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## GENERALIZATION OF THE SECOND TRACE FORM OF CENTRAL SIMPLE ALGEBRAS IN CHARACTERISTIC TWO\*

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### Abstract

*Let  $F$  be a field with characteristic two. We generalize the second trace form for central simple algebras with odd degree over  $F$ . We determine the second trace form and the Arf invariant and Clifford invariant for tensor products of central simple algebras.*

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## 1. Introduction

Let  $A$  be a central simple algebra over a field  $F$ . For each  $a \in A$ , let  $\text{Prd}_{A,a}(x) = x^n - t_1(a)x^{n-1} + t_2(a)x^{n-2} + \cdots + (-1)^n t_n(a)$  be the reduced polynomial of  $a$  (so  $t_1(a)$  is the reduced trace  $\text{Trd}_A(a)$  and  $t_n(a)$  is the reduced norm  $\text{Nrd}_A(a)$  of  $a$ .) Put  $q_1(x) = t_1(x^2)$  and  $q_2(x) = t_2(x)$ . When the characteristic of  $F$  is not equal to 2, the trace form  $(A, q_1)$  and the second trace form  $(A, q_2)$  are nonsingular quadratic forms (see [L], [LM] and [T]). If the characteristic is 2, however, then the trace form  $(A, q_1)$  has rank zero (and is therefore singular). In this situation, the second trace form  $(A, q_2)$  is nonsingular if the degree of  $A$  is even (see BF) but it is necessarily singular if the degree is odd.

In this article, we extend the definition of the second trace form to the case that the degree of  $A$  is odd and study the behavior with respect to tensor products. Our definition is similar to the way in which Revoy defined the trace form  $T_{E=F}$  for a field extension  $E$  of  $F$  in [R].

One reason for wanting to have the notion of a second trace form for central simple algebras of odd degree is the following. When we have two central simple algebras,  $A$  with even degree and  $B$  with odd degree, the tensor product  $A \otimes B$  has even degree. Hence, we have two nonsingular second trace forms  $(A, q_2)$  and  $(A \otimes B, q_2)$ . Now, if we extend the definition of the second trace form to a nonsingular quadratic form over the central simple algebras with odd degree, then it is possible to obtain  $(A \otimes B, q_2)$  through  $(A, q_2)$  and  $T_{B/F}$ , where  $T_{B/F}$  is the second trace form for odd component  $B$ . Decompositions of this type naturally appear when decomposing central simple algebras in terms of a matrix algebra and a division algebra, and in the primary decomposition of division algebras. We show (cf. Theorem 2) that our definition of the second trace form for central simple algebras of odd degree is compatible with such tensor product decompositions.

Let us now describe the contents of the paper in more detail. In Section 2, we define the second trace form  $T_{A/F}$  for central simple algebras  $A$  (with any parity) over a field  $F$  of characteristic two. We prove that it is a nonsingular quadratic form over  $F$ . In Section 3, we calculate this form for a crossed product. In particular, we prove that—given a Galois field extension  $E = F$  of odd degree—the second trace form for the crossed product algebra  $(E, G, \Phi)$  is Witt equivalent to the second trace form  $T_{E=F}$  of Revoy. We conclude that section with a relatively simple criterion that uses the trace form in order to recognize in many cases that a given field extension is not Galois. Note that in general this is not easy for fields

of characteristic two. In section 4, we study the second trace form for tensor products of central simple algebras over  $F$  (for any parity). As an application, we determine the Arf invariant and the Clifford invariant for tensor products in Section 5 (see Refs. [A], [B] and [Sah] for properties of these invariants).

## 2. Second trace form

In this section we will define the second trace form for central simple algebra over a field of characteristic two. We will prove that this is a nonsingular quadratic form. First, we give some properties and notations.

Given a central simple algebra  $A$  over  $F$ , the degree of  $A$ ,  $\text{deg}A$  for short, is the integer  $n$  such that  $\dim_F A = n^2$  [P, p. 36]. A splitting field for  $A$  is a field  $E$  containing  $F$  such that  $A \otimes_F E$  is isomorphic to the matrix algebra  $M_n(E)$ , and a splitting representation is a  $F$ -algebra isomorphism  $\phi : A \rightarrow M_n(E)$ . For each  $a \in A$  the reduced polynomial  $\text{Prd}_{A,a}(x)$  is defined as

$$\text{Prd}_{A,a}(x) := \det(xI_n - \phi(a)) = x^n - t_1(a)x^{n-1} + t_2(a)x^{n-2} + \dots + (-1)^n t_n(a). \tag{2.1}$$

This reduced polynomial  $\text{Prd}_{A,a}(x)$  has coefficients in  $F$  (i.e. it lies in  $F[x]$ ) and it is independent of  $E$  and  $\phi$  (see [P, p. 295]).

For each central simple algebra  $A$  over  $F$ ,  $(A, t_2)$  is a quadratic space. If in particular the characteristic of  $F$  is two, then it is easy to prove that for each  $a \in A$

$$t_2(a) = \sum_{1 \leq i < j \leq n} (\phi(a)_{ii} \phi(a)_{jj} + \phi(a)_{ij} \phi(a)_{ji}) \tag{2.2}$$

(where  $\phi$  is a splitting representation) and, furthermore, that the associated bilinear form  $b_{t_2}$  satisfies

$$b_{t_2}(x, y) = t_1(xy) + t_1(x)t_1(y) \quad (\text{for each } x, y \in A). \tag{2.3}$$

Since the quadratic space  $(E, t_2)$  is necessarily singular when  $\text{deg} A$  is odd, we will define a reduced second trace form  $T_{A/F}$  in the spirit of Revoy's definition in [R] for quadratic forms over extension fields (see section 3). To this end we first note that, by Eq. (2.3) and the linearity of  $t_1$ , the spaces  $F$  and  $A_0 := \text{Ker } t_1$  are mutually orthogonal. The reduced trace form  $T_{A/F}$  is now defined as

$$T_{A/F} = \begin{cases} (A; t_2) & \text{if } \text{deg } A \text{ is even;} \\ (A_0; t_2) & \text{if } \text{deg } A \text{ is odd;} \end{cases} \tag{2.4}$$

**Remark 1.** Let  $A$  and  $B$  be isomorphic central simple algebras over  $F$ . Then, by Eqs. (2.2) and (2.3), it is clear that  $T_{A/F}$  and  $T_{B/F}$  are isometric.

Let us denote the quadratic form  $ax^2 + xy + by^2$  as  $[a, b]$  and the hyperbolic plane  $[0, 0]$  as  $\mathbf{H}$ .

**Proposition 1 :** For a matrix algebra  $A = M_n(F)$  with  $n > 1$ , the reduced trace form  $T_{A/F}$  is Witt equivalent to

$$(2.5) \quad \begin{cases} \mathbf{H} & \text{if } n \equiv 0; 1; 2; 7 \pmod{8}; \\ [1, 1] & \text{if } n \equiv 3; 4; 5; 6 \pmod{8}; \end{cases}$$

**Proof.** We write the canonical base for  $M_n(F)$  as  $\{E_{ij}\}$  and put  $e_i := E_{ii}$ . For each triple of natural numbers  $0 < k_1 \leq n$  and  $0 < k_2, k_3 \leq \frac{n+3}{4}$  we define:

$$\begin{aligned} V_{k_1} &:= \text{span} \cup_{1 \leq i < j \leq k_1} \{\mathbf{E}_{ij}, \mathbf{E}_{ji}\}; \\ V_{k_2} &:= \text{span} \cup_{0 \leq i \leq k_2} \{\mathbf{e}_{4i+1} + \mathbf{e}_{4i+2}; \mathbf{e}_{4i+2} + \mathbf{e}_{4i+3}\}; \\ V_{k_3} &:= \text{span} \cup_{1 \leq i \leq k_3} \left\{ \sum_{t=1}^{4i} e_t; e_{4i} + e_{4i+1} + e_{i+2} + e_{4i+3} \right\}; \end{aligned}$$

It is clear that each pair of these subspaces has trivial intersection. It moreover follows from Eq. (2.3) that the above bases for the subspaces  $V_{k_1}$ ,  $V'_{k_2}$  and  $W_{k_3}$  are in fact symplectic. Hence, using Eqs. (2.2) and (2.3) we obtain that  $(V_{k_1}, t_2) = \frac{k_1(k_1-1)}{2}\mathbf{H}$ ,  $(V'_{k_2}, t_2) = (k_2 + 1)[1, 1]$ , and  $(W_{k_3}, t_2) = k_3\mathbf{H}$ . Furthermore by Eq. (2.3) the subspaces in question are mutually orthogonal. Defining  $S_{k_1, k_2, k_3}$  as the direct sum of those spaces, i.e.  $S_{k_1, k_2, k_3} = V_{k_1} \oplus V'_{k_2} \oplus W_{k_3}$ , we have that

$$(2.6) \quad (S_{k_1, k_2, k_3}, t_2) = (k_2 + 1)[1, 1] + \left( \frac{k_1(k_1 - 1)}{2} + k_3 \right) \mathbf{H}.$$

Defining  $W_0 = \{\theta\}$ , where  $\theta$  is the null matrix, we extend the definition of  $S_{k_1, k_2, k_3}$  to the case  $k_3 = 0$ . With this notation, we have that

$$A = \begin{cases} \mathbf{S}_{n, k-1, k-1} \oplus^\perp \left\langle \sum_{i=1}^{4k} \mathbf{e}_i, \mathbf{e}_{4k} \right\rangle & \text{if } n = 4k, \\ \mathbf{V}_2 \oplus^\perp \left\langle \sum_{i=1}^2 \mathbf{e}_i, \mathbf{e}_2 \right\rangle & \text{if } n = 2, \\ \mathbf{S}_{n, k-1, k-1} \oplus^\perp \left\langle \sum_{i=1}^{4k} \mathbf{e}_i, \mathbf{e}_{4k} + \mathbf{e}_{4k+1} \right\rangle \perp \left\langle \sum_{i=1}^{4k+2} \mathbf{e}_i, \mathbf{e}_{4k+2} \right\rangle & \\ \text{if } \begin{cases} n = 4k + 2; \\ n \neq 2; \end{cases} & \end{cases}$$

and

$$A_0 = \begin{cases} \mathbf{S}_{n,k-1,k-1} \oplus^\perp \langle \sum_{i=1}^{4k} e_i, e_{4k} + e_{4k+1} \rangle & \text{if } n = 4k + 1, \\ \mathbf{S}_{n;k;k} & \text{if } n = 4k + 3, \end{cases}$$

where  $\langle x, y \rangle$  denotes the space generated by  $x$  and  $y$ . Using Eq. (2.6), the fact that  $[0, 0] = [1, 0]$  (see [Sah, p. 150]), and replacing  $n = 2m + 1$  with  $m = 2k$  or  $m = 2k + 1$ , we obtain that  $T_{A/F}$  can be written as

$$\begin{cases} \left[ \frac{n}{4} \right] [1, 1] \perp (2m^2 - \frac{m}{2}) \mathbf{H} & \text{if } n = 2m; \\ \left[ \frac{n+1}{4} \right] [1, 1] \perp (2m^2 + \left[ \frac{3m}{2} \right]) \mathbf{H} & \text{if } n = 2m + 1; \end{cases}$$

where  $\left[ \frac{x}{y} \right]$  denotes the integer part of  $\frac{x}{y}$ . Hence, using that  $[1, 1] + [1, 1] = 2\mathbf{H}$ , the proof is complete.  $\square$

**Proposition 2 :** For each central simple algebra  $A$  over  $F$  with  $A \neq F$ , the second trace form  $T_{A/F}$  is a nonsingular quadratic form over  $F$ .

**Proof.** Let  $A$  be a central simple algebra over  $F$  with  $A \neq F$ . Let  $E$  be a splitting field of  $A$  with  $\phi : A \rightarrow M_n(E)$  the splitting representation. By [P, p. 238], we extend  $\phi$  to  $E$ -algebra isomorphism  $\phi : A \otimes_F E \rightarrow M_n(E)$  given by  $\phi(a \otimes e) = \phi(a)e$ . Clearly  $\phi(A_0 \otimes_F E) = M_n(E)_0$ , whence the quadratic form  $(A_0 \otimes_F E, t_2) = (M_n(E)_0, t_2)$  is nonsingular for  $\deg A$  odd (by Proposition 1). As a consequence,  $(A_0, t_2)$  is nonsingular. Similarly, if  $\deg A$  is even then  $(A \otimes_F E, t_2)$  is nonsingular (again by Proposition 1). Hence  $(A, t_2)$  is nonsingular.  $\square$

**Remark 2 :** Observe that we give here an alternative proof for the even case, already established by Berhuy and Frings in [BF, p. 4,5].

### 3. Crossed product algebra

In this section we will compute the second trace form for a crossed product.

Given a field extension  $E = F$ , we denote by  $T_{E=F}$  the second trace form due to Revoy [R], that is

$$(3.1) \quad T_{E/F} = \begin{cases} (E; T_2) & \text{if } [E : F] \text{ is even;} \\ (\text{Ker } T_1; T_2) & \text{if } [E : F] \text{ is odd;} \end{cases}$$

where for  $a \in E$ ,  $T_1(a)$  and  $T_2(a)$  denote the coefficients of  $x^{[E:F]-1}$  and  $x^{[E:F]-2}$ , respectively, in the characteristic polynomial of  $a$  (so  $T_1(a)$  is the trace  $tr_{E/F}(a)$  of  $a$ ). Note that there are two different definitions for the second trace forms in the literature. The second trace form, due to Bergé and Martinet [BM], increases the dimension of the space by 1 using the étale  $F$ -algebra. In [M] we proved that the two definitions are Witt equivalent.

**Proposition 3 :** Let  $E/F$  be a Galois field extension, with  $Gal(E/F) = G$ . Let  $A = (E, G, \Phi) = \sum_{\sigma \in G} u_\sigma E$  be the crossed product, where  $\Phi$  is normalized and for each  $\sigma, \tau \in G$  and  $c \in E$

$$(3.2) \quad u_\sigma^{-1} c u_\sigma = \sigma(c) \text{ and } \Phi(\sigma, \tau) = u_{\sigma\tau}^{-1} u_\sigma u_\tau \in E.$$

Then

- i)  $(E, t_2) = (E, T_2)$ , where  $T_2$  as above and  $t_2$  as in (2.1) ;
- ii)  $t_1(\sum_{\sigma \in G} u_\sigma c_\sigma) = tr_{E/F}(c_{id})$ , where  $c_{id}$  is the coefficient that correspond to  $id$  (the identity) ;
- iii) if  $\sigma \neq id$ , then for each  $c \in E$ ,  $u_\sigma c \in A_0$  ;
- iv) if  $\sigma\tau \neq id$ , then for each  $c, d \in E$ ,  $b_{t_2}(u_\sigma c, u_\tau d) = 0$  ;
- v) the subspaces  $E$  and  $\langle u_\sigma c \mid c \in E, \sigma \neq id \rangle$  are mutually orthogonal ;
- vi) if  $\rho^2 \neq id$ , then for each  $c \in E$ ,  $t_2(u_\rho c) = 0$ .

**Proof.** For i) see [P, p. 297]. There is a good splitting representation  $\phi$  of  $A = (E, G, \Phi)$  in [P, p. 298], given by

$$(3.3) \quad \phi \left( \sum_{\rho \in G} u_\rho c_\rho \right) = [d_{\sigma\tau}] \quad \text{where } d_{\sigma\tau} = \Phi(\sigma\tau^{-1}, \tau) c_{\sigma\tau^{-1}},$$

where for  $\tau \in G$  and  $c \in E$ , the notation  $c^\tau$  corresponds to  $\tau(c)$ . Using that  $t_1$  is the trace of the matrix  $[d_{\sigma\tau}]$ , we have

$$(3.4) \quad t_1 \left( \sum_{\rho \in G} u_\rho c_\rho \right) = \sum_{\sigma=\tau} d_{\sigma\tau} = \sum_{\sigma \in G} c_{id}^\sigma = tr_{E/F}(c_{id}).$$

Hence ii) and iii) are true. Furthermore, using Eq. (2.3) together with Eq. (3.2) we obtain iv) (and v) as a particular case). Now, by Eqs. (3.3) and (2.2), we have

$$(3.5) t_2 \left( \sum_{\rho \in G} u_\rho c_\rho \right) = \sum_{\sigma \neq \tau} (c_{id}^\sigma c_{id}^\tau + u_\sigma^{-1} u_{\sigma\tau^{-1}} u_\tau c_{\sigma\tau^{-1}}^\tau u_\tau^{-1} u_{\tau\sigma^{-1}} u_\sigma c_{\tau\sigma^{-1}}^\sigma).$$

Using the fact that if  $\rho^2 \neq id$  and  $\rho = \sigma\tau^{-1}$  for some  $\sigma$  and  $\tau$  in  $G$ , then  $\rho \neq \tau\sigma^{-1}$ , we obtain in Eq. (3.5) that for  $c \in E$ ,  $t_2(u_\rho c) = 0$ .  $\square$

The following theorem characterizes the second trace form  $T_{A/F}$  for a crossed product  $(E, Gal(E/F), \Phi)$  in terms of the second trace form  $T_{E/F}$  of Revoy.

**Theorem 1 :** Let  $E/F$  be a Galois extension with  $G = Gal(E/F)$ . Let  $A$  be the crossed product  $(E, G, \Phi) = \sum_{\sigma \in G} u_\sigma E$ . Then  $T_{A/F}$  is Witt equivalent to

$$\begin{cases} T_{E/F} \perp (B; t_2) & \text{if } [E : F] \text{ is even ;} \\ T_{E/F} & \text{if } [E : F] \text{ is odd ;} \end{cases}$$

where  $B$  is the subspace

$$B := \langle u_\rho e \mid e \in E, \rho \in G, \rho^2 = id, \rho \neq id \rangle$$

**Proof .** We can suppose that  $\Phi$  is normalized [P, p. 252]. Let us write  $G$  as

$$G = \{\rho_1 = id, \rho_2, \dots, \rho_s, \sigma_1, \sigma_2, \dots, \sigma_t, \sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_t^{-1}\},$$

where  $s+2t = [E : F]$  and for  $i \leq j \leq s$ ,  $\rho_j^2 = id$ , and for  $1 \leq i \leq t$ ,  $\sigma_i^2 \neq id$ . Let  $B' := \langle u_{\sigma_i} e \mid e \in E, 1 \leq i \leq t \rangle$ . It is clear that the sum  $E + B'$  is direct (when  $[E : F]$  is even,  $E + B' + B$  is direct). Furthermore, by Proposition 3,  $E$ ,  $B'$  and  $B$  are mutually orthogonal, and  $B'$  is a totally isotropic subspace (with dimension  $t[E : F]$ ). Hence, by combining the fact that  $\dim_F B = (s-1)[E : F]$  and that  $[E : F] = s + 2t$ , we see that :

- if  $[E : F]$  is even, then  $T_{AF} = T_{E/F} \perp (B, t_2) \perp t[E : F] \mathbf{H}$  (see [B, p. 17]),

- if  $[E : F]$  is odd, then  $s = 1$ , whence  $T_{A/F} = T_{E/F} \perp t[E : F] \mathbf{H}$ .  $\square$

**Corollary 1 :** Let  $E/F$  be a Galois extensions with degree  $n = 2m + 1$ . Let  $A$  be the crossed product  $(E, Gal(E/F), \Phi)$ . Then

$$[T_{A/F} = \begin{cases} \mathbf{mH} & \text{if } n \equiv 1; 7 \pmod{8} \\ [1, 1] + (\mathbf{m} - 1)\mathbf{H} & \text{if } n \equiv 3; 5 \pmod{8} \end{cases}$$

**Proof.** The algebra  $B = End_F(E) \cong M_n(F)$  is the crossed product  $(E, G, I)$  [P, p. 252]. Hence, by Theorem 1,  $T_{A/F} = T_{E/F} = T_{B/F}$ . Using Eq. (2.5), we obtain the result.  $\square$

In general it is not easy to decide whether a given field extension is Galois. The following by-product of our work on the trace form permits an elegant partial solution to this problem.

**Corollary 2 :** Let  $E/F$  be a field extension with odd degree  $n$ . If  $T_{E/F}$  is not Witt equivalent to  $\mathbf{H}$  when  $n \equiv 1, 7 \pmod{8}$ , or not equivalent to  $[1, 1]$  when  $n \equiv 3, 5 \pmod{8}$ , then the extension  $E/F$  is not Galois.

**Proof.** Immediate, by Corollary 1.  $\square$

**Example 1 :** Let  $F = \mathbf{F}_2(a)$  and  $E = F(b)$ , where  $a^2 + a + 1 = 0$  and  $b^3 + b + a = 0$ . It is easy to prove that  $T_{E/F} = [1, a]$ . Note that  $[1, 1] \neq [1, a]$ , because  $1 \in \wp(F) = \{x^2 + x | x \in F\}$  and  $a \notin \wp(F)$ . Hence,  $E/F$  is not Galois by corollary 2.

#### 4. Second trace form of a tensor product.

Let  $C.S(F)$  be the set of central simple algebras over  $F$ . For each  $A \in C.S(F)$  and any  $a \in A$ , we remember that the reduced trace of  $a$  is  $Trd_A(a) := t_1(a)$ .

**Proposition 4 :** Let  $A$  and  $B$  be central simple algebras over  $F$ . Then

- i)  $1_A \otimes_F B_0$  and  $A_0 \otimes_F 1_B$  are mutually orthogonal (as quadratic subspaces of  $(A \otimes_F B, t_2)$ ),



ii) for each  $a, a' \in A$  and  $b, b' \in B$  we have that

$$(4.1) \quad \text{Trd}_{A \otimes_F B}(a \otimes b) = \text{Trd}_A(a) \text{Trd}_B(b)$$

$$(4.2) \quad t_2(a \otimes b) = (\text{Trd}_A(a))^2 t_2(b) + (\text{Trd}_B(b))^2 t_2(a)$$

$$(4.3) \quad \begin{aligned} b_{t_2}(a \otimes b, a' \otimes b') &= \text{Trd}_A(aa') b_{t_2}(b, b') + \text{Trd}_B(b) \text{Trd}_B(b') b_{t_2}(a, a') \\ &= \text{Trd}_B(bb') b_{c_2}(a; a') + \text{Trd}_A(a) \text{Trd}_A(a') b_{c_2}(b; b'); \end{aligned}$$

iii) if  $\{a, a'\} \cap A_0 \neq \phi$  and  $\{b, b'\} \cap B_0 \neq \phi$ , then

$$(4.4) \quad b_{t_2}(a \otimes b, a' \otimes b') = b_{t_2}(a, a') b_{t_2}(b, b').$$

**Proof.** Let  $A, B \in C.S(F)$ , and let  $E$  and  $L$  be splitting fields for  $A$  and  $B$ , respectively, with splitting representations  $\phi : A \rightarrow M_m(E)$  and  $\varphi : B \rightarrow M_n(L)$ . Clearly  $EL$  is a splitting field for  $A \otimes_F B$  ( $EL$  is the minimal field that contains  $E$  and  $L$ ). A corresponding splitting representation is given by  $\Phi : A \otimes_F B \rightarrow M_{mn}(EL)$  of the form  $\Phi(a \otimes b)_{ij} = \phi(a)_{kl} \varphi(b)_{st}$ , where  $i = (k-1)n + s$  and  $j = (l-1)n + t$ , with  $1 \leq k, l \leq m+1$  and  $0 \leq t \leq n-1$ . Hence, by using the definition of the reduced trace and Eqs. (2.2), (2.3), we obtain the equations that appear in ii). Furthermore, by Eq. (4.3) and the fact that  $F$  and  $\text{Ker } t_1$  are mutually orthogonal, we obtain i). In order to obtain iii), we use that  $\text{Trd}_A(a) \text{Trd}_A(a') = 0 = \text{Trd}_B(b) \text{Trd}_B(b')$ .  $\square$

The following theorem provides the second trace form of a tensor product in terms of the second trace forms of its constituents.

**Theorem 2 :** Let  $A_1, A_2 \in C.S(F)$ , with  $\deg A_i = n_i$  for  $i = 1, 2$ .  $T_{A_1 \otimes A_2 / F}$  can be represented as

$$\left\{ \begin{array}{ll} T_{A_1/F} + T_{A_2/F} + \frac{(n_1^2-1)(n_2^2-1)}{2} \mathbf{H} & \text{if } n_1, n_2 \text{ are odd;} \\ [1; 1] + \left( \frac{n_1^2 n_2^2}{2} - 1 \right) \mathbf{H} & \text{if } n_1, n_2 \equiv 2 \pmod{4}; \\ \frac{n_1^2 n_2^2}{2} \mathbf{H} & \text{if } n_1 \equiv 0 \pmod{4} \text{ and } n_2 \text{ is even;} \\ T_{A_i/F} + \frac{n_i^2(n_j^2-1)}{2} \mathbf{H} & \text{if } \begin{cases} n_i \equiv 0 \pmod{4} \text{ and } n_j \text{ is odd;} \\ n_i \equiv 2 \pmod{4} \text{ and } n_j \equiv 1 \pmod{4} \end{cases} \\ [1; 1] + T_{A_i/F} + \left( \frac{n_i^2(n_j^2-1)}{2} - 1 \right) \mathbf{H} & \text{if } n_i \equiv 2 \pmod{4} \text{ and } n_j \equiv 3 \pmod{4}; \end{array} \right.$$

**Proof.** For  $j = 1, 2$ , let

$\{e_i^{(j)}, f_i^{(j)}\}_{i \in I^{(j)}}$ , be a symplectic basis of  $(A_j)_0$  if  $n_j$  is odd, and  $\{1_{A_j}, f^{(j)}\} \cup \{e_i^{(j)}, f_i^{(j)}\}_{i \in I^{(j)}}$  be a symplectic basis of  $A_j$  if  $n_j$  is even.

Put

$$(4\mathfrak{H}) = \bigoplus_{i \in I^{(1)}, j \in I^{(2)}}^{\perp} \left( \langle e_i^{(1)} \otimes e_j^{(2)}, f_i^{(1)} \otimes f_j^{(2)} \rangle \perp \langle e_i^{(1)} \otimes f_j^{(2)}, f_i^{(1)} \otimes e_j^{(2)} \rangle \right).$$

In view of (4.2),  $(W, t_2)$  is hyperbolic.

Using Eqs. (4.1), (4.3) and (4.4) we obtain decompositions of  $(A_1 \oplus A_2)_0$  and  $(A_1 \otimes A_2)_0$ , respectively in the following cases:

-  $A_1$  and  $A_2$  have odd degree :

$$(A_1 \otimes A_2)_0 = \bigoplus_{i \in I^{(1)}}^{\perp} \langle e_i^{(1)} \otimes 1_{A_2}, f_i^{(1)} \otimes 1_{A_2} \rangle \perp \bigoplus_{j \in I^{(2)}}^{\perp} \langle 1_{A_1} \otimes e_j^{(2)}, 1_{A_1} \otimes f_j^{(2)} \rangle \perp W.$$

-  $A_1$  and  $A_2$  have even degree

$$A_1 \otimes A_2 = \langle 1_{A_1} \otimes f^{(2)}, f^{(1)} \otimes 1_{A_2} \rangle \perp W \perp V',$$

where

$\langle 1_{A_1} \otimes 1_{A_2}, 1_{A_1} \otimes e_j^{(2)}, 1_{A_1} \otimes f_j^{(2)}, e_i^{(1)} \otimes 1_{A_2}, f_i^{(1)} \otimes 1_{A_2} \rangle_{\{i \in I^{(1)}, j \in I^{(2)}\}}$  is a totally isotropic subspace of  $V'$  with dimension  $\dim_F A_1 + \dim_F A_2 - 3 = \frac{1}{2} \dim_F V'$ . Hence  $(V', t_2) = (\dim_F A_1 + \dim_F A_2 - 3)\mathbf{H}$ .

-  $A_1$  and  $A_2$  have different parity :

Suppose that the degree of  $A_1$  is even. Then  $A_1 \otimes A_2$  can be written as

$$\langle 1_{A_1} \otimes 1_{A_2}, f^{(1)} \otimes 1_{A_2} \rangle \perp \bigoplus_{\{i \in I^{(1)}\}}^{\perp} \langle e_i^{(1)} \otimes 1_{A_2}, f_i^{(1)} \otimes 1_{A_2} \rangle \perp W \perp V'',$$

where  $\langle 1_{A_1} \otimes e_j^{(2)}, 1_{A_1} \otimes f_j^{(2)} \rangle_{j \in J^{(2)}}$  is a totally isotropic subspace of  $V''$  with dimension  $(\dim_F A_2 - 1) = \frac{1}{2} \dim_F V''$ . Hence,  $(V'', t_2) = (\dim_F A_2 - 1)\mathbf{H}$ .

Finally, we obtain the claim using Eq. (4.2) and the fact that for each  $A \in C.S(F)$ ,  $t_2(1_A)$  is given by

$$t_2(1_A) = \begin{cases} 0 & \text{if } \deg A \equiv 0; 1 \pmod{4}; \\ 1 & \text{if } \deg A \equiv 2; 3 \pmod{4}; \end{cases} \quad \square$$

**Corollary 3 :** Let  $A$  be a central simple algebra with even degree. If the second trace form  $T_{A/F}$  is not Witt equivalence to  $[1, 1]$  or to  $\mathbf{H}$ , then  $A = M_k(D)$  with  $k$  odd and  $D$  a division algebra.

**Proof.** Since for each  $A \in C.S(F)$ ,  $A \simeq M_k(D) \simeq M_k(F) \otimes D$  for some division algebra  $D$ , by Theorem 2 we obtain that  $T_{A/F}$  is Witt equivalent to  $T_{D/F}$ ,  $[1, 1] + T_{D/F}$ ,  $\mathbf{H}$  or  $[1, 1]$ , depending on  $k$  and  $\deg D$ . Now, if  $k$  is even then  $T_{A/F}$  is Witt equivalent to  $\mathbf{H}$  or to  $[1, 1]$ .  $\square$

**Corollary 4:** Let  $A$  be a central simple algebra over  $F$ .

$$[T_{A \oplus A/F} \text{ is Witt equivalent to } \begin{cases} [1, 1] & \text{if } \deg A \equiv 2 \pmod{4} \\ \mathbf{H} & \text{otherwise} \end{cases}$$

### 5. Invariants

In this section we will determine the Arf invariant and the Clifford invariant for the tensor product of algebras as an application of Theorem 2.

For  $a \in F^*$  and  $b \in F$  the quaternion algebra  $(a, b) \in C.S(F)$  is defined as the algebra with basis  $\{1, e, f, ef\}$  satisfying  $e^2 = a$ ,  $f^2 + f = b$  and  $ef + fe = e$ . Now, given a non degenerated quadratic form  $(V, q)$  over  $F$ , we can rewrite  $q$  as  $q = \langle a_1 \rangle [1, b_1] \perp \cdots \perp \langle a_n \rangle [1, b_n]$ , with  $a_i \in F^*$  and  $b_i \in F$ . The Arf invariant is given by  $Arf(q) := b_1 + b_2 + \cdots + b_n \pmod{\wp(F)}$ , where  $\wp(F) := \{x^2 + x | x \in F\}$ , considered as additive subgroup of  $F$ . The Clifford invariant  $C(V, q)$  is the class of the tensor products of quaternion algebras  $(a_1, b_1) \otimes \cdots \otimes (a_n, b_n)$  in the Brauer group  $Br(F)$ . Note that if  $a \neq 0$  then

$$C([a, b]) = C(\langle a \rangle [1, ab]) = (a, ab) = ((a, b))_F,$$

where  $((a, b))_F$  [KMRT, p. 25] is the algebra generated by  $r$  and  $s$  satisfying

$$r^2 = a, \quad s^2 = b, \quad rs + sr = 1.$$

We need the following result of Berhuy and Frings [BF].

**Theorem 3 :** (Berhuy-Frings [BF]).

Let  $F$  be a field of characteristic two,  $n \geq 2$  an even integer and  $A \in C.S(F)$  an algebra of degree  $n$  over  $F$ . Then we have  $Arf(T_{A/F}) = \left[ \frac{n}{4} \right]$

and  $C(A, T_{A/F}) = [A]^{\frac{n}{2}}$ , where  $[\frac{n}{4}]$  denotes the integer part of  $\frac{n}{4}$  and  $[A]$  denotes the class of  $A$  in the Brauer group  $Br(F)$ .

The following theorem gives the Arf invariant and the Clifford invariant of the second trace form of a tensor product in terms of the corresponding invariants of its constituents.

**Theorem 4 :** Let  $A_1, A_2 \in C.S(F)$ , with  $\deg A_i = n_i$ . Then

$$[\text{Arf}(T_{A_1 \otimes A_2/F}) = \frac{\text{Arf}(T_{A_1/F}) + \text{Arf}(T_{A_2/F})}{4} \quad \text{if } n_1 n_2 \text{ is odd,} \\ \text{otherwise ;}$$

and

$$C(T_{A_1 \otimes A_2/F}) = \begin{cases} C(T_{A_1/F}) \cdot C(T_{A_2/F}) & \text{if } n_1, n_2 \text{ are odd ;} \\ ((1; 1))_F & \text{if } n_1, n_2 \equiv 2 \pmod{4} ; \\ 1 & \text{if } n_i \equiv 0 \pmod{4} \text{ and } n_j \text{ is even ;} \\ [A_i]^{\frac{n_i}{2}} & \text{if } \begin{cases} n_i \equiv 0 \pmod{4} \text{ and } n_j \text{ is odd ;} \\ \text{or} \\ n_i \equiv 2 \pmod{4} \text{ and } n_j \equiv 1 \\ \pmod{4} ; \end{cases} \\ ((1; 1))_F \cdot [A_i]^{\frac{n_i}{2}} & \text{if } n_i \equiv 2 \pmod{4} \text{ and } n_j \equiv 3 \pmod{4} ; \end{cases}$$

where  $\{i; j\} = \{1; 2\}$ .

**Proof.** When  $A_1$  or  $A_2$  have even degree, we have that  $A_1 \otimes A_2$  also has even degree. Hence, by Theorem 3,  $\text{Arf}(T_{A_1 \otimes A_2/F}) = [\frac{n_1 n_2}{2}]$ . But, if  $A_1$  and  $A_2$  have odd degree, then by Theorem 2,  $T_{A_1 \otimes A_2/F} = T_{A_1/F} + T_{A_2/F}$ . Hence, in this case  $\text{Arf}(T_{A_1 \otimes A_2/F}) = \text{Arf}(T_{A_1/F}) + \text{Arf}(T_{A_2/F})$ . In order to obtain the Clifford invariants, we need Theorem 2, the fact that  $C(\mathbf{H}) = 1 = [F]$ , and that  $C(T_{A/F}) = [A]^{n/2}$  if  $A$  has even degree (cf. Theorem 3).  $\square$

**Remark 3:** It follows from a comment given by Berhuy and Frings [BF, p. 4], that if  $A \in C : S(F)$  with odd degree  $n$ , then the second trace from  $T_{A/F}$  is Witt equivalent to  $T_{Mn(F)}$  (the second trace form for the matrix algebra of dimension  $n$  over  $F$ ). As a consequence, the statements in Theorem 2 and Theorem 3 can be made more explicit in the case that the algebras  $A_1; A_2 \in C : S(F)$  both have odd degrees  $n_1$  and  $n_2$ , respectively.

Indeed, we get upon invoking Proposition 1 that in this case

$$T_{A_1 \otimes A_2 / F} = \begin{cases} \frac{n_1^2 n_2^2 - 1}{2} \mathbf{H} & \text{if } \mathbf{n}_1 \mathbf{n}_2 \equiv 1; 7 \pmod{8} ; \\ [1, 1] + \frac{n_1^2 n_2^2 - 3}{2} \mathbf{H} & \text{if } \mathbf{n}_1 \mathbf{n}_2 \equiv 3; 5 \pmod{8} : \end{cases}$$

The corresponding invariants of  $T_{A_1 \otimes A_2 / F}$  thus become of the form

$$[(\mathbf{Arf}(T_{A_1 \otimes A_2 / F}); \mathbf{C}(T_{A_1 \otimes A_2 / F}))] = \begin{cases} (0, 1) & \text{if } \mathbf{n}_1 \mathbf{n}_2 \equiv 1; 7 \pmod{8} ; \\ ([1; 1]; ((1; 1))_F) & \text{if } \mathbf{n}_1 \mathbf{n}_2 \equiv 3; 5 \pmod{8} : \end{cases}$$

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