{(a)})^- \subset (A_{[a]})^- \subset cl(A)_{[a]}$;
- (3) $cl(A) = \bigwedge_{a \in L} (a \vee (A^{(a)})^-) = \bigwedge_{a \in L} (a \vee (A^{[a]})^-)$;
- (4) $cl(A)^{(a)} \subset (A^{(a)})^- \subset (A^{[a]})^- \subset cl(A)^{[a]}$;
- (5) $int(A) = \bigvee_{a \in L} (a \wedge (A_{(a)})^\circ) = \bigvee_{a \in L} (a \wedge (A_{[a]})^\circ)$;
- (6) $int(A)_{(a)} \subset (A_{(a)})^\circ \subset (A_{[a]})^\circ \subset int(A)_{[a]}$;
- (7) $int(A) = \bigwedge_{a \in L} (a \vee (A^{(a)})^\circ) = \bigwedge_{a \in L} (a \vee (A^{[a]})^\circ)$;
- (8) $int(A)^{(a)} \subset (A^{(a)})^\circ \subset (A^{[a]})^\circ \subset int(A)^{[a]}$,

where $(A_{(a)})^-$ and $(A_{(a)})^\circ$ denote respectively the closure and the interior of $A_{(a)}$ in (X, τ) , and $cl(A)$ and $int(A)$ denote respectively the closure and the interior of A in $(X, \omega_L(\tau))$.

Lemma 5.7 : Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ) . If A is an β -open L -set in (X, τ) , Then \mathcal{X}_A is an β -open set in $(X, \omega_L(\tau))$. If B is an β -open L -set in $(X, \omega_L(\tau))$, Then $B_{(a)}$ is an β -open set in (X, τ) for every $a \in L$.

Proof. If A is a β -open set in (X, τ) , then $A \subseteq ((A^-)^\circ)^-$. Thus we have that

$$\mathcal{X}_A \leq \mathcal{X}_{((A^-)^\circ)^-} = cl(\mathcal{X}_{(A^-)^\circ}) = cl(int(\mathcal{X}_{A^-})) = cl(int(cl(A))).$$

This shows that A is β -open in $(X, \omega_L(\tau))$.

If B is a β -open L -set in $(X, \omega_L(\tau))$, then $B \leq cl(int(cl(B)))$. From Theorem 4.2 we have that

$$B_{(a)} \subseteq cl(int(cl(B)))_{(a)} \subseteq (int(cl(B))_{(a)})^- \subseteq ((cl(B)_{(a)})^\circ)^- \subseteq (((B_{(a)})^-)^\circ)^-.$$

This shows that $B_{(a)}$ is a β -open set in (X, τ) .

The following two theorems show that β -compactness, countable β -compactness and the β -Lindelöf property are good extensions.

Theorem 5.8 : Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ) . Then $(X, \omega_L(\tau))$ is (countably) β -compact if and only if (X, τ) is (countably) β -compact.

Proof : (Necessity) Let \mathcal{A} be an (a countable) β -open cover of (X, τ) . Then $\{A \mid A \in \mathcal{A}\}$ is a family of β -open L -sets in $(X, \omega_L(\tau))$ with

$\bigwedge_{x \in X} \left(\bigvee_{A \in \mathbf{U}} A(x) \right) = 1$. From (countable) β -compactness of $(X, \omega_L(\tau))$ we know that

$$1 \geq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left(\bigvee_{A \in \mathcal{V}} A(x) \right) \geq \bigwedge_{x \in X} \left(\bigvee_{A \in \mathcal{U}} A(x) \right) = 1.$$

This implies that there exists $\mathcal{V} \in 2^{\mathcal{U}}$ such that $\bigwedge_{x \in X} \left(\bigvee_{A \in \mathcal{V}} A(x) \right) = 1$.

Hence \mathcal{V} is a cover of (X, τ) . Therefore (X, τ) is (countably) β -compact.

(Sufficiency) Let \mathcal{U} be a (countable) family of β -open L -sets in $(X, \omega_L(\tau))$ and let $\bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{U}} B(x) \right) = a$. If $a = 0$, then obviously we have that

$$\bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{U}} B(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left(\bigvee_{A \in \mathcal{V}} B(x) \right).$$

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a) \setminus \{0\}$ we have that

$$b \in \beta \left(\bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{U}} B(x) \right) \right) \subseteq \bigcap_{x \in X} \beta \left(\bigvee_{B \in \mathcal{U}} B(x) \right) = \bigcap_{x \in X} \bigcup_{B \in \mathcal{U}} \beta(B(x)).$$

From Lemma 5.7 this implies that $\{B_{(b)} \mid B \in \mathcal{U}\}$ is a β -open cover of (X, τ) . From (countable) β -compactness of (X, τ) we know that there exists $\mathcal{V} \in 2^{\mathcal{U}}$ such that $\{B_{(b)} \mid B \in \mathcal{V}\}$ is a cover of (X, τ) . Hence

$$b \leq \bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{V}} B(x) \right).$$

Further we have that

$$b \leq \bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{V}} B(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{V}} B(x) \right).$$

This implies that

$$\bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{U}} B(x) \right) = a = \bigvee \{b \mid b \in \beta(a)\} \leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{V}} B(x) \right).$$

Therefore $(X, \omega_L(\tau))$ is (countably) β -compact.

Analogously we have the following result.

Theorem 5.1. *Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ) . Then $(X, \omega_L(\tau))$ has the β -Lindelöf property if and only if (X, τ) has the β -Lindelöf property.*

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