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ORLICZ - PETTIS THEOREMS FOR MULTIPLIER CONVERGENT OPERATOR VALUED SERIES

CHARLES SWARTZ
New State University, Las Cruces - USA

Abstract

Let X, Y be locally convex spaces and $L(X, Y)$ the space of continuous linear operators from X into Y . We consider 2 types of multiplier convergent theorems for a series $\sum T_k$ in $L(X, Y)$. First, if λ is a scalar sequence space, we say that the series $\sum T_k$ is λ multiplier convergent for a locally convex topology τ on $L(X, Y)$ if the series $\sum t_k T_k$ is τ convergent for every $t = \{t_k\} \in \lambda$. We establish conditions on λ which guarantee that a λ multiplier convergent series in the weak or strong operator topology is λ multiplier convergent in the topology of uniform convergence on the bounded subsets of X . Second, we consider vector valued multipliers. If E is a sequence space of X valued sequences, the series $\sum T_k$ is E multiplier convergent in a locally convex topology η on Y if the series $\sum T_k x_k$ is η convergent for every $x = \{x_k\} \in E$. We consider a gliding hump property on E which guarantees that a series $\sum T_k$ which is E multiplier convergent for the weak topology of Y is E multiplier convergent for the strong topology of Y .

1. INTRODUCTION

The original Orlicz-Pettis Theorem which asserts that a series $\sum x_k$ in a normed space X which is subseries convergent in the weak topology of X is actually subseries convergent in the norm topology of X can be interpreted as a theorem about multiplier convergent series. Let λ be a scalar sequence space which contains the space of all sequences which are eventually 0 and let (E, τ) be a Hausdorff topological vector space. The series $\sum x_k$ in E is λ multiplier convergent with respect to τ if the series $\sum t_k x_k$ is τ convergent for every $t = \{t_k\} \in \lambda$; the elements of λ are called multipliers. Multiplier convergent series where the multipliers come from some of the classical sequence spaces, such as l^p , have been considered by various authors ([B],[FP]); in particular, c_0 multiplier convergent series have been used to characterize Banach spaces which contain no copy of c_0 ([D]). Thus, a series $\sum x_k$ is subseries convergent if and only if $\sum x_k$ is m_0 multiplier convergent, where m_0 is the sequence space of all scalar sequences which have finite range. Several Orlicz-Pettis Theorems have been established for multiplier convergent series where the multipliers are from various classical sequence spaces ([LCC], [SS],[WL]). In this note we consider Orlicz-Pettis Theorems for multiplier convergent series of linear operators where the multipliers are both scalar and vector valued. In general, as Example 2.1 below shows, a series of operators which is m_0 multiplier convergent in the weak operator topology is not m_0 multiplier convergent in the topology of uniform convergence on bounded subsets; however, if the space of multipliers satisfies certain conditions an Orlicz-Pettis result of this type does hold. One of our results contains a gliding hump assumption on the multiplier space which is of independent interest and also yields an Orlicz-Pettis result for series in the strong topology of a general locally convex space.

We begin by fixing the notation and terminology which we use. Let X, Y be real Hausdorff locally convex spaces and let $L(X, Y)$ be the space of continuous linear operators from X into Y . If $x \in X, y' \in Y'$, let $x \otimes y'$ be the linear functional on $L(X, Y)$ defined by $\langle x \otimes y', T \rangle = \langle y', Tx \rangle$ and let $X \otimes Y'$ be the linear subspace spanned by $\{x \otimes y' : x \in X, y' \in Y'\}$. The weak operator topology on $L(X, Y)$ is the weak topology from the duality between $L(X, Y)$ and $X \otimes Y'$ ([DS]VI.1). The strong operator topology on $L(X, Y)$ is the topology of pointwise convergence on X ([DS]VI.1). Let $L_b(X, Y)$ be $L(X, Y)$ with the topology of uniform convergence on the bounded subsets of X ; the topology of $L_b(X, Y)$ is generated by the seminorms $p_A(T) = \sup\{p(Tx) : x \in A\}$, where p is a continuous semi-norm on

Y and A is a bounded subset of X ([DS],VI.1).

2. SCALAR MULTIPLIERS

In this section we establish Orlicz-Pettis type theorems for multiplier convergent series of operators with respect to the weak operator topology and the topology of $L_b(X, Y)$. As the following example illustrates, an operator valued series which is subseries or m_0 multiplier convergent in the strong (weak) operator topology needn't be m_0 multiplier convergent in $L_b(X, Y)$.

Example 1. Define $T_k : l^1 \rightarrow l^1$ by $T_k t = \langle e^k, t \rangle = t_k e^k$, where e^k is the sequence with 1 in the k th coordinate and 0 in the other coordinates.. Since for every subsequence $\{n_k\}$ the series $\sum_{k=1}^{\infty} T_{n_k} t = \sum_{k=1}^{\infty} t_{n_k} e^{n_k}$ converges in l^1 for every $t \in L^1$, the series $\sum T_k$ is m_0 multiplier convergent in the strong operator topology. But, $\|T_k\| = 1$ so $\sum T_k$ is not m_0 multiplier convergent in the operator norm.

However, as the following two theorems show, if the space of multipliers λ satisfies some additional conditions, an operator valued series which is λ multiplier convergent in the weak(strong) operator topology can indeed be λ multiplier convergent in $L_b(X, Y)$.

The sequence space λ is an AK space if λ has a Hausdorff locally convex topology such that the coordinate maps $t = \{t_k\} \rightarrow t_k$ from λ to \mathbf{R} are continuous and each t has a series expansion $t = \sum_{k=1}^{\infty} t_k e^k$ which converges in the topology of λ ([BL]).

Theorem 2. If λ is a barrelled AK space and $\sum T_k$ is λ multiplier convergent in the weak operator topology of $L(X, Y)$, then $\sum T_k$ is λ multiplier convergent in $L_b(X, Y)$.

Proof: From Corollary 2.4 of [SS] it follows that $\sum T_k$ is λ multiplier convergent in the strong topology $\beta(L(X, Y), X \otimes Y')$. Thus, it suffices to show that if a sequence(net) $\{S_k\}$ in $L(X, Y)$ converges in $\beta(L(X, Y), X \otimes Y')$, then $\{S_k\}$ converges in $L_b(X, Y)$; that is, the strong topology $\beta(L(X, Y), X \otimes Y')$ is stronger than the topology of $L_b(X, Y)$.

Let $A \subset X$ be bounded, $B \subset Y'$ be equicontinuous and let $C = \{x \otimes y' : x \in A, y' \in B\}$. It is easily checked that C is $\sigma(X \otimes Y', L(X, Y))$ bounded so $\sup\{|\langle x \otimes y', S_k \rangle| : x \in A, y' \in B\} \rightarrow 0$ which implies that $S_k x \rightarrow 0$ uniformly for $x \in A$ or $S_k \rightarrow 0$ in $L_b(X, Y)$.

We next consider an analogue of Theorem 2 for other spaces of multipliers which are described by purely algebraic conditions in contrast to the AK and barrelledness assumptions of Theorem 2.

Let E be a vector space of X valued sequences which contains the space of all X valued sequences which are eventually 0. If $t = \{t_j\}$ is a scalar sequence and $x = \{x_j\}$ is either a scalar or X valued sequence, $tx = \{t_j x_j\}$ will be the coordinatewise product of t and x . A sequence of interval $\{I_j\}$ in \mathbf{N} is increasing if $\max I_j < \min I_{j+1}$ for all j ; if I is an interval, then C_I will be the characteristic function of I . The space E has the *infinite gliding hump property* (∞ -GHP) if whenever $x \in E$ and $\{I_k\}$ is an increasing sequence of intervals, there exist a subsequence $\{n_k\}$ and $a_{n_k} > 0, a_{n_k} \rightarrow \infty$ such that every subsequence of $\{n_k\}$ has a further subsequence $\{p_k\}$ such that $\sum_{k=1}^{\infty} a_{p_k} C_{I_{p_k}} x \in E$ (coordinate sum). [The term ∞ -GHP is used to suggest that the "humps" $C_I x$ are multiplied by a sequence which tends to ∞ ; there are other gliding hump properties where the "humps" are multiplied by elements of classical sequence spaces ([Sw3])].

We now give several examples of spaces with the ∞ -GHP. The space E is normal (solid in the scalar case) if $l^\infty E = E$ ([KG]2.1); E is c_0 -invariant if $x \in E$ implies that there exist $t \in c_0, y \in E$ such that $x = ty$ ([G]; the term c_0 -factorable might be more descriptive).

Example 3. If E is normal and is c_0 -invariant, then E has ∞ -GHP. Let $x \in E$ with $x = ty$ where $t \in c_0, y \in E$ and let $\{I_k\}$ be an increasing sequence of intervals. Pick an increasing sequence $\{n_k\}$ such that $\sup\{|t_j| : j \in I_{n_k}\} = b_{n_k} > 0$ (if this choice is impossible there is nothing to do). Note that $b_{n_k} \rightarrow 0$ so $a_{n_k} = 1/b_{n_k} \rightarrow \infty$. Define $v_j = t_j a_{n_k}$ if $j \in I_{n_k}$ and $v_j = 0$ otherwise; then $v \in l^\infty$ so $vy \in E$ since E is normal. We have $\sum_{j=1}^{\infty} (vy)_j e^j = \sum_{k=1}^{\infty} a_{n_k} C_{I_{n_k}} x \in E$. Since the same argument can be applied to any subsequence of $\{n_k\}$, E has ∞ -GHP.

We now give some examples of spaces which satisfy the conditions of Example 3.

Example 4. Let X be a normed space and let $c_0(X)$ be the space of all null sequences in X . Then $c_0(X)$ is normal and c_0 -invariant and so has ∞ -GHP.

Example 5. Let $0 < p < \infty$ and let X be a normed space. Let $l^p(X)$ be the space of all p^{th} power summable sequences in X . Let $x \in l^p(X)$ and pick an increasing sequence $\{n_k\}$ such that $\sum_{j=n_k+1}^{n_{k+1}} \|x_j\|^p < 1/2^{k(p+1)}$.

Set $I_k = [n_k + 1, n_{k+1}]$, $t = \sum_{k=1}^{\infty} 2^{-k} C_{I_k}$, $y = \sum_{k=1}^{\infty} 2^k C_{I_k} x$ so $x = ty$ with $t \in c_0$ and $y \in l^p$. Hence, $l^p(X)$ is c_0 -invariant and is obviously normal.

Likewise, it is easily checked that the spaces $d = \{t : \sup_k |t_k|^{1/k} < \infty\}$ and $\delta = \{t : \lim |t_k|^{1/k} = 0\}$ (see [KG] p.48 and 68) are normal and c_0 -invariant and, hence, have ∞ -GHP.

There exist non-normal sequence spaces with ∞ -GHP.

Example 6. Let cs be the space of convergent series ([KG]). Let $t \in cs$ and $\{I_k\}$ be an increasing sequence of intervals. Choose a subsequence $\{n_k\}$ such that $|\sum_{j \in I_{n_k} \cap I} t_j| < 1/k2^k$ for any interval I . Consider $s = \sum_{k=1}^{\infty} k C_{I_{n_k}} t$. If I is any interval contained in the interval $[\min I_{n_k}, \infty)$, then

$$\left| \sum_{j \in I} s_j \right| = \left| \sum_{i=k}^{\infty} \sum_{j \in I_{n_i} \cap I} i t_j \right| \leq \sum_{i=k}^{\infty} 1/2^i = 2^{-k+1}$$

so the partial sums of the series generated by s are Cauchy and $s \in cs$. Since the same argument applies to any subsequence of $\{n_k\}$, cs has ∞ -GHP.

Note that the argument above shows that any Banach AK space has ∞ -GHP; e.g., bv_0 ([KG]).

The spaces l^∞ , m_0 , bs and bv do not have ∞ -GHP.

We next consider a result analogous to Theorem 2 except that we use the strong operator topology.

Theorem 7. Let λ have ∞ -GHP. If $\sum T_j$ is λ multiplier convergent in the strong operator topology, then $\sum T_j$ is λ multiplier convergent in $L_b(X, Y)$.

Proof: If the conclusion fails, there exist $\varepsilon > 0$, $t \in \lambda$, $A \subset X$ bounded, a continuous semi-norm p on Y and subsequences $\{m_k\}, \{n_k\}$ such that $m_1 < n_1 < m_2 < \dots$ and $p_A(\sum_{l=m_k}^{n_k} t_l T_l) > \varepsilon$. For every k there exists $x_k \in A$ such that

$$(1) \quad p(\sum_{l=m_k}^{n_k} t_l T_l x_k) > \varepsilon.$$

Set $I_k = [m_k, n_k]$. Since λ has ∞ -GHP, there exist $\{p_k\}, a_{p_k} > 0, a_{p_k} \rightarrow \infty$ such that every subsequence of $\{p_k\}$ has a further subsequence $\{q_k\}$ such that $s = \sum_{k=1}^{\infty} a_{q_k} C_{I_{q_k}} t \in \lambda$. Let $M = [m_{ij}] = [\sum_{l=m_j}^{n_j} (t_l a_{p_j}) T_l (x_i / a_{p_i})]$.

We use the Antosik-Mikusinski Matrix Theorem ([Sw2]2.2.2) to show that the diagonal of M converges to 0; this will contradict (1). First, the columns of M converge to 0 since $x_i/a_{p_i} \rightarrow 0$ and each T_l is continuous. Next, given a subsequence there is a further subsequence $\{q_k\}$ such that $s = \sum_{k=1}^{\infty} a_{q_k} C_{I_{q_k}} t \in \lambda$. The series $\sum_{l=1}^{\infty} s_l T_l$ converges in the strong operator topology to an operator $T \in L(X, Y)$. Hence, $\sum_{j=1}^{\infty} m_{iq_j} = \sum_{j=1}^{\infty} \sum_{l \in I_{q_j}} s_l T_l(x_i/a_{p_i}) = T(x_i/a_{p_i}) \rightarrow 0$. It follows that M is a K matrix. By the Antosik-Mikusinski Matrix Theorem the diagonal of M converges to 0 which contradicts (1).

Remark 8. If the multiplier space λ in Theorem 7 is normal, we may replace the assumption that $\sum T_j$ is λ multiplier convergent in the strong operator topology with the assumption that the series is λ multiplier convergent in the weak operator topology. For if λ is normal and $\sum t_j T_j$ is convergent in the weak operator topology for every $t \in \lambda$, then the series is subseries convergent in the weak operator topology and, therefore, convergent in the strong operator topology by the classical Orlicz-Pettis Theorem.

The proof of Theorem 7 also establishes the following version of the Orlicz-Pettis Theorem for multiplier convergent series which should be compared to the version given in Corollary 2.4 of [SS] where it is assumed that λ is a barrelled AK-space.

Theorem 9.(Orlicz-Pettis) If λ has ∞ -GHP and if $\sum_j x_j$ is λ multiplier convergent in the weak topology $\sigma(X, X')$, then $\sum_j x_j$ is λ multiplier convergent in the strong topology $\beta(X, X')$.

The spaces $d = \{t : \sup_k |t_k|^{1/k} < \infty\}$ and $\delta = \{t : \lim |t_k|^{1/k} = 0\}$ ([KG]) give examples of spaces to which Theorems 7 and 9 apply but to which Theorem 2 and its scalar counterpart, Corollary 2.4 of [SS], do not apply. (The natural metric on d does not give a vector topology ([KG]p. 68).)

The scalar versions of Theorems 2 and 7 are both of interest. That is, the case when the space Y is the scalar field.

Corollary 10. Assume that λ has ∞ -GHP or that λ is a barrelled AK space.

If $\sum_j x'_j$ is λ multiplier convergent in the weak* topology $\sigma(X', X)$ of X' , then $\sum_j x'_j$ is λ multiplier convergent in the strong topology $\beta(X', X)$.

Corollary 10 should be compared to the Diestel-Faires Theorem concerning subseries convergence in the weak* topology of the dual of a Banach space. In the Diestel-Faires result the emphasis is on conditions on the space while in Corollary 10 the conditions are on the space of multipliers.

3. VECTOR MULTIPLIERS

We now consider vector valued multipliers for operator valued series. Let $\{T_k\} \subset L(X, Y)$ and let E be a vector space of X valued sequences containing the space of all sequences which are eventually 0. If τ is a locally convex Hausdorff topology on Y , we say that the series $\sum T_k$ is E multiplier convergent with respect to τ if the series $\sum T_k x_k$ is τ convergent for every $x = \{x_k\} \in E$. We establish Orlicz-Pettis Theorems for multiplier convergent series analogous to Theorems 2 and 7 for the weak and strong topologies of Y .

As in section 2 the following example shows that if the space of multipliers does not satisfy some condition an Orlicz-Pettis Theorem does not in general hold for the weak and strong topologies of Y .

Example 1. Let $l^\infty(X)$ be the space of all bounded X valued sequences. Assume that l^∞ has the weak topology $\sigma(l^\infty, l^1)$. Define $P_k : l^\infty \rightarrow l^\infty$ by $P_k x = x_k e^k$. Let $E = l^\infty(l^\infty)$. If $x = \{x^k\} \in E$, then $\sum P_k x^k = \sum x_k^k e^k$ is $\sigma(l^\infty, l^1)$ convergent, but if $x = \{e^k\} \in E$, then $\sum P_k e^k = \sum e^k$ is not $\beta(l^\infty, l^1) = |||_\infty$ convergent.

We begin with the analogue of Theorem 2.7 since it is straightforward to state and prove.

Theorem 2. Let E have ∞ -GHP and $\{T_k\} \subset L(X, Y)$.

If $\sum T_k$ is E multiplier convergent with respect to the weak topology $\sigma(Y, Y')$, then $\sum T_k$ is E multiplier convergent with respect to the strong topology $\beta(Y, Y')$.

Proof: If the conclusion fails, there exist $x \in E$, $\{y'_k\} \sigma(Y', Y)$ bounded, $\varepsilon > 0$ and subsequences $\{m_k\}, \{n_k\}$ with $m_1 < n_1 < m_2 < \dots$ and

$\left| \sum_{l=m_k}^{n_k} \langle y'_k, T_l x_l \rangle \right| > \varepsilon$ for all k . Set $I_k = [m_k, n_k]$. Since E has ∞ -GHP, there exist $\{p_k\}, a_{p_k} > 0, a_{p_k} \rightarrow \infty$ such that every subsequence of $\{p_k\}$ has a further subsequence $\{q_k\}$ such that $\sum a_{q_k} C_{I_{q_k}} x \in E$. Define an infinite matrix $M = [m_{ij}] = [\sum_{l \in I_{p_j}} \langle y'_{p_i}/a_{p_i}, T_l(a_{p_j} x_l) \rangle]$. We show that M is a K matrix so the diagonal of M converges to 0 by the Antosik-Mikusinski Matrix Theorem ([Sw2]2.2.2) and this will contradict the inequality above. First, the columns of M converge to 0 since $\{y'_i\}$ is $\sigma(Y', Y)$ bounded and $a_{p_i} \rightarrow \infty$. Next, given a subsequence there is a further subsequence $\{q_j\}$ such that $y = \sum a_{q_k} C_{I_{q_k}} x \in E$. Let $\sum_{l=1}^{\infty} T_l y_l = \sum_{j=1}^{\infty} \sum_{l \in I_{q_j}} T_l(a_{q_j} x_l) = z$ be the $\sigma(Y, Y')$ sum of this series. Then $\sum_{j=1}^{\infty} m_{iq_j} = \langle y'_{q_i}/a_{q_i}, z \rangle \rightarrow 0$ so M is a K matrix and the result follows.

We next establish the analogue of Theorem 2.2. If $u \in X$, $e^k \otimes u$ is the sequence with u in the k^{th} coordinate and 0 in the other coordinates. If τ is a Hausdorff locally convex topology on E , (E, τ) , or E if τ is understood, is an AK space if $x = \tau\text{-}\lim_n \sum_{k=1}^n e^k \otimes x_k = \sum_{k=1}^{\infty} e^k \otimes x_k$ for every $x \in E$ ([BL]).

Let $\sum T_k$ be E multiplier convergent with respect to $\sigma(Y, Y')$. Define a linear map $\hat{T} : E \rightarrow Y$ by setting $\hat{T}x = \sigma(Y, Y') - \lim_n \sum_{k=1}^n T_k x_k = \sum_{k=1}^{\infty} T_k x_k$. Note that $\hat{T}(e^k \otimes u) = T_k u$ for $u \in X, k \in \mathbf{N}$. Recall that the (scalar) β -dual of E is defined to be $E^\beta = \{\{y'_i\} : \sum_{i=1}^{\infty} \langle y'_i, x_i \rangle \text{ converges } \forall x \in E\}$ ([BL]). If $y' = \{y'_i\} \in E^\beta$ and $x = \{x_i\} \in E$, we set $y' \cdot x = \sum_{i=1}^{\infty} \langle y'_i, x_i \rangle$ and note that E and E^β are in duality with respect to this bilinear pairing. If $y' \in Y', x \in E$, then $\langle y', \hat{T}x \rangle = \langle y', \sum T_k x_k \rangle = \sum \langle y', T_k x_k \rangle = \sum \langle T'_k y', x_k \rangle$ so $\{T'_k y'\} \in E^\beta$ and $\langle y', \hat{T}x \rangle = \{T'_k y'\} \cdot x$. Hence, \hat{T} is $\sigma(E, E^\beta) - \sigma(Y, Y')$ continuous and, therefore, $\beta(E, E^\beta) - \beta(Y, Y')$ continuous ([W]11.2.3, [Sw1]26.15) so we have

Theorem 3. If $(E, \beta(E, E^\beta))$ is an AK space and $\sum T_k$ is E multiplier convergent with respect to $\sigma(Y, Y')$, then $\sum T_k$ is E multiplier convergent with respect to $\beta(Y, Y')$.

Proof: If $x \in E$, $\hat{T}x = \hat{T}(\sum e^k \otimes x_k) = \beta(Y, Y') - \lim_n \sum_{k=1}^n \hat{T}(e^k \otimes x_k) = \beta(Y, Y') - \lim_n \sum_{k=1}^n T_k x_k$ by the strong continuity of \hat{T} established above.

If E is a barrelled AK space, then the argument in Lemma 3.9 of [KG] shows that $E' = E^\beta$ and since E is barreled, the original topology of E is just $\beta(E, E^\beta)$ so Theorem 3 is applicable. If X is a Banach space, the spaces $c_0(X)$ of null sequences with the sup-norm and $l^p(X)$ of absolutely p -summable sequences with the l^p -norm give examples of barrelled (B-spaces), AK spaces to which Theorem 3 applies.

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Charles Swartz

Mathematics Department
New Mexico state University
Las Cruces
NM 8800
USA
e-mail : cswartz@nmsu.edu