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CLASSES OF FORMS WITT EQUIVALENT TO A SECOND TRACE FORM OVER FIELDS OF CHARACTERISTIC TWO*

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Abstract

Let F be a field of characteristic two. We determine all non-hyperbolic quadratic forms over F that are Witt equivalent to a second trace form.

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1. Introduction

Let E/F be a finite separable field extension. We define the trace form for this extension by $q(x) = tr_{E/F}(x^2)$. When the characteristic of F is not equal to 2, the trace form (E, q) is non-degenerate. However, if the characteristic is 2 then (E, q) is degenerate and splits as $[1] \perp V$, with V totally isotropic. It is therefore natural to introduce a modified "second trace form". To this end one considers for each $a \in E$ its characteristic polynomial

$$p(x, a) = x^n - T_1(a)x^{n-1} + T_2(a)x^{n-2} + \cdots + (-1)^n T_n(a) \tag{1.1}$$

(whence $T_1(a) = tr_{E/F}(a)$ and $T_n(a)$ is the norm of a). It is clear that (E, T_2) is a quadratic form. When the degree n of the extension is odd this form is necessarily singular. To arrive at a non-degenerate form, two methods have been proposed in the literature. One method, due to Bergé and Martinet [BM], increases the dimension of the space by 1 using the étale F-algebra. The other method, due to Revoy [R], reduces the dimension of the space by 1. In this note we will adopt the second method and call such forms *2-trace forms*.

We consider the following problem: Which elements $[q] \neq 0$ of the Witt-group $W_q(F)$ are represented by 2-trace forms?. Our theorem 3 fully answers this question. Moreover, we will partially answer the same question for $[q] = 0$ (see Prop. 1 and Prop. 2). For fields of characteristic not equal to 2 this problem seems quite more complicated: for partial results concerning generic fields one may consult [CP] and [EHP]; a complete solution for Hilbertian fields is given in [Sch], [KS] and [Wat].

2. The second trace form

As we remarked in the introduction, there are two ways to define a second trace form. In this section we will prove that in fact the corresponding forms are Witt equivalent.

Let E/F be a finite separable field extension. The second trace form $T_{E/F}$ of the extension E/F was defined by Revoy [R] as (E, T_2) if the degree $[E : F]$ is even, and as (E_0, T_2) if the degree is odd, where T_1, T_2 are given by (1.1) and $E_0 = Ker\ T_1$. It is important to remark that the bilinear form b_q associated to $T_{E/F}$ satisfies the following relations:

$$b_q(x, y) = T_2(x + y) - T_2(x) - T_2(y) = T_1(xy) - T_1(x)T_1(y), \tag{2.1}$$

(2.2) $$T_1(x^2) = (T_1(x))^2 \quad \text{and} \quad b_q(x^2, y^2) = b_q(x, y)^2$$

On the other hand, Bergé and Martinet defined in [BM] the second trace form as Revoy if the degree $[E : F]$ is even and as $(E \times F, T_2)$ if not.

Theorem 1. *The Revoy form and the Bergé-Martinet form are Witt equivalent.*

Proof. If $[E : F]$ is odd, then the form $(E \times F, T_2)$ of Bergé and Martinet [BM, p. 14] splits as follows $(F(1,0) + F(0,1)) \perp E_0 \times \{0\}$. Since $F(1,0) + F(0,1)$ is an hyperbolic plane, the claim follows immediately. □

3. 2-algebraic forms

In this section we determine all non-hyperbolic quadratic forms over F that are Witt equivalent to some second trace form. Furthermore we give fields where hyperbolic forms are Witt equivalent to a second trace form.

Theorem 2. *Let E/F be a finite separable field extension, with $[E : F] = 2n + 1$ or $[E : F] = 2n$. Then $T_{E/F} = (n-1)\mathbf{H} \perp [1, a]$, for some $a \in F$.*

Proof. The assertion is deduced from Theorem 1 above and Theorem 3.5 in [BM, p. 13-14]. In order to illustrate the ideas, we will give the proof in case $[E : F] = 2n + 1$.

Since (E_0, T_2) is nonsingular, there exists a symplectic basis $\{e_i, f_i\}_{1 \le i \le n}$ of E_0. Hence for each $1 \le i \le n$ there exists $x_i,\ y_i \in Fe_i + Ff_i$ such that $T_2(x_i) \neq 0$ and $b_q(x_i, y_i) = 1$ (note that $[0,0] = [1,0]$ [Sa1, p. 150]).

Put $e_i' := T_2(x_i)^{-1} x_i^2$ and $f_i' := T_2(x_i) y_i^2$. We have by (2.2) that $e_i',\ f_i' \in E_0$ and $b_q(e_i', f_i') = 1$. Since $T_2(e_i') = 1$, we obtain a symplectic basis such that the quadratic form decomposes as follows

$$T_{E/F} = [1, a_1] + [1, a_2] + \cdots + [1, a_n],$$

where $a_i = T_2(f_i')$. Using the relation $[1, b] + [1, c] \cong [1, b + c] + \mathbf{H}$ over F, we obtain $T_{E/F} \cong (n-1)\mathbf{H} + [1, a]$ with $a \equiv a_1 + a_2 + \cdots a_n \ \wp(F)$, where $\wp(F) := \{x^2 + x \mid x \in F\}$. □

A form T over F is called 2-algebraic if it is Witt equivalent to some second trace form.

Corollary 1. *A non hyperbolic quadratic form (V,q) over F is 2-algebraic if and only if $(V,q) = r\mathbf{H} \perp (V_a, q_a)$, with V_a an anisotropic plane representing 1.*

Proof: $\Rightarrow$) Is clear by Theorem 2.

$\Leftarrow$) Let $(V,q) = r\mathbf{H} \perp (V_a, q_a)$, with (V_a, q_a) 2-dimensional anisotropic representing 1. Then we can rewrite $(V_a, q_a) = [1, b]$, where $b \notin \wp(F)$. Let $E = F(\alpha)$ with $\alpha \in \overline{F}$ and $\alpha^2 + \alpha + b = 0$. We have that $(E, T_{E/F}) = [1, b]$. □

Example 1. *Let $F = \mathbf{F}_2(a)$ and $E = F(b)$, where $a^2 + a + 1 = 0$ and $b^3 + b + a = 0$. Then $T_{E/F} = [1, a] \neq [1, 1]$.*

In fact, using (2.1), we see that $\{b^2, (1+a)b\}$ is a symplectic basis for $E_0 = Ker\ T_1$. Since $p(x, (1+a)b) = x^3 + ax + a$, $1 \in \wp(F)$ and $a \notin \wp(F)$, we obtain the form $(E_0, T_2) = [1, a] \neq [1, 1]$.

Corollary 2. *If a non hyperbolic quadratic form (V,q) over F is 2-algebraic then there exists a quadratic extension field E of F such that the extension $(V \otimes_F E, q_E)$ is hyperbolic.*

Proof: See the proof of Theorem3 and note that $[1, b] = \mathbf{H}$ over $E = F(\alpha)$, with $\alpha^2 + \alpha + b = 0$ (see [Sa1, p. 150]) . □

Theorem 3. *Let $F = \mathbf{F}_2$ or $F = \mathbf{F}_2(t)$ with t transcendental over $\mathbf{F}_2$. Then the hyperbolic quadratics form over F are 2-algebraic.*

Proof. We only need to find an extension E of F such that $T_{E/F}$ is hyperbolic. We first remark that $p(x) := x^4 + x^3 + 1$ is irreducible over $\mathbf{F}_2$ and also over $\mathbf{F}_2(t)$. Let α be a root of p and $E = F(\alpha)$. We decompose the trace form $T_{E/F}$ with respect to the basis $\{\alpha,\ 1+\alpha^3\} \cup \{\alpha^2, \alpha + \alpha^2 + \alpha^3\}$. Noting that this basis has the elements conjugate to α, it is easy to recognise that each vector basis is isotropic, and furthermore by (2.1) we see that it is a symplectic basis. Hence, the space is hyperbolic. □

Theorem 4. *Let F be a field. If there exists $a \in F^*$ and n odd such that the polynomial $x^n - a$ is irreducible over $F[x]$, then the hyperbolic quadratics space over F are 2-algebraic.*

Proof. Let $E = F(\alpha)$, where $\alpha \in \overline{F}$ and $\alpha^n = a$. For $1 \leq k \leq n-1$, the linear transformation $f_{\alpha^k} : x \mapsto x\alpha^k$ is given by the matrix $c_{ij}(k)$, where

$$c_{ij}(k) = \begin{cases} 1 & \text{if } j = i - k \\ a & \text{if } j = n + i - k \\ 0 & \text{otherwise} \end{cases}$$

Then for $1 \leq k \leq n-1$, $\alpha^k \in E_0$, because $c_{ii}(k) = 0$ for each i. Noting that $T_1(a) = a$ we obtain the decomposition

$$E_0 = \langle \alpha, \alpha^{n-1} \rangle \perp \langle \alpha^2, \alpha^{n-2} \rangle \perp \cdots \perp \langle \alpha^{\frac{n-1}{2}}, \alpha^{\frac{n+1}{2}} \rangle,$$

where $\langle x, y \rangle$ is the space generate by x and y. Hence, using that $n \neq 2k$, we deduce that $T_2(\alpha^k) = 0$ for $1 \leq k \leq n-1$,so $(E_0, T_2) = (\frac{n-1}{2})\mathbf{H}$. $\square$

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