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SOME RESULTS ON PRIME AND PRIMARY SUBMODULES

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Abstract

This article is devoted to study some properties of prime and primary submodules. First we characterize prime submodules of free modules and give a primality condition for certain submodules in terms of associated prime ideals. Furthermore, by using symmetric algebra of modules we describe Rees algebras associated to prime submodules and provide a computational method to check if some primary submodules of a free module have prime radical.

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1. INTRODUCTION

To extend the concepts of prime and primary ideal from the category of rings to the category of modules has stimulated several authors to show that many, but not all, of the results in the theory of rings are also valid for modules. (See, for example, Refs. [1],[5],[7] .)

Before we state some results let us introduce some notation and terminology. Throughout this note all rings are commutative with identity and all modules are unital. Let R be a ring and let M be an R-module. A proper submodule N of M is said to be prime (resp. primary) if for every $a \in R$, the induced homothety $h_a : M/N \to M/N$, $h_a(\overline{n}) = a{\cdot}\overline{n}$, is either injective or null (resp. nilpotent). In light of this definition, it turns out that if N is a prime (resp. primary) submodule of M then the set of homotheties of R vanishing on M/N, i.e., $(N : M) = \{a \in R \ / \ aM \subseteq N\} = Ann(M/N)$ is a prime (resp. primary) ideal of R. Furthermore, if N is a primary submodule of M, the radical of the primary ideal $(N : M)$, denoted by $\sqrt{(N : M)}$, is a prime ideal of R formed by all nilpotent homotheties of R on M/N, i.e., $\sqrt{(N : M)} = \{a \in R \ / \ a^n M \subseteq N \text{ for some } n > 0\}$. Thus if N is a prime submodule of M with $p = (N : M)$ we shall call N p-prime submodule and if N is a primary submodule of M being $p = \sqrt{(N : M)}$ we will say that N is a p-primary submodule. Note that a p-primary submodule N of M is p -prime if and only if $(N : M) = p \in SpecR$.

It is easily seen that a submodule N of M is called to be p -prime (resp. p-primary) if $N \neq M$ and, given $r \in R, m \in M$, then $rm \in N$ implies $m \in N$ or $r \in p = (N : M)$ (resp. $r \in p = \sqrt{(N : M)}$). A special class of prime submodules of a module over a domain is given by the ones whose prime ideal in R is the ideal (0). According to [2, Definition 2.1], these prime submodules are called $0-$submodules and it can be proved that a proper submodule N of a module M over a domain R is a $0-$submodule if and only if M/N is a torsion-free module, or equivalently if and only if for every non-zero $a \in R$, $n \in M$, the relation $a \cdot n \in N$ implies that $n \in N$.

For any submodule N of an R-module M the radical, $rad_M(N)$, of N is defined to be the intersection of all prime submodules of M containing N and $rad_M(N) = M$ if N is not contained in any prime submodule of M. The radical of the module M is defined to be $rad_M(0)$. The set of all prime submodules of M is called the Spectrum of M and denoted by $Spec(M)$.

The purpose of this paper is to explore some basic facts of these class of submodules. First we turn our attention to prime submodules of free modules of finite rank. Assume that R is a domain and let K be its quotient field. Let N be a submodule of the free R-module $F = R^n$. In [5] and [1], based upon the relationship between KN and KF, several results concerning prime submodules of free modules were obtained. By applying an useful relationship between $Spec(M)$ and $Spec(M_S)$ we provide a characterization of prime submodules of free modules in order to show how is their structure. In section 2 we prove a primality condition for some submodules of a finitely generated module in terms of associated prime ideals of the corresponding quotient module.

In the next sections we apply our study of prime submodules, begun in [3], to investigate two interesting questions: on the one hand, given a prime submodule of a finitely generated module we describe the Rees algebra of the corresponding quotient module and, on the other, we give a simple condition for detecting whether a primary submodule of a finitely generated module has prime radical and it is then used to determine, by using the computer algebra system Macaulay2, when some primary submodules of free modules have prime radical.

2. A CHARACTERIZATION OF PRIME SUBMODULES OF FREE MODULES

This section is devoted to describing prime submodules of free modules and to do so we first need to introduce some notation.

Given an R-module M and a multiplicative set S, let us consider the canonical map $f : M \longrightarrow M_S$. If N is a submodule of M then the submodule $f^{-1}(N_S)$ is denoted by $\mathcal{S}\,(N)$ and called the saturation of N with respect to S. Clearly $N \subset \mathcal{S}\,(N)$ and $\mathcal{S}\,(N) = \{m \in M : sm \in N \text{ for some } s \in S\}$. Note that if N is a submodule of M such that $(N : M) = p \in Spec(R)$ then the saturation $\mathcal{S}_p(N)$ of N with respect to p, i.e., $S = R \setminus p$, is a p-prime submodule of M.

Theorem 1. Let R be a ring, let F be a free R–module of finite rank and let M be a submodule of F. Then M is p-prime if and only if $M = \mathcal{S}(N + pF)$, where N is the inverse image of a direct summand of F_p by the canonical map $F \to F_p$.

Proof. Clearly $\mathcal{S}(N + pF)$ is a p-prime submodule of F (See [6, Lemma 1.7]). Conversely, assume that M is a p-prime submodule of F. Thus we have $pF \subseteq M$. Let us consider the following composition of morphisms:

$$F \xrightarrow{f} F_p \xrightarrow{g} \frac{F_p}{pF_p}$$

where as above f stands for the canonical map. Choose a basis $\{n_1, ..., n_s\}$ of $g(M_p)$ and a set of elements $m_1, ..., m_s \in M_p \subseteq F_p$ such that $g(m_i) = n_i$ for $1 \leq i \leq s$. Then it is esay to check that $M_p = (m_1, ..., m_s) + pF_p$. Because $pF_p \subseteq M_p$ we have an exact sequence

$$0 \longrightarrow pF_p \longrightarrow M_p \longrightarrow M_p/pF_p \longrightarrow 0.$$

Now by applying Nakayama's lemma it follows that $(m_1, ..., m_s)$ must be a direct summnad of F_p. If we set $N = f^{-1}((m_1, ..., m_s))$, it is clear that $M \subseteq N + F \subseteq \mathcal{S}(N + pF) \in Spec(F)$. Finally, by applying [1, Corollary 3] there exists a bijective correspondence between the p-prime submodules of F and those F_p. Since $M_p = N_p + pF_p = \mathcal{S}(N + pF)_p$, it is obtained, undoubtedly, that $M = \mathcal{S}(N + pF)$ and the theorem is proved. □

3. A PRIMALITY CONDITION FOR SOME SUBMODULES IN TERMS OF ASSOCIATED PRIME IDEALS

Let R be a Noetherian ring and M a finitely generated R-module. Let $Ass(M)$ be the set of associated prime ideals of M, that is

$$Ass(M) = \{p \in Spec(R) \ / \ p = Ann(m), \ m \in M \setminus \{0\}\}$$

As it is well-known a submodule $Q \subset M$ is called p-primary if $Ass(M/Q)$ consists of the p ideal only (See[4, Theorem 6.6]). The aim of this section is to obtain an analogue result for some prime submodules.

Theorem 2. Let R be a Noetherian ring, P a prime ideal of R and let M be a finitely generated R-module. Assume that N is a submodule of M such that $PM \subset N$. Then N is a p-prime submodule of M if and only if $Ass(M/N) = p$

Proof. Since every prime submodule is also primary the sufficient condition is clear. Assume that $Ass(M/N)$ consists of a unique ideal p. Let $b \in R$ an element which induces a non injective homothety on M/N. This implies that there exists an element $m \in M$ such that $bm = 0$. Let now q be a minimal prime ideal among all prime ideals containing $Ann(m)$. Since $Supp(m) = \{p \in Spec(R) \ / \ m_p \neq 0\}$ coincides with the closed subset $V(Ann(m)) = \{p \in Spec(R) \ / \ p \supset Ann(m)\}$ it follows that q is also a minimal prime ideal of the set $Supp(m)$. From [4, Theorem 6.5] q is an associated prime ideal to (m). By definition there exists an element $m' \in (m)$ such that $(m') \simeq R/q$. This implies that q is an associated prime ideal of

M/N, so that $q = p$. Hence $b \in p$ and since $pM \subset N$ it follows that b induces a vanishing homothety on M/N. Therefore, N is a prime submodule of M.

4. REES ALGEBRAS ASSOCIATED TO PRIME SUBMODULES

Throughout this section we deal with the definition of the Rees algebra of a module suggested by W. Vasconcelos in [8, Pag. 3]. As it is well-known, for an ideal I of a commutative ring R there is a canonical epimorphism

$$\phi : S(I) \longrightarrow \Re(I) = R[It]$$

between the symmetric algebra of I and its Rees algebra. If, further, the ring R is a domain, then the Kernel of ϕ is just the R-torsion submodule of $S(I)$. This suggests the definition of the Rees algebra $\Re(M)$ of an R-module M as $S(M)/T$, where T is the prime ideal of the R-torsion elements of $S(M)$.

Given a prime submodule of a finitely generated module, the aim of this section is to describe the Rees algebra of the corresponding quotient module but to do so we need some results obtained in [3, Section 2]. Let M be a finitely generated $R-$module. Then to each $p-$prime submodule N of M, we associate a prime ideal of the symmetric algebra of M, called the *expansion* $\mathcal{E}_N$ of N and defined to be the set of all elements $b \in S(M)$ for which there exists an $a \in R$, $a \notin p$ such that $a \cdot b \in (p, N) \cdot S(M)$.

Let N be a p-prime submodule of a finitely generated $R-$module M. The following theorem describes the Rees algebra of the quotient module M/N over R/p.

Theorem 3. Letting $\overline{R}$ denote $\overline{R} = R/p$, then $\Re_{\overline{R}}(M/N) \simeq S_R(M)/\mathcal{E}_N$.

Proof. As it is well-known, the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

yields to another one

$$0 \longrightarrow N \cdot S_R(M) \longrightarrow S_R(M) \longrightarrow S_R(M/N) \longrightarrow 0.$$

Hence, taking into account the natural map

$$S_R(M/N) \longrightarrow S_{\overline{R}}(M/N),$$

it follows the exact sequence

$$0 \longrightarrow (p,N)\cdot S_R(M) \longrightarrow S_R(M) \xrightarrow{\pi} S_{\overline{R}}(M/N) \longrightarrow 0$$

Now it turns out that the $\overline{R}-$ torsion elements of $S_{\overline{R}}(M/N)$ are just the image of the expansion $\mathcal{E}_N$ of N by the map π, which will be denoted by $\overline{\mathcal{E}}_N$. Indeed, if $\overline{m}$ is a $\overline{R}-$ torsion element of $S_{\overline{R}}(M/N)$, there exists an $\overline{a}\in\overline{R}$ such that $\overline{a}\cdot\overline{m}=\overline{0}$. But this implies that $a\cdot m\in(p,N)\cdot S_R(M)$, so that $m\in\mathcal{E}_N$.

Conversely, if $\overline{m}\in\overline{\mathcal{E}_N}$ we choose an inverse image $m\in E_N$ by π. Then by definition there exists an $a\in R-p$ such that $a\cdot m\in(p,N)\cdot S_R(M)$. Therefore, $\overline{a\cdot m}=\overline{0}$ en $S_{\overline{R}}(M/N)$, that is, $\overline{m}$ is an $\overline{R}$-torsion element of $S_{\overline{R}}(M/N)$.

Finally, since $(p,N)\cdot S_R(M)\subseteq\mathcal{E}_N$, it follows that $\Re_{\overline{R}}(M/N)\simeq S_R(M)/\mathcal{E}_N$and the theorem is proved.

5. A CONDITION FOR DETECTING WHEN A PRIMARY SUBMODULE HAS PRIME RADICAL

Contrary to what happens in rings, the radical of a primary submodule is not, in general, a prime submodule (See [7, Example 1.12]). So the aim of this section is to give a condition which allows us to determine whether primary submodules of a module have prime radical.

Proposition 1. *Let R be a Noetherian ring, let M be an R-module and let Q be a p-primary submodule of M. Then $rad_M(Q)$ is a p -prime submodule of M if and only if $\mathcal{S}_p(rad_M(Q))=rad_M(Q)$*

Proof. One direction is obvious. Assume now that $\mathcal{S}_p(rad_M(Q))=rad_M(Q)$. Since $(rad_M(Q))_p=rad_{M_p}(Q_p)$ it follows that $(rad_M(Q))_p\in Spec(M)$ (See [7, Lemma1.4]). Hence by applying [1, Corollary 3] and taking into account that $(\mathcal{S}_p(rad_M(Q)))_p=(rad_M(Q))_p$ it must be $\mathcal{S}_p(rad_M(Q))\in Spec(M)$. □

The importance of this condition is given by the fact that since we know to compute some radicals of submodules of free modules (See [3, Section 3, 3.2]) we also are able to determine if $\mathcal{S}_p(rad_M(Q))=rad_M(Q)$ by using the computer algebraic system Macaulay2. Therefore we can decide if some radicals of submodules of free modules have prime radical. To illustrate this we present the following example:

Example 1. *Let* $R = (\mathbf{Z}/101\mathbf{Z})[x,y]$ *and let* $F = R \oplus R$ *a free* $R-$*module with basis* $\{e_1, e_2\}$. *Let* Q *be the submodule of* F *defined by* $Q = \langle xe_1 + y^2e_2, xe_2, x^2e_1\rangle$. *It can easily be checked that* $Q = \{(u,v) \in F : y^2u - xv \in p^2\}$ *where* $p = Rx$. *By applying [7, Proposition 1.2] it follows that* Q *is a* p*-primary submodule of* F. *According [3, Section 3.2], to compute* $rad_F(Q)$ *we shall replace* $\{e_1, e_2\}$ *by* $\{z, t\}$ *and consider the ideal* $J = \langle x_z + y^2t, xt, x^2z\rangle$. *By using computer algebra system we obtain that*

$\sqrt{J} = \langle yt, xz, yzt, xyz, y^2t + xz, xt, x^2z\rangle$. *Thus taking into account* $\sqrt{J}(1)$ *we have that* $rad_F(Q) = \langle ye_2, xe_1, xye_1, y^2e_2, xe_1, xe_2, x^2e_1\rangle$. *On the other hand, by using Macaulay2 we obtain that* $(rad_F(Q) :_M y) = \langle e_1, xe_2\rangle \neq rad_F(Q)$, *so that* Q *has not prime radical.*

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