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NONRESONANCE BETWEEN TWO EIGENVALUES NOT NECESSARILY CONSECUTIVE

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Abstract

In this paper we study the existence of solutions for a semilinear elliptic problem in case two eigenvalues are not necessarily consecutive.

Résumé : *Dans cet article, nous étudions l'existence des solutions entre deux valeurs propres non nécessairement consecutives d'un problème semi-linéaire elliptique.*

Key words : *Variational elliptic problems - Resonance.*

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$, and let $g : \Omega \times \mathbf{R} \to \mathbf{R}$ be a nonlinear function satisfying the Carathéodory conditions. We consider the Dirichlet problem

$$\begin{cases} -\Delta u = g(x,u) + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $h \in L^2(\Omega)$. Given $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots \leq \lambda_k \leq \ldots$ the sequence of eigenvalues of the problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$.

Let us denote by $G(x,s)$, the primitive $\int_0^s g(x,t)\,dt$, and write

$$l_\pm(x) = \liminf_{s\to\pm\infty} \frac{g(x,s)}{s}, \qquad k_\pm(x) = \limsup_{s\to\pm\infty} \frac{g(x,s)}{s}$$

$$L_\pm(x) = \liminf_{s\to\pm\infty} \frac{2G(x,s)}{s^2}, \qquad K_\pm(x) = \limsup_{s\to\pm\infty} \frac{2G(x,s)}{s^2}$$

with, for an autonomous nonlinearity $g(x,s) = g(s)$, $l_\pm$ instead of $l_\pm(x)$. Assume that

$$\lambda_k \leq l_\pm(x) \leq k_\pm(x) \leq \lambda_{k+1} \tag{1.2}$$

uniformly for a.e. $x \in \Omega$.

As is well known, in the special case when g is linear, i.e. $g(x,s) = \lambda s$, the problem (1.1) is completely solved by the Fredholm alternative, namely (1.1) has a solution for each h, if and only if λ is not an eigenvalue of the linear operator $-\Delta$. For instance, we recall that, according to Dolph [8], the solvability of (1.1)), for any $h \in L^2(\Omega)$, is ensured when

$$\lambda_k < \nu_k \leq l_\pm(x) \leq k_\pm(x) \leq \nu_{k+1} < \lambda_{k+1} \tag{1.3}$$

However, the situation where $l_\pm(x) \equiv \lambda_k$ or $k_\pm(x) \equiv \lambda_{k+1}$ was considered in several works, (see [12], [1] [4], [2], [14], [7], [9], [13]).

In [6], Costa and Oliviera extended the result of [8], allowing equality in both sides of (1.3) for every $x \in \Omega$, and assumed the following condition

$$\lambda_k \leq L_\pm(x) \leq K_\pm(x) \leq \lambda_{k+1} \tag{1.4}$$

uniformly for a.e. $x \in \Omega$, with strict inequalities $\lambda_k < L_\pm(x), K_\pm(x) < \lambda_{k+1}$ holding on subset of positive measure.

More recently, the author and Moussaoui, in [10], proved an existence result in situation $L_\pm(x) \equiv \lambda_k$ for a.e. $x \in \Omega$ and $K_\pm(x) \equiv \lambda_{k+1}$ for a.e.

$x \in \Omega$. They showed that (1.1) is solvable when $\frac{g(x,s)}{s}$ stays "between" λ_k and λ_{k+1} for large values of $|s|$ and they replaced (1.4) by classical resonance conditions of Ahmad-Lazer-Paul on two sides of (1.4).

In this paper, our main objective is to study the solutions of problem (1.1) when the nonlinearity g lies asymptotically between two eigenvalues not necessarily consecutive. It is clear that is such situations the solvability of (1.1) cannot be guaranteed without further assumption on the potential G.

To state our main result, let us denote by $E(\lambda_j)$ the λ_j-eigenspace. For every $u \in H_0^1(\Omega)$ write $u^j = P_j u$, where P_j is orthogonal projection onto $E(\lambda_j)$.

Theorem 1.1. *Let $k \geq 2$ and make the following assumptions:*

G_0*)* $\sup_{|s|\leq R} |g(x,s)| \in L^2(\Omega)$ for all $R > 0$,

G_1)
$$\lambda_{k-1} < \nu_{k-1} \leq l_{\pm}(x) \leq k_{\pm}(x) \leq \lambda_{k+1} \text{ uniformly on } \Omega$$

G_2*) whenever $u_n \subset H_0^1(\Omega)$ is such that $\frac{u_n}{\|u_n\|} \rightharpoonup z \not\equiv 0$ $\frac{u_n^k}{\|u_n\|} \to z^k \not\equiv 0$ as $n \to \infty$, then*

$$0 < \limsup_{n\to\infty} \int [g(x,u_n(x)) - \lambda_k u_n(x)] \frac{u_n^k(x)}{\|u_n\|^2}\, dx.$$

G_3)
$$\int_{z>0} (\lambda_{k+1} - K_+(x)) z^2\, dx + \int_{z<0} (\lambda_{k+1} - K_-(x)) z^2\, dx > 0,$$

for every $z \in E(\lambda_{k+1})$.

G_4) $\lambda_k \leq L\pm(x)$ and $\displaystyle\int_{z>0} (L_+(x) - \lambda_k) z^2\, dx + \int_{z<0} (L_-(x) - \lambda_k) z^2\, dx > 0,$

for every $z \in E(\lambda_k)$.

Then, for any $h \in L^2(\Omega)$, problem (1.1) has at least one solution.

Remark 1. *Note that the assumptions G_3) and G_4) are weaker than condition on the potential G assumed in [6]. Indeed,*

1. *G_3) occurs if G verified $K_\pm(x) \le \lambda_{k+1}$ and the following condition:*

$$\begin{cases} \text{there exists a subset } \Omega' \text{ of } \Omega \text{such that} \\ K_+(x) = \limsup_{s\to\infty} \dfrac{2G(x,s)}{s^2} (resp. K_-(x) \\ = \limsup_{s\to-\infty} \dfrac{2G(x,s)}{s^2}) < \lambda_{k+1} \text{ a.e. in } \Omega'. \end{cases}$$

2. *Furthermore G_4) is satisfied if $\lambda_k < L_+(x)$ or $\lambda_k < L_-(x)$ holds on the subset of positive measure.*

Next, we are interested in situations where $\frac{g(x,s)}{s}$ is less than λ_2 and both $l_\pm(x), L_\pm(x)$ can be greater than λ_1.

Theorem 1.2. *Assume that G_2), $k = 1$ and*

G_5) $|g(x,s)| \le A|s| + b(x)$, for all *$s \in \mathbf{R}$ and all every $x \in \Omega$, $A > 0, b \in L^2(\Omega)$.*

G_6)
$$k_\pm(x) \le \lambda_2 \text{ uniformly on } \Omega$$

G_7)
$$\int_{z>0} (\lambda_2 - K_+(x)) z^2\, dx + \int_{z<0} (\lambda_2 - K_-(x)) z^2\, dx > 0,$$

for every $z \in E_{\lambda_2}$.

Then, for any $h \in L^2(\Omega)$, problem (1.1) has at least one solution.

The proofs of theorem 1.1 and 1.2 use the general minimax theorem proved by Bartolo et al. in [3].

In section 4, we present several examples where our results apply and where, as far as we can see, previously known results do not hold.

2. Preliminaries. A compactness condition

By a solution of (1.1) we mean a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \nabla v - \int_\Omega g(x,u) v - \int_\Omega h(x) v = 0, \text{ for all } v \in H_0^1(\Omega)$$

where $H_0^1(\Omega)$ is the dual space obtained through completion of $C_c^\infty(\Omega)$ with respect to the norm induced by the inner product

$$< u, v >= \int_\Omega \nabla u \nabla v, \quad u, v \in H_0^1(\Omega)$$

If g is Hölder continuous, then the regularity arguments imply that any solution of (1.2) is, in fact, in $C^2(\Omega) \cap C(\overline{\Omega})$, and satisfies the equation (1.1) for every $x \in \Omega$.

Define, for all $u \in H_0^1(\Omega)$, the functional

$$\Phi(u) = \int_\Omega |\nabla u|^2 - \int G(x,u) - \int h(x)u.$$

Under the growth condition on g, it is well know that Φ is well defined on $H_0^1(\Omega)$, weakly lower semicontinuous and continuously Fréchet differentiable, with derivative given by

$$\Phi'(u)v = \int_\Omega \nabla u \nabla v - \int g(x,u)v - \int h(x)v, \text{ for all } u, v \in H_0^1(\Omega)$$

Thus, finding solutions of (1.1) is equivalent to finding critical points of the functional Φ.

In order to apply minimax methods for finding critical points of Φ, we need to verify that Φ satisfies a compactness condition of the Palais-Smale type which was introduced by Cerami.

A functional $\Phi \in C^1(E, \mathbf{R})$, where E is a real Banach space, is said to satisfy condition $(C)_c$ at the level $c \in \mathbf{R}$ if the following holds:

$(C)_c$ **i)** any bounded sequence $(u_n) \subset E$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ possesses a convergent subsequence;

ii) there exist constants δ, R,$\alpha > 0$ such that

$$\|\Phi'(u)\|\|u\| \geq \alpha \text{ for any } u \in \Phi^{-1}([c-\delta, c+\delta]) \text{ with } \|u\| \geq R.$$

It was shown in [3] that condition (C) actually is sufficient to get a deformation theorem and then, by standard minimax arguments (see [3]), the following result was proved.

Theorem 2.1. *: Suppose that $\Phi \in C^1(E, \mathbf{R})$, E is a real Banach space and satisfies condition $(C)_c$ $\forall c \in \mathbf{R}$ and that there exists a closed subset $S \subset E$ and $Q \subset E$ with boundary ∂Q satisfying the following conditions :*

i) $\sup_{u \in \partial Q} \Phi(u) \leq \alpha < \beta \leq \inf_{u \in S} \Phi(u)$ for some $0 \leq \alpha < \beta$;

ii) S and ∂Q link;

iii) $\sup_{u \in Q} \Phi(u) < \infty$.

Then Φ possesses a critical value $c \geq \beta$.

Since we are going to apply the variational characterization of the eigenvalues, we will decompose the space $H_0^1(\Omega)$ as $E = E_- \oplus E_k \oplus E_+$, where E_- is the subspace spanned by the λ_j- eigenfunctions with $j < k$ and E_j is the eigenspace generated by the λ_j-eigenfunctions and E_+ is the orthogonal complement of $E_- \oplus E_k$ in $H_0^1(\Omega)$. We will also decompose for any $u \in H_0^1(\Omega)$, as $u = u^- + u^k + u^+$ where $u^- \in E_-$, $u^k \in E_k$, and $u^+ \in E_+$.

3. Proofs of theorems

To apply theorem 2.1, we shall do separate studies of the "compactness" of Φ and its "geometry". First, we prove that Φ satisfies the Cerami condition.

Lemma 3.1. *Φ satisfies the $(C)_c$ condition on $H_0^1(\Omega)$, for all $c \in \mathbf{R}$.*

Proof: Let us initially verify that the Palais-Smale condition is satisfied on the bounded subset of $H_0^1(\Omega)$. Let $(u_n)_n \subset H_0^1(\Omega)$, be bounded and such that $\Phi'(u_n) \to 0$ in $H^{-1}(\Omega)$. If we identify $L^2(\Omega)$ with its dual, one has that

$$-\Delta u_n - g(x, u_n) - h(x) \to 0 \qquad in\ H^{-1}(\Omega).$$

This implies that

$$u_n - (-\Delta)^{-1}[g(x, u_n) + h] \to 0 \qquad in\ H_0^1(\Omega).$$

Since (u_n) is bounded we can select a subsequence noted also (u_n) weakly converging to $u_0 \in H_0^1(\Omega)$ and on the other hand, we have $u \mapsto g(x, u) + h$ is completely continuous from $H_0^1(\Omega) \to H^{-1}(\Omega)$ then,

$$(-\Delta)^{-1}[g(x, u_n) + h] \to (-\Delta)^{-1}[g(x, u_0) + h].$$

It obvious that the subsequence (u_n) converges in $H_0^1(\Omega)$.

Let us now prove that $(C)_c ii)$ is satisfied for every $c \in \mathbf{R}$. Assume by contradiction, Let $c \in \mathbf{R}$ and $(u_n)_n \subset H_0^1(\Omega)$ such that:

$$\Phi(u_n) \to c \tag{3.1}$$

$$\|u_n\|| < \Phi'(u_n), v > | \le \epsilon_n \|v\| \qquad \forall v \in H_0^1(\Omega) \tag{3.2}$$

$$\|u_n\| \to \infty, \epsilon_n \to 0,\ \text{as } n \to \infty.$$

Set $z_n = \frac{u_n}{\|u_n\|}$, we have $\|z_n\| = 1$ and, passing if necessary to a subsequence, we may assume that: $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$, $z_n \to z$ strongly in $L^2(\Omega)$ and $z_n(x) \to z(x)$ a.e. in Ω.

We consider $(\frac{g(.,u_n(.))}{\|u_n\|})$ which, by the linear growth of g, remains bounded in L^2. Thus , for a subsequence $(\frac{g(.,u_n(.))}{\|u_n\|})$ converges weakly in L^2 to some $\tilde{g} \in L^2$ and by standard arguments based on $G_0) - G_1)$, $\tilde{g}$ can be written as

$$\tilde{g}(x) = m(x)z(x)$$

where the L^∞-function m satisfy

$$\lambda_{k-1} < \nu_{k-1} \leq m(x) \leq \lambda_{k+1}. \tag{3.3}$$

Now, by (3.2), we have

$$\frac{< \Phi'(u_n), u_n >}{\|u_n\|^2} \to 1 - \int \tilde{g}(x)z(x)\, dx = 0.$$

So that, $z \not\equiv 0$. In other words , we verify easily that z satisfied

$$(I) \begin{cases} -\Delta z &= m(x)z \quad \text{in } \Omega \\ z &= 0 \quad \text{on } \partial\Omega \end{cases}$$

We now distinguish two cases : i) $m(x) < \lambda_{k+1}$ on subset of positive measure; ii) $m(x) \equiv \lambda_{k+1}$.

Case i). First, we claim that $z^k \not\equiv 0$. Assume by contradiction that $z^k \equiv 0$. Multiplying the first equation of (I) by $z^- - z^+$ and integrating over Ω, we obtain

$$\int |\nabla z^+|^2 - m(x){z^+}^2\, dx = \int |\nabla z^-|^2 - m(x){z^-}^2\, dx \tag{3.4}$$

From (3.3) and (3.4), it is obvious that

$$0 \leq \int |\nabla z^+|^2 - m(x){z^+}^2\, dx \leq (\lambda_{k-1} - \nu_{k-1}) \int {z^-}^2\, dx \leq 0.$$

This leads to

$$z^- \equiv 0 \text{ and } \int |\nabla z^+|^2 - m(x){z^+}^2\, dx = 0.$$

Define the functional $\mu : E^+ \to \mathbf{R}$ by

$$\mu(v) = \int |\nabla v|^2 - m(x)v^2\,dx = 0, \text{ for all } v \in E^+.$$

We first show that $\mu(v) = 0$ implies that $v \equiv 0$. Indeed, since $\int |\nabla v|^2 \geq \lambda_{k+1} \int |v|^2$ for $v \in E^+$, we have

$$\mu(v) \geq \int [\lambda_{k+1} - m(x)]v^2\,dx \geq 0, \text{ for all } v \in E^+.$$

Thus, if $\mu(v) = 0$ then $v = 0$ on the set $\Omega_0 = \{x \in \Omega : m(x) < \lambda_{k+1}\}$

We also get

$$0 = \mu(v) \geq \int |\nabla v|^2 - \lambda_{k+1} \int |v|^2 \geq 0.$$

Thus v is an eigenfunction for λ_{k+1}. Therefore, since $v = 0$ on a set of positive measure, the unique continuation implies that $v \equiv 0$. Therefore, we conclude that $z^+ \equiv 0$. This contradicts $z \not\equiv 0$. So that, $z^k \not\equiv 0$.

Therefore, from G_2) we obtain

$$\limsup_{n\to\infty} \int [g(x, u_n(x)) - \lambda_k u_n(x) + h(x)]\, u_n^k(x)\,dx = \infty.$$

On the other hand, we have

$$\limsup_{n\to\infty} \|\Phi'(u_n)\|\|u_n\| \geq \limsup_{n\to\infty} \int [g(x, u_n(x)) - \lambda_k u_n(x) + h(x)] u_n^k(x)\,dx| > 0.$$

this contradicts (3.2).

Case ii). If $m(x) \equiv \lambda_{k+1}$

Dividing (3.1) by $\|u_n\|^2$, then we have

$$\frac{\Phi(u_n)}{\|u_n\|^2} \to 0, \text{ as } n \to \infty.$$

Since $z_n \to z$ strongly in $H_0^1(\Omega)$, we get

$$\int \frac{G(x, u_n(x))}{\|u_n\|^2}\,dx \to \frac{1}{2} \int |\nabla z|^2\,dx$$

and using the Fatou's lemma, we also have

$$\lambda_{k+1}\int z^2 \le \int \limsup \frac{2G(x,u_n(x))}{|u_n|^2}\frac{u_n^2}{\|u_n\|^2}\,dx$$

$$\le \int_{z>0} \limsup \frac{2G(x,u_n(x))}{|u_n|^2} z^2\,dx + \int_{z<0} \limsup \frac{2G(x,u_n(x))}{|u_n|^2} z^2\,dx.$$

Therefore, we obtain

$$\int_{z>0}(\lambda_{k+1} - K_+(x))z^2\,dx + \int_{z<0}(\lambda_{k+1} - K_-(x))z^2\,dx \le 0.$$

But this gives us once more a contradiction from G_3). The proof is complete.

Lemma 3.2. : *Under hypothesis of Theorem 1.1, the functional Φ has the following properties:*

i) $\Phi(w) \to \infty$, *as* $\|w\| \to \infty$, $w \in E_+$.

ii) $\Phi(v) \to -\infty$, *as* $\|v\| \to \infty$, $v \in E_k \oplus E_-$

Proof i) The proof is by contradiction. Suppose that

$$\Phi(w_n) = \frac{1}{2}\int |\nabla w_n|^2\,dx - \int G(x,w_n) - \int h w_n\,dx \le B \tag{3.5}$$

for some constant B and some sequence $(w_n) \subset E_+$ with $\|w_n\| \to \infty$.

Let $\varepsilon > 0$, from G_0)-G_1) there exists $B_\varepsilon(x) \in L^1(\Omega)$ such that

$$G(x,s) \le \lambda_{k+1}\frac{s^2}{2} + \varepsilon s^2 + B_\varepsilon(x) \text{ a.e. in } \Omega,\ \forall\ s \in \mathbf{R}. \tag{3.6}$$

However, by (3.5) and (3.6) we get that $\|w_n\|_2 \to \infty$, as $n \to \infty$, otherwise, we would obtain

$$\|w_n\|^2 \le \lambda_{k+1}\|w_n\|_2^2 + 2\epsilon\|w_n\|_2^2 + 2\int B_\epsilon(x)\,dx + \int |hw_n|\,dx + 2B. \tag{3.7}$$

If we take $0 < \varepsilon < \frac{1}{2}$, we obtain

$$\|w_n\| \le \textit{constant}.$$

Letting $z_n = \dfrac{w_n}{\|w_n\|_2}$ and dividing (3.7) by $\|w_n\|_2^2$, we obtain in view of Poincaré inequality that

$$\|z_n\|^2 - \lambda_{k+1} \le 2\frac{\varepsilon}{\lambda_1}\|z_n\|^2 + \frac{2\int B_\epsilon(x)\,dx + 2B}{\|w_n\|_2} + \frac{\int |hz_n|\,dx}{\|w_n\|_2}.$$

As $\|w_n\|_2 \to \infty$, there exist constants $M, N > 0$ such that

$$\|z_n\|^2 - \lambda_{k+1} \le \epsilon M \|z_n\|^2 + N. \tag{3.8}$$

If we take $0 < \varepsilon < \min(\frac{1}{2}, \frac{1}{M})$, we get

$$\|z_n\| \le cte. \tag{3.9}$$

Passing to a subsequence if necessary, we obtain

$$z_n \to z \text{ weakly in } H_0^1(\Omega), z_n \to z \qquad a.e.\ on\ \Omega \text{ and in } \ L^2$$

for some $z \in H_0^1(\Omega)$ with $\|z\|_2 = 1$ (since $\|z_n\|_2 = 1$).

As $z \in E_{k+1} \oplus E_+$ we have necessarily, from (3.8) and (3.9), that z is λ_{k+1}-eigenfunction. since $w_n \in E^+$, inequality (3.5) becomes

$$\lambda_{k+1}\int w_n^2\,dx \le \int 2G(x, w_n) - 2\int hw_n\,dx + 2B$$

Dividing the above estimate by $\|w_n\|_2^2$ and using Fatou's lemma, we get

$$\lambda_{k+1}\int z^2\,dx \le \int_{z>0} K_+(x)z^2\,dx + \int_{z<0} K_-(x)z^2\,dx.$$

Hence

$$\int_{z>0}(\lambda_{k+1} - K_+)z^2\,dx + \int_{z<0}(\lambda_{k+1} - K_-(x))z^2\,dx \le 0.$$

But this yields us a contradiction.

Proof of (ii). This part of the proof is also by contradiction. Assume that there exist a constant B and a sequence $(v_n) \subset V$ with $\|v_n\| \to \infty$ such that

$$B \le \Phi(v_n) = \frac{1}{2}\int |\nabla v_n|^2\,dx - \int G(x, v_n) - \int hv_n\,dx,$$

and so

$$2B + \int 2G(x, v_n) + 2\int hv_n\,dx \leq \lambda_k \int v_n^2\,dx. \tag{3.10}$$

Set $z_n = \dfrac{v_n}{\|v_n\|}$, and passing to a subsequence if necessary, we obtain $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$, $z_n \to z$ strongly in $L^2(\Omega)$ and $z_n(x) \to z(x)$ a.e. in Ω.

Proceeding as in *ii*) and using $\lambda_k \leq L_\pm(x)$, we obtain z is a λ_k-eigenfunction. Dividing (4.0) by $\|v_n\|_2^2$ and using Fatou's lemma, one has that

$$\int_{z>0} (L_+(x) - \lambda_k) z^2\,dx + \int_{z<0} (L_-(x) - \lambda_k) z^2\,dx \leq 0.$$

This is a contradiction with assumption G_4).

Proof of theorem 1.1. In view of lemmas 3.1 and 3.2, we may apply theorem 2.1 letting $S = E_+$ and $Q = \{v \in E_- \oplus E_k : \|v\| \leq R\}$, with $R > 0$ being such that

$$\alpha = \max_{\partial Q} \Phi < \inf_{E_+} \Phi = \beta$$

It follows that the functional Φ has a critical value $c \geq \beta$ and, hence, problem (1) has a solution $u \in H_0^1$.

Proof of theorem 1.2. In the similar way of lemma 3.1 we prove that Φ satisfies the $(C)_c$ condition, for every $c \in \mathbf{R}$. In the second step, we establish that Φ has the following properties :

i) $\Phi(w) \to \infty$, as $\|w\| \to \infty, w \in E_+$,

ii) $\Phi(v) \to -\infty$, as $\|v\| \to \infty, v \in E_1$.

Let us prove the anticoercivness on Φ on E_1. Since E_1 is one-dimensional, we set $E_1 = \{t\varphi_1 \mid t \in \mathbf{R}\}$, where φ_1 is the normalized λ_1-eigenfunction (i.e. $\|\varphi_1\| = 1$). We note that φ_1 does not change sign in Ω. Letting $h(x,s) = g(x,s) + h(x)$ and $H(x,s) = \int_0^s h(x,t)\,dt$, we have for all $R > 0$,

$$\begin{aligned} \int H(x, t\varphi_1)\,dx &= \int H(x, R\varphi_1)\,dx + \int \left(\int_R^t h(x, s\varphi_1)\varphi_1\,ds \right) dx \\ &= \int H(x, R\varphi_1)\,dx + \int_R^t \frac{1}{s} \left(\int h(x, s\varphi_1) s\varphi_1\,dx \right) ds \end{aligned} \tag{3.11}$$

On the other hand, there exist $\gamma,\ R > 0$ such that

$\int[h(x,s\varphi_1)-\lambda_1 s]s\varphi_1\,dx \geq \gamma s^2$ for all $|s|\geq R$. If not, there is a sequence $s_n \in \mathbf{R}$ such that

$$\limsup_{n\to\infty}\int \frac{h(x,s_n\varphi_1)-\lambda_1 s_n\varphi_1}{s_n^2} s_n\varphi_1\,dx \leq 0.$$

This contradicts G_2). We conclude that from (15),

$$\begin{aligned}\int H(x,t\varphi_1)\,dx &\geq \int_R^t \frac{1}{s}(\gamma s^2)\,ds + \int H(x,R\varphi_1)\,dx \\ &= \frac{t^2}{2}\gamma - \frac{R^2}{2} + \int H(x,R\varphi_1)\,dx\end{aligned}$$

Hence, $\Phi(t\varphi_1) = \frac{t^2}{2} - \int G(x,t\varphi_1)\,dx - \int h(x)t\varphi_1\,dx \to -\infty$, as $|t|\to\infty$. Since $E_1 = \{t\varphi_1 \mid t\in\mathbf{R}\}$, Φ is anticoercive in E_1.

We verify easily as in i) of lemma 3.2 that Φ is coercive on E^+. Then theorem 1.2 follows from theorem 2.1. The proof is complete.

4. EXAMPLES

First, we establish the following result

Claim There exist $\Omega_1 \subset \Omega$ such that $meas(\Omega_1) > 0$ and

$$\int_{\Omega_1} zz^k\,dx < \int_{\Omega\setminus\Omega_1} zz^k\,dx, \qquad \forall z\in H_0^1(\Omega), \|z^k\| = 1.$$

If not, for every sequence (Ω_n) such that $meas(\Omega_n) > 0$ there exist $(z_n) \subset H_0^1(\Omega)$, with $\|z_n^k\| = 1$ and

$$\int_{\Omega_n} z_n z_n^k\,dx \geq \int_{\Omega\setminus\Omega_n} z_n z_n^k\,dx = \int_\Omega (z_n^k)^2\,dx - \int_{\Omega_n} z_n z_n^k\,dx. \tag{4.1}$$

From a sequence (Ω_n) satisfying

$$\Omega_{n+1}\subset\Omega_n, \quad meas(\Omega_n) = \frac{1}{n}, \forall n\geq 1.$$

Thus, we have

$$\chi_{\Omega_n} \to 0 \text{ in } L^\infty.$$

On the other hand, there exists $z \in H_0^1(\Omega)$ such that $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$, $z_n \to z$ strongly in $L^2(\Omega)$ and $z_n^k \to z^k$ strongly in E_k.

From (4.2), we obtain

$$\int_{\Omega} z^{k^2}\, dx \le 0,$$

and hence $z^k \equiv 0$.This a contradiction, since $\|z^k\| = 1$.

Example 1: Consider two-point boundary value problem

$$\begin{cases} -u" &= g(x,u) + h(x) \quad 0 < x < \pi \\ u(0) &= u(\pi) = 0 \end{cases}$$

where $h \in L^2(0,\pi)$. Let g the continuous function defined by

$$g(s) = \begin{cases} s(k^2 + 2(k-1)\sin(s)) + \frac{3}{2}s(1+\sin(s)) & \text{if} \\ s \ge 1 & \\ as + b & \text{if} \\ -1 \le s \le 1 & \\ s\sin(\ln(1-ks)) - \frac{s^2}{2}\cos(\ln(1-ks))\frac{1}{1-ks} + (k^2+k)s & \text{if} \\ s \le -1 & \end{cases}$$

A simple computation of the primitive $G(s) = \int_0^s g(t)\,dt$ gives

$$G(s) = \begin{cases} (k^2 + \frac{3}{2})\frac{s^2}{2} - (k - \frac{1}{2})[s\cos s - \sin s] & \text{if} \quad s \ge 1 \\ a\frac{s^2}{2} + bs & \text{if} \quad -1 \le s \le 1 \\ k\frac{s^2}{2}\sin(\ln(1-ks)) + \frac{k^2+k}{2}s^2 & \text{if} \quad s \le -1 \end{cases}$$

Let, $k \ge 2$ and

$$g(x,s) = \begin{cases} g(s) & a.e. \quad x \in \Omega_1, \forall s \in \mathbf{R} \\ (k^2 + 2k - 2)s & a.e. \quad x \in \Omega \setminus \Omega_1, \forall s \in \mathbf{R} \end{cases}$$

Set $h(x,s) = g(x,s) - k^2 s$. For every $(u_n) \subset H_0^1(0,\pi)$ is such that $\frac{u_n}{\|u_n\|} \rightharpoonup z \not\equiv 0$ $\frac{u_n^k}{\|u_n\|} \to z^k \not\equiv 0$ as $n \to \infty$, and since $\liminf\limits_{|s|\to\infty} \frac{g(x,s)}{s} \ge (k-1)^2 + 1$ the dominated convergence theorem can be used to show that

$$\limsup_{n\to\pm\infty}\int h(x,u_n(x))\frac{u_n^k(x)}{\|u_n\|^2}\,dx$$

$$\geq\limsup_{n\to\pm\infty}\int_{\Omega_1}(2-2k)\frac{u_nu_n^1}{\|u_n\|^2}+\int_{\Omega\setminus\Omega_1}(2k-2)\frac{u_nu_n^1}{\|u_n\|^2}$$

$$=\left\|z^k\right\|^2(2k-2)\left[\int_{\Omega\setminus\Omega_1}\frac{z}{\|z^k\|}\frac{z^k}{\|z^k\|}-\int_{\Omega_1}\frac{z}{\|z^k\|}\frac{z^k}{\|z^k\|}\right]$$

$$>0$$

and thus condition $G_2)$ is satisfied. It is clear that $g(x,.)$ and $G(x,.)$ satisfies

$\liminf\limits_{s\to\pm\infty}\frac{g(x,s)}{s}=(k-1)^2+1,\liminf\limits_{s\to\pm\infty}\frac{2G(x,s)}{s^2}$
$=k^2+\frac{3}{2},\limsup\limits_{s\to\pm\infty}\frac{g(x,s)}{s}=(k+1)^2$

$\liminf\limits_{s\to\pm\infty}\frac{g(x,s)}{s}=k^2-1,\liminf\limits_{s\to\pm\infty}\frac{2G(x,s)}{s^2}=k^2,\limsup\limits_{s\to\pm\infty}\frac{2G(x,s)}{s^2}$
$=k^2+2k-2,\limsup\limits_{s\to\pm\infty}\frac{g(x,s)}{s}$

$=(k+1)^2.$

Theorem 1.1 thus implies that problem (1) has at least one solution for any $h\in L^2(0,\pi)$.

Example 2: Consider

$$g(x,s)=\begin{cases}as & a.e.\quad x\in\Omega_1,\forall s\in\mathbf{R}\\(2\lambda_1-a)s & a.e.\quad x\in\Omega\setminus\Omega_1,\forall s\in\mathbf{R}\end{cases}$$

with $2\lambda_1-\lambda_2<a<\lambda_1$, and put $h(x,s)=g(x,s)-\lambda_1 s$.

For every $(u_n)\subset H_0^1(\Omega)$ is such that $\dfrac{u_n}{\|u_n\|}\rightharpoonup z\not\equiv 0$ $\dfrac{u_n^1}{\|u_n\|}\to z^1\not\equiv 0$ as $n\to\infty$, then

$$\limsup_{n\to\pm\infty} \int h(x, u_n(x)) \frac{u_n^1(x)}{\|u_n\|^2}\, dx$$

$$= \limsup_{n\to\infty} \int_{\Omega_1} (a-\lambda_1) \frac{u_n u_n^1}{\|u_n\|^2} + \int_{\Omega\backslash\Omega_1} (\lambda_1 - a) \frac{u_n u_n^1}{\|u_n\|^2}$$

$$= (\lambda_1 - a) \left[\int_{\Omega\backslash\Omega_1} zz^1 dx - \int_{\Omega_1} zz^1 dx\right]$$

$$> 0$$

and thus condition G_2) is satisfied. It is clear that g and G satisfies

$$\liminf_{s\to\pm\infty} \frac{g(x,s)}{s} = a, \liminf_{s\to\pm\infty} \frac{2G(x,s)}{s^2} = a, \limsup_{s\to\pm\infty} \frac{g(x,s)}{s} = 2\lambda_1 - a$$

Theorem 1.2 thus implies that problem (1) has at least one solution for any $h \in L^2$.

Note that these examples is not covered by the results in (3.2) and (3.5).

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