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JORDAN NILALGEBRAS OF DIMENSION 6

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Abstract

It is known the classification of commutative power-associative nilalgebras of dimension ≤ 4 *(see,* [4]*). In* [2]*, we give a description of commutative power-associative nilalgebras of dimension* 5*. In this work we describe Jordan nilalgebras of dimension* 6*.*

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1. Preliminaries

Let A be a commutative algebra over a field K. If x is an element of A, we define $x^1 = x$ and $x^{k+1} = x^k x$ for all $k \geq 1$.

A is called power-associative, if the subalgebra of A generated by any element $x \in A$ is associative. An element $x \in A$ is called nilpotent, if there is an integer $r \geq 1$ such that $x^r = 0$. If any element in A is nilpotent, then A is called a nilalgebra. Now A is called a nilalgebra of nilindex $n \geq 2$, if $y^n = 0$ for all $y \in A$ and there is $x \in A$ such that $x^{n-1} \neq 0$.

If B, D are subspaces of A, then BD is the subspace of A spanned by all products bd with b in B, d in D. Also we define $B^1 = B$ and $B^{k+1} = B^k B$ for all $k \geq 1$. If there exists an integer $n \geq 2$ such that $B^n = 0$ and $B^{n-1} \neq 0$, then B is nilpotent of index n.

A is a Jordan algebra, if it satisfies the Jordan identity $x^2(yx) = (x^2y)x$ for all x, y in A. It is known that any Jordan algebra is power-associative, and also that any finite-dimensional Jordan nilalgebra (of characteristic $\neq 2$) is nilpotent (see, [5]).

We will use the following result which we give in [2] :

Proposition 1.1 If A is a Jordan nilalgebra of nilindex $n \geq 3$ with $\dim_K(A) = m \geq n$, then $n - 2 \leq \dim_K(A^2) \leq m - 2$.

Throughout, A will denote a commutative nilalgebra of nilindex $n \geq 3$ over a field K of characteristic $\neq 2, 3$. We will denote by $< x_1, ..., x_j >_K$ the subspace generated over K by the elements $x_1, ..., x_j$ in A. Also we will denote by α, β,....,etc., the elements of field K. If $x \in A$ with $x^{n-1} \neq 0$, then we will denote by X the subspace $< x, x^2, ..., x^{n-1} >_K$. It is clear that $x, x^2, ..., x^{n-1}$ are linearly independent and so $\dim_K(A^2) \geq n - 2$ and $\dim_K(A^3) \geq n - 3$.

2. COMMUTATIVE NILALGEBRAS OF NILINDEX 3 AND DIMENSION 6

In this section, A will denote a commutative nilalgebra of nilindex 3. It is well known that a commutative nilalgebra of nilindex 3 is a Jordan algebra (see [6], page 114).

Since $x^3 = 0$ for all xA, then by linearization method we obtain that the following identities are valid in A :

$$x^2y + 2\,(xy)\,x = 0, \;\; (xy)\,z + (yz)\,x + (zx)\,y = 0 \tag{2.1}$$

It is clear that the identity $x^4 = \left(x^2\right)^2 = 0$ is valid in A, which implies that for all x, y, z in A we have :

$$x^2\left(yx\right) = \left(x^2 y\right)x = 0,\ 2\left(xy\right)^2 + x^2y^2 = 0 \tag{2.2}$$

Lemma 2.1 If $\left(A^2\right)^2 \neq 0$, then $\dim_K\left(A\right) \geq 8$.

Proof. If $(A^2)^2 \neq 0$, then there exist $x, y \in A$ such that $x^2y^2 \neq 0$. We note first that using (1) and (2), we obtain that: $x^2(yx^2) = -2(x(yx^2)x = 0$, $x^2(xy^2) = 0$, $x^2(x^2y^2) = -2((x^2y^2)x)x = 0$, $x^2y^2 + 2(y^2x)x = 0$ and $x^2y^2 + 2(x^2y)y = 0$. We will prove that the elements y, x, x^2, y^2, yx^2, xy^2, xy, x^2y^2 are linearly independent. Let $\alpha_1 y + \alpha_2 x + \alpha_3 x^2 + \alpha_4 y^2 + \alpha_5 yx^2 + \alpha_6 xy^2 + \alpha_7 xy + \alpha_8 x^2y^2 = 0$. Multiplying by x^2 we obtain that $\alpha_1 yx^2 + \alpha_4 x^2y^2 = 0$. Thus $0 = 2y(\alpha_1 yx^2 + \alpha_4 x^2y^2) = 2\alpha_1 y(yx^2) = -\alpha_1 x^2y^2$ implies $\alpha_1 = 0$. Clearly also $\alpha_4 = 0$. Similarly we prove that $\alpha_2 = \alpha_3 = 0$. Now we have that $\alpha_5 yx^2 + \alpha_6 xy^2 + \alpha_7 xy + \alpha_8 x^2y^2 = 0$. Multiplying by x we get $\alpha_6 x(xy^2) + \alpha_7 x(xy) = 0$. Hence $0 = 2y(\alpha_6 x(xy^2) + \alpha_7 x(xy)) = -\alpha_6 y(x^2y^2) - \alpha_7 y(yx^2) = \frac{1}{2}\alpha_7 x^2y^2$ which implies $\alpha_7 = 0$. Finally it is clear that $\alpha_6 = 0$, and also that $\alpha_5 yx^2 + \alpha_8 x^2y^2 = 0$ implies $\alpha_5 = \alpha_8 = 0$. This proves what we wanted.

Lemma 2.2 If $A^4 \neq 0$, then $\dim_K\left(A\right) \geq 7$.

Proof. By Lemma 2.1, we can suppose that $(A^2)^2 = 0$. Since $A^4 \neq 0$, there exist elements y, x, z in A such that $z(yx^2) \neq 0$. Now using relation (1), we obtain that $2z((yx)x) = -z(yx^2) \neq 0$. We will prove that y, x, z, yx^2, yx, x^2, $z(yx^2)$ are linearly independent. Let (**1**): $\alpha_1 y + \alpha_2 x + \alpha_3 z + \alpha_4 yx^2 + \alpha_5 yx + \alpha_6 x^2 + \alpha_7 z(yx^2) = 0$. Multiplying by yx^2 we get $0 = \alpha_1 y(yx^2) + \alpha_3 z(yx^2) = -\frac{1}{2}\alpha_1 y^2x^2 + \alpha_3 z(yx^2) = \alpha_3 z(yx^2) = 0$ which implies $\alpha_3 = 0$. Multiplying (**1**) by x^2 we obtain $\alpha_1 = 0$. We note that using (1) we get $x(z(yx^2)) = -z(x(yx^2)) - (yx^2)(xz) = 0$ and $y(z(yx^2)) = -z(y(yx^2)) - (yx^2)(yz) = 0$. Similarly $z(z(yx^2)) = 0$. Now multiplying (**1**) by $2x$ we obtain $0 = 2\alpha_2 x^2 + 2\alpha_5 x(yx) = 2\alpha_2 x^2 - \alpha_5 yx^2$. So $0 = y(2\alpha_2 x^2 - \alpha_5 yx^2) = 2\alpha_2 yx^2 - \alpha_5 y(yx^2) = 2\alpha_2 yx^2 + \frac{1}{2}\alpha_5 y^2x^2 = 2\alpha_2 yx^2$ implies $\alpha_2 = 0$. It is clear that also $\alpha_5 = 0$. Finally it is possible to prove that $\alpha_4 yx^2 + \alpha_6 x^2 + \alpha_7 z(yx^2) = 0$ implies $\alpha_4 = \alpha_6 = \alpha_7 = 0$. Therefore we conclude that $\dim_K(A) \geq 7$, as desired.

We see that Lemmas 2.1 and 2.2 imply the following result:

Corollary 2.3 If $\dim_K(A) \leq 6$, then $(A^2)^2 = A^4 = 0$.

Now if $A^3 \neq 0$, then there exist elements y, x in A such that $yx^2 \neq 0$. In this case it is easy to prove that y, x, yx^2, x^2, yx are linearly independent. Therefore we obtain the following result:

Lemma 2.4 If $A^3 \neq 0$, then $\dim_K(A) \geq 5$.

We observe that when $\dim_K(A) = 6$, then by Proposition 1.1 we have that $1 \leq \dim_K\left(A^2\right) \leq 4$. Moreover, if $A^3 \neq 0$ and $\dim_K(A) = 6$, then $3 \leq \dim_K(A) \leq 4$.

Proposition 2.5 If $\dim_K(A) = 6$, $A^3 \neq 0$ and $\dim_K(A^2) = 4$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$of Asuch that $u_1^2 = u_6$, $u_1u_2 = u_4$, $u_1u_5 = u_3$, $u_2^2 = u_5$, $u_2u_4 = -\frac{1}{2}u_3$, all other products being zero.

Proof. We know that $(A^2)^2 = A^4 = 0$. Since $A^3 \neq 0$, then there exist y, $x \in A$ such that y, x, yx^2, yx, x^2 are linearly independent. Clearly y, x are not elements in A^2, and thus there exists $z \in A$ such that $\{y, x, yx^2, yx, x^2, z^2\}$ is a basis of A. As $z = \alpha_1 y + \alpha_2 x + \alpha_3 yx^2 + \alpha_4 yx + \alpha_5 x^2 + \alpha_6 z^2$, then $z^2 = (z - \alpha_6 z^2)^2 \in < y^2, x^2, yx, y^2x, yx^2 >_K$. From this we see that if $y^2 \in < x^2, yx, yx^2 >_K$, then $z^2 \in < x^2, yx, yx^2 >_K$, which is a contradiction. Hence $y^2 \notin < x^2, yx, yx^2 >_K$, and so $\{y, x, yx^2, yx, x^2, y^2\}$ is a basis of A. Since $xy^2 \in A^2$, then $xy^2 = \alpha yx^2 + \beta yx + \gamma x^2 + \delta y^2$. Multiplying by $2x$ we get $2\beta x(yx) + 2\delta xy^2 = -\beta yx^2 + 2\delta xy^2 = 0$. Thus $-\beta yx^2 + 2\delta(\alpha yx^2 + \beta yx + \gamma x^2 + \delta y^2) = 0$, implies $\beta = \delta = 0$, and so $xy^2 = \alpha yx^2 + \gamma x^2$. But $0 = y(xy^2) = y(\alpha yx^2 + \gamma x^2) = \gamma yx^2$ implies $\gamma = 0$, and therefore $xy^2 = \alpha yx^2$. Finally, if we define $u_1 = y + \alpha x$, $u_2 = x$, $u_3 = yx^2$, $u_4 = yx + \alpha x^2$, $u_5 = x^2$, $u_6 = y^2 + 2\alpha yx + \alpha^2 x^2$, we get $u_1^2 = u_6$, $u_1u_2 = u_4$, $u_1u_5 = u_3$, $u_2^2 = u_5$, $u_2u_4 = -\frac{1}{2}u_3$, all other products zero.

Proposition 2.6 If $\dim_K(A) = 6$, $A^3 \neq 0$ and $\dim_K(A^2) = 3$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1u_2 = u_5$, $u_1u_6 = u_4$, $u_2^2 = u_6$, $u_2u_3 = -\beta u_6$, $u_2u_5 = -\frac{1}{2}u_4$, $u_3^2 = \delta u_4$, $u_3u_5 = \beta u_4$, all other products being zero.

Proof. We know that $(A^2)^2 = A^4 = 0$. Since $A^3 \neq 0$, there exist y, $x \in A$ such that y, x, yx^2, yx, x^2 are linearly independent, and thus there exists an element $z \in A$ such that $\{y, x, z, yx^2, yx, x^2\}$ is a basis of A. As $y^2 \in A^2$, then $y^2 = \sigma_1 yx^2 + \sigma_2 yx + \sigma_3 x^2$. If $y_0 = y - \frac{1}{2}\sigma_2 x - \frac{1}{2}\sigma_1 x^2$ we obtain that $y_0^2 = (\sigma_3 + \frac{1}{4}\sigma_2^2)x^2$, and so $0 = y_0^3 = (\sigma_3 + \frac{1}{4}\sigma_2^2)yx^2$ which implies $\sigma_3 + \frac{1}{4}\sigma_2^2 = 0$. Thus $y_0^2 = 0$ and clearly $\{y_0, x, z, y_0x^2, y_0x, x^2\}$ is a basis of A. Since $zx \in A^2$, then $zx = \alpha_1 y_0 x^2 + \alpha_2 y_0 x + \alpha_3 x^2$. If $z_0 = z + 2\alpha_1 y_0 x - \alpha_2 y_0 - \alpha_3 x$, we get that $\{y_0, x, z_0, y_0x^2, y_0x, x^2\}$ is a basis of A with $z_0 x = 0$. Let $y_0 z_0 = \beta_1 y_0 x^2 + \beta_2 y_0 x + \beta_3 x^2$. If $z_1 = z_0 - \beta_1 x^2$, we obtain that $\{y_0, x, z_1, y_0x^2, y_0x, x^2\}$ is a basis of A with $z_1 x = 0$ and $y_0 z_1 = \beta_2 y_0 x + \beta_3 x^2$. Now $0 = y_0^2 z_1 = -2y_0(y_0 z_1) = -2y_0(\beta_2 y_0 x + \beta_3 x^2) = \beta_2 y_0^2 x - 2\beta_3 y_0 x^2 = -2\beta_3 y_0 x^2$ implies $\beta_3 = 0$. Therefore we can suppose that in the basis $\{y, x, z, yx^2, yx, x^2\}$ of A, we have $y^2 = 0$, $zx = 0$ and $yz = \beta yx$. Let $z^2 = \delta yx^2 + \varepsilon yx + \theta x^2$. Now we have that: $z(yx) = -x(zy) - y(xz) = -x(zy) = -\beta x(yx) = \frac{1}{2}\beta yx^2$, $0 = 4(xz)z = -2xz^2 = -2x(\delta yx^2 + \varepsilon yx + \theta x^2) = -2\varepsilon x(yx) = \varepsilon yx^2$ implies $\varepsilon = 0$, and $\theta yx^2 = y(\delta yx^2 + \varepsilon yx + \theta x^2) = yz^2 = -2(yz)z = -2\beta(yx)z = -\beta^2 yx^2$ implies $\theta = -\beta^2$. Thus $z^2 = \delta yx^2 - \beta^2 x^2$. Finally, if we define: $u_1 = y$, $u_2 = x$, $u_3 = z - \beta x$, $u_4 = yx^2$, $u_5 = yx$, $u_6 = x^2$, we obtain that $u_1 u_2 = u_5$, $u_1 u_6 = u_4$, $u_2^2 = u_6$, $u_2 u_3 = -\beta u_6$, $u_2 u_5 = -\frac{1}{2}u_4$, $u_3^2 = \delta u_4$, $u_3 u_5 = \beta u_4$, and other products zero.

We note that when $\dim_K(A) = 6$, then Proposition 1.1 implies $1 \leq \dim_K(A^2) \leq 4$. Suppose moreover that $A^3 = 0$ and $\dim_K(A^2) = 4$. Then there exists a subspace A_0 of A such that $A = A_0 \oplus A^2$. Since $\dim_K(A_0) = 2$ and $A^2 = A_0^2$ we conclude that $\dim_K(A^2) \leq 3$, a contradiction. Therefore $\dim_K(A) = 6$ and $A^3 = 0$ imply $1 \leq \dim_K(A^2) \leq 3$.

Proposition 2.7 Suppose that $\dim_K(A) = 6$, with $\dim_K(A^2) = 3$ and $A^3 = 0$.

(a) If for all x, $y \in A$ we have that x^2, y^2, xy are linearly dependent, then there exist a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = u_4$, $u_1 u_2 = \frac{1}{8}\delta^{-1}\varepsilon^{-1}u_4 + 2\delta\varepsilon u_5$, $u_1 u_3 = \frac{1}{4}\delta^{-1}u_4 + \delta u_6$, $u_2^2 = u_5$, $u_2 u_3 = \varepsilon u_5 + \frac{1}{4}\varepsilon^{-1}u_6$, $u_3^2 = u_6$ with $\delta\varepsilon \neq 0$, all other products zero.

(b) If there exist elements y, x in A such that x^2, y^2, xy are linearly independent, then there exist a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = \alpha_1 u_4 + \beta_1 u_5 + \gamma_1 u_6$, $u_1 u_2 = \beta u_5$, $u_1 u_3 = \alpha_0 u_4 + \beta_0 u_5 + \gamma_0 u_6$, $u_2^2 = u_4$, $u_2 u_3 = u_6$, $u_3^2 = u_5$, all other products zero.

Proof. To prove (a), we consider x, y, z in A such that x^2, y^2, z^2 are linearly independent. We will prove that x, y, z, x^2, y^2, z^2 are linearly independent. If $\delta_1 x + \delta_2 y + \delta_3 z + \delta_4 x^2 + \delta_5 y^2 + \delta_6 z^2 = 0$, then $\delta_1 x = -(\delta_2 y +$

$\delta_3 z + \delta_4 x^2 + \delta_5 y^2 + \delta_6 z^2)$ which implies that $\delta_1^2 x^2 = \delta_2^2 y^2 + 2\delta_2\delta_3 yz + \delta_3^2 z^2$. By hypothesis $yz \in < y^2, z^2 >_K$, and so $\delta_1 = 0$. Similarly we prove that $\delta_2 = \delta_3 = 0$, and clearly $\delta_4 = \delta_5 = \delta_6 = 0$. Therefore $\{x, y, z, x^2, y^2, z^2\}$ is a basis of A. By hypothesis $xy = \alpha x^2 + \beta y^2$, $xz = \gamma x^2 + \delta z^2$, $yz = \varepsilon y^2 + \theta z^2$, and also for all α_1, α_2, α_3, β_1, β_2, β_3 in K, the vectors $(\alpha_1 x + \alpha_2 y + \alpha_3 z)^2$, $(\alpha_1 x+\alpha_2 y+\alpha_3 z)(\beta_1 x+\beta_2 y+\beta_3 z)$, $(\beta_1 x+\beta_2 y+\beta_3 z)^2$ are linearly dependent. We have that $(\alpha_1 x + \alpha_2 y + \alpha_3 z)^2 = (\alpha_1^2 + 2\alpha_1\alpha_2\alpha + 2\alpha_1\alpha_3\gamma)x^2 + (\alpha_2^2 + 2\alpha_1\alpha_2\beta + 2\alpha_2\alpha_3\varepsilon)y^2 + (\alpha_3^2 + 2\alpha_1\alpha_3\delta + 2\alpha_2\alpha_3\theta)z^2$, $(\alpha_1 x + \alpha_2 y + \alpha_3 z)(\beta_1 x + \beta_2 y + \beta_3 z) = (\alpha_1\beta_1 + \alpha_1\beta_2\alpha + \alpha_2\beta_1\alpha + \alpha_1\beta_3\gamma + \alpha_3\beta_1\gamma)x^2 + (\alpha_2\beta_2 + \alpha_1\beta_2\beta + \alpha_2\beta_1\beta + \alpha_2\beta_3\varepsilon + \alpha_3\beta_2\varepsilon)y^2 + (\alpha_3\beta_3 + \alpha_1\beta_3\delta + \alpha_3\beta_1\delta + \alpha_2\beta_3\theta + \alpha_3\beta_2\theta)z^2$, and $(\beta_1 x + \beta_2 y + \beta_3 z)^2 = (\beta_1^2 + 2\beta_1\beta_2\alpha + 2\beta_1\beta_3\gamma)x^2 + (\beta_2^2 + 2\beta_1\beta_2\beta + 2\beta_2\beta_3\varepsilon)y^2 + (\beta_3^2 + 2\beta_1\beta_3\delta + 2\beta_2\beta_3\theta)z^2$. We conclude that for all α_1, α_2, α_3, β_1, β_2, β_3 in K, the vectors $(\alpha_1^2 + 2\alpha_1\alpha_2\alpha + 2\alpha_1\alpha_3\gamma,\ \alpha_2^2 + 2\alpha_1\alpha_2\beta + 2\alpha_2\alpha_3\varepsilon,\ \alpha_3^2 + 2\alpha_1\alpha_3\delta + 2\alpha_2\alpha_3\theta)$, $(\alpha_1\beta_1 + \alpha_1\beta_2\alpha + \alpha_2\beta_1\alpha + \alpha_1\beta_3\gamma + \alpha_3\beta_1\gamma,\ \alpha_2\beta_2 + \alpha_1\beta_2\beta + \alpha_2\beta_1\beta + \alpha_2\beta_3\varepsilon + \alpha_3\beta_2\varepsilon,\ \alpha_3\beta_3 + \alpha_1\beta_3\delta + \alpha_3\beta_1\delta + \alpha_2\beta_3\theta + \alpha_3\beta_2\theta)$, $(\beta_1^2 + 2\beta_1\beta_2\alpha + 2\beta_1\beta_3\gamma,\ \beta_2^2 + 2\beta_1\beta_2\beta + 2\beta_2\beta_3\varepsilon,\ \beta_3^2 + 2\beta_1\beta_3\delta + 2\beta_2\beta_3\theta)$ in K^3 are linearly dependent, which implies that $\beta = 2\delta\varepsilon$, $\delta = 2\beta\theta$, $\varepsilon = 2\gamma\beta$, $\theta = 2\alpha\delta$, $\gamma = 2\alpha\varepsilon$, $\alpha = 2\gamma\theta$. We observe that if $0 \in \{\alpha, \beta, \gamma, \delta, \varepsilon, \theta\}$, then $\alpha = \beta = \gamma = \delta = \varepsilon = \theta = 0$. In this case $(x+y)^2$, $(x+z)^2$ and $(x+y)(x+z)$ are linearly independent, a contradiction. Therefore α, β, γ, δ, ε, θ are not zero and we get $\alpha = \frac{1}{8}\delta^{-1}\varepsilon^{-1}$, $\beta = 2\delta\varepsilon$, $\gamma = \frac{1}{4}\delta^{-1}$, $\theta = \frac{1}{4}\varepsilon^{-1}$. Finally, if we define $u_1 = x$, $u_2 = y$, $u_3 = z$, $u_4 = x^2$, $u_5 = y^2$, $u_6 = z^2$, we obtain (a).

Suppose now that there exist y, x in A such that x^2, y^2, xy are linearly independent. In this case it is easy to prove that x, y, x^2, y^2, xy are linearly independent. Let u be an element in A such that $\{u, x, y, x^2, y^2, xy\}$ is a basis of A. Since $ux \in A^2$, then $ux = \alpha x^2 + \beta y^2 + \gamma xy$. If $u_0 = u - \alpha x - \gamma y$, then $u_0 x = \beta y^2$. Finally, if we define $u_1 = u_0$, $u_2 = x$, $u_3 = y$, $u_4 = x^2$, $u_5 = y^2$, $u_6 = xy$, we get (b). □

Proposition 2.8 If $\dim_K(A) = 6$, $A^3 = 0$ and $\dim_K(A^2) = 2$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = \alpha_1 u_5 + \alpha_2 u_6$, $u_1 u_2 = \alpha_3 u_5 + \alpha_4 u_6$, $u_1 u_4 = \alpha_5 u_5 + \alpha_6 u_6$, $u_2^2 = \alpha_7 u_5 + \alpha_8 u_6$, $u_2 u_4 = \alpha_9 u_5 + \alpha_{10} u_6$, $u_3^2 = u_5$, $u_3 u_4 = u_6$, $u_4^2 = \varepsilon u_5$, and other products zero.

Proof. It is possible to prove that there exist elements y, x in A such that x, y, x^2, yx are linearly independent, and $y^2 = \varepsilon x^2$ (see, [4]). We consider u, $v \in A$ such that $\{u, v, x, y, x^2, yx\}$ is a basis of A. Since ux and vx are elements in A^2, then $ux = \alpha x^2 + \beta xy$ and $vx = \gamma x^2 + \delta xy$. If $u_0 = u - \alpha x - \beta y$ and $v_0 = v - \gamma x - \delta y$, then $\{u_0, v_0, x, y, x^2, yx\}$ is a basis of A with $u_0 x = v_0 x = 0$. If we define $u_1 = u_0$, $u_2 = v_0$, $u_3 = x$, $u_4 = y$,

$u_5 = x^2$, $u_6 = yx$, we obtain that $u_1^2 = \alpha_1 u_5 + \alpha_2 u_6$, $u_1 u_2 = \alpha_3 u_5 + \alpha_4 u_6$, $u_1 u_4 = \alpha_5 u_5 + \alpha_6 u_6$, $u_2^2 = \alpha_7 u_5 + \alpha_8 u_6$, $u_2 u_4 = \alpha_9 u_5 + \alpha_{10} u_6$, $u_3^2 = u_5$, $u_3 u_4 = u_6$, $u_4^2 = \varepsilon u_5$, and other products zero. □

Proposition 2.9 If $\dim_K(A) = 6$, $A^3 = 0$ and $\dim_K(A^2) = 1$, then there exists a basis $\{u_1, u_2, u_{3,} u_4, u_5, u_6\}$ such that $u_1^2 = u_6$, $u_2^2 = \beta u_6$, $u_3^2 = \gamma u_6$, $u_4^2 = \delta u_6$, $u_5^2 = \varepsilon u_6$, all other products being zero.

Proof. There is an element u_1 in A such that $u_1^2 \neq 0$, and so $A^2 =< u_1^2 >_K$. We can write A as a direct sum $A = Ku_1^2 \oplus A_0$, where $A_0 = Ku_1 \oplus W$ for some subespace W. The map $f : A_0 \times A_0 \to K$ defined by $xy = f(x, y)u_1^2$ for all x, y in A_0 is a symmetric bilinear form. It is known that there is a basis $\{u_1, u_{2,} u_3, u_4, u_5\}$ of A_0 such that $f(u_i, u_j) = 0$, if $i \neq j$. Finally, if $u_6 = u_1^2$ we have that $\{u_1, u_2, u_{3,} u_4, u_5, u_6\}$ is a basis of A such that $u_1^2 = u_6$, $u_2^2 = \beta u_6$, $u_3^2 = \gamma u_6$, $u_4^2 = \delta u_6$, $u_5^2 = \varepsilon u_6$, all other products being zero. □

3. JORDAN NILALGEBRAS OF NILINDEX 4 AND DIMENSION 6

In this section, A is a Jordan nilalgebra of nilindex 4 and dimension 6. Therefore the identities $x^2(yx) = (x^2y)x$ and $x^4 = (x^2)2 = 0$ are valid in A. By linearization we obtain that also are valid in A the following identities:

$$x^2y^2 + 2(xy)^2 = 0 \tag{3.1}$$

$$x^2(yx) = (x^2y)x = 0 \tag{3.2}$$

In [3], we prove that any Jordan nilalgebra of nilindex $n \geq 4$ and dimension k with $n + 1 \leq k \leq n + 2$, is nilpotent of index n. From this we conclude that $A^4 = 0$.

Proposition 3.1 If $(A^2)2 \neq 0$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = u_3$, $u_2^2 = u_4$, $u_6^2 = -\frac{1}{2}u_5$, $u_1 u_2 = u_6$, $u_3 u_4 = u_5$, all other products being zero.

Proof. Since $(A^2)^2 \neq 0$, there exist x, $y \in A$ such that $x^2y^2 \neq 0$. We know that $2(xy)^2 = -x^2y^2 \neq 0$, $A^4 = 0$ and moreover $A(A^2)^2 \subset AA^3 = A^4 = 0$. We will prove that x, y, x^2, y^2, x^2y^2, xy are linearly independent. It is easy to prove that x^2, y^2, x^2y^2, xy are linearly independent. Now if $\alpha x + \beta y + \gamma x^2 + \delta y^2 + \varepsilon x^2y^2 + \theta xy = 0$, then $\alpha x + \beta y = -(\gamma x^2 + \delta y^2 +$

$\varepsilon x^2y^2 + \theta xy$) which implies $\alpha^2 x^2 + 2\alpha\beta xy + \beta^2 y^2 = \theta^2(xy)^2 + 2\gamma\delta x^2y^2 = (-\frac{1}{2}\theta^2 + 2\gamma\delta)x^2y^2$. Thus we conclude that $\alpha = \beta = 0$, and clearly $\gamma = \delta = \varepsilon = \theta = 0$. Therefore $\{x, y, x^2, y^2, x^2y^2, xy\}$ is a basis of A, and moreover Proposition 1.1 implies that $A^2 =< x^2, y^2, x^2y^2, xy >_K$. Now we will prove that $A^3 =< x^2y^2 >_K$. If $z \in A^3$, then $z = \gamma_1 x^2 + \delta_1 y^2 + \varepsilon_1 x^2y^2 + \theta_1 xy$. So $0 = x^2 z = \delta_1 x^2y^2$ implies $\delta_1 = 0$, $0 = y^2 z = \gamma_1 x^2y^2$ implies $\gamma_1 = 0$, and $0 = (xy)z = \theta_1(xy)^2 = -\frac{1}{2}\theta_1 x^2y^2$ implies $\theta_1 = 0$. Hence $z = \varepsilon_1 x^2y^2$, and thus $A^3 =< x^2y^2 >_K$. Therefore $yx^2 = \delta_0 x^2y^2$, $xy^2 = \delta x^2y^2$, $x(xy) = \gamma(xy)^2$, $y(xy) = \gamma_0(xy)^2$, $x^3 = \alpha x^2y^2$, $y^3 = \alpha_0 x^2y^2$. If $x_0 = x - \delta x^2 - \gamma xy - \alpha y^2$, $y_0 = y - \delta_0 y^2 - \gamma_0 xy - \alpha_0 x^2$ and we define $u_1 = x_0$, $u_2 = y_0$, $u_3 = x_0^2$, $u_4 = y_0^2$, $u_5 = x_0^2y_0^2$, $u_6 = x_0y_0$, then we get that $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ is a basis of A such that $u_1^2 = u_3$, $u_2^2 = u_4$, $u_6^2 = -\frac{1}{2}u_5$, $u_1u_2 = u_6$, $u_3u_4 = u_5$, all other products being zero. □

Lemma 3.2 $1 \leq \dim_K (A^3) \leq 2$

Proof. Since $2 \leq \dim_K(A^2) \leq 4$, then $1 \leq \dim_K(A^3) \leq 3$. Suppose that $\dim_K(A^3) = 3$. Then there exist elements y, z, u, v, x in A such that $A^3 =< uy^2, vz^2, x^3 >_K$. Clearly $x^2 \notin A^3$, and so $A^2 =< x^2, uy^2, vz^2, x^3 >_K$. Hence $y^2 = \alpha x^2 + \beta uy^2 + \gamma vz^2 + \delta x^3$ and $z^2 = \alpha_0 x^2 + \beta_0 uy^2 + \gamma_0 vz^2 + \delta_0 x^3$. Since $A^4 = 0$, we obtain $uy^2 = \alpha ux^2$ and $vz^2 = \alpha_0 vx^2$ with $\alpha \neq 0$ y $\alpha_0 \neq 0$. Therefore $A^2 =< x^2, ux^2, vx^2, x^3 >_K$. Now it is easy to prove that u, v, x, x^2, ux^2, vx^2, x^3 are linearly independent, a contradiction. Therefore $1 \leq \dim_K(A^3) \leq 2$, as desired. □

By Proposition 3.1 we know that there is a unique nilalgebra such that $(A^2)^2 \neq 0$. In the following, we assume that $(A^2)^2 = 0$.

Proposition 3.3 Suppose that $\dim_K (A^2) = 4$ and $\dim_K (A^3) = 2$.

(*a*) If for all y, $x \in A$ we have that yx^2, x^3 are linearly dependent, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = u_2$, $u_1u_2 = u_3$, $u_1u_4 = \gamma u_2 + \delta u_3 + \varepsilon u_5 + \theta u_6$, $u_1u_5 = u_6$, $u^2u_4 = u_3$, $u_4^2 = u_5$, $u_4u_5 = u_6$, all other products being zero.

(*b*) If there exist elements y, x in A such that yx^2, x^3 are linearly independent, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = u_6$, $u_1u_2 = \alpha u_3 + \beta u_5 + \gamma u_6$, $u_1u_3 = u_4$, $u_1u_6 = \delta u_4 + \varepsilon u_5$, $u_2^2 = u_3$, $u_2u_3 = u_5$, $u_2u_6 = \theta u_4 + \sigma u_5$, all other products being zero.

Proof. To prove (a), we consider an element $x \in A$ with $x^3 \neq 0$. By hypothesis, we have that for all $y \in A : yx^2 \in < x^3 >_K$. As $A^4 = 0$, we have that $J = < x^2, x^3 >_K$ is an ideal of A, and moreover A^3 is not a subset of J. Now if $y^3 \in J$ for all $y \in A$, then the quotient algebra $\overline{A} = A/J$ is a nilalgebra of nilindex 3 with $\dim_K(\overline{A}) = 4$ and $\overline{A}^3 \neq \overline{0}$ which is a contradiction, since by Lemma 2.4 we know that $\dim_K(\overline{A}) \geq 5$. Therefore there exists $y \in A$ such that $y^3 \notin < x^2, x^3 >_K$. By hypothesis $yx^2 = \alpha x^3$, $xy^2 = \beta y^3$. Now it is possible to prove that x, y, x^2, x^3, y^2, y^3 are linearly independent, and so $xy = \gamma_0 x^2 + \delta_0 x^3 + \varepsilon_0 y^2 + \theta_0 y^3$. By hypothesis for all γ_1, δ_1, α_1, β_1 in K, we have that the vectors $(\gamma_1 x + \delta_1 y)^3$, $(\alpha_1 x + \beta_1 y)(\gamma_1 x + \delta_1 y)^2$ are linearly dependent. Now we have that $(\gamma_1 x + \delta_1 y)^3 = (\gamma_1^3 + 2\gamma_1^2\delta_1\gamma_0 + \gamma_1^2\delta_1\alpha + 2\gamma_1\delta_1^2\gamma_0\alpha)x^3 + (\gamma_1\delta_1^2\beta + 2\gamma_1^2\delta_1\beta\varepsilon_0 + \delta_1^3 + 2\gamma_1\delta_1^2\varepsilon_0)y^3$ and $(\alpha_1 x + \beta_1 y)(\gamma_1 x + \delta_1 y)^2 = (\gamma_1^2\alpha_1 + 2\gamma_1\alpha_1\delta_1\gamma_0 + \gamma_1^2\beta_1\alpha + 2\gamma_1\delta_1\beta_1\alpha\gamma_0)x^3 + (\delta_1^2\alpha_1\beta + 2\gamma_1\alpha_1\delta_1\beta\varepsilon_0 + \delta_1^2\beta_1 + 2\gamma_1\delta_1\beta_1\varepsilon_0)y^3$. We conclude that for all γ_1, δ_1, α_1, β_1 in K the vectors $(\gamma_1^3 + 2\gamma_1^2\delta_1\gamma_0 + \gamma_1^2\delta_1\alpha + 2\gamma_1\delta_1^2\gamma_0\alpha,\ \gamma_1\delta_1^2\beta + 2\gamma_1^2\delta_1\beta\varepsilon_0 + \delta_1^3 + 2\gamma_1\delta_1^2\varepsilon_0)$ and $(\gamma_1^2\alpha_1 + 2\gamma_1\alpha_1\delta_1\gamma_0 + \gamma_1^2\beta_1\alpha + 2\gamma_1\delta_1\beta_1\alpha\gamma_0,\ \delta_1^2\alpha_1\beta + 2\gamma_1\alpha_1\delta_1\beta\varepsilon_0 + \delta_1^2\beta_1 + 2\gamma_1\delta_1\beta_1\varepsilon_0)$ in K^2 are linearly dependent, which implies that $\alpha\beta = 1$. Finally, if $u_1 = x$, $u_2 = x^2$, $u_3 = x^3$, $u_4 = \beta y$, $u_5 = \beta^2 y^2$, $u_6 = \beta^3 y^3$, we obtain (a). To prove (b), we consider y, $x \in A$ such that yx^2, x^3 are linearly independent. Then $A^3 = < yx^2, x^3 >_K$ and x^2, yx^2, x^3 are linearly independent. As $\dim_K(A^2) = 4$, there exists $z \in A$ such that $A^2 = < x^2, yx^2, x^3, z^2 >_K$. It is easy to prove that $\{y, x, x^2, yx^2, x^3, z^2\}$ is a basis of A. Now if $z = \alpha_1 y + \alpha_2 x + \alpha_3 x^2 + \alpha_4 yx^2 + \alpha_5 x^3 + \alpha_6 z^2$, then $z^2 - (\alpha_1^2 y^2 + 2\alpha_1\alpha_2 xy + \alpha_2^2 x^2) \in A^3$ which implies $\alpha_1 \neq 0$. If $y_0 = \alpha_1 y + \alpha_2 x$, then $y_0^2 \notin < x^2, yx^2, x^3 >_K = < x^2, y_0x^2, x^3 >_K$, and so $A^2 = < x^2, y_0x^2, x^3, y_0^2 >$. If $y_0 x = \alpha x^2 + \lambda y_0 x^2 + \beta x^3 + \gamma y_0^2$ and $x_0 = x - \lambda x^2$, then $y_0 x_0 = \alpha x^2 + \beta x^3 + \gamma y_0^2 \in < x_0^2, x_0^3, y_0^2 >_K$. Therefore we can assume that there exist elements y, x in A such that $\{y, x, x^2, yx^2, x^3, y^2\}$ is a basis of A with $yx = \alpha x^2 + \beta x^3 + \gamma y^2$, $y^3 = \delta yx^2 + \varepsilon x^3$ and $xy^2 = \theta yx^2 + \sigma x^3$. If we define $u_1 = y$, $u_2 = x$, $u_3 = x^2$, $u_4 = yx^2$, $u_5 = x^3$, $u_6 = y^2$, we obtain (b).

Proposition 3.4 If $\dim_K(A^2) = 4$ and $\dim_K(A^3) = 1$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = u_5$, $u_1u_2 = u_6$, $u_1u_5 = \beta u_4$, $u_1u_6 = \gamma u_4$, $u_2^2 = u_3$, $u_2u_3 = u_4$, $u_2u_5 = \delta u_4$, $u_2u_6 = \varepsilon u_4$, all other products being zero.

Proof. We consider $x \in A$ such that $x^3 \neq 0$. Since $\dim_K(A^2) = 4$, there are y, $z \in A$ such that $A^2 = < x^2, x^3, y^2, z^2 >_K$. We have that y, x, x^2, x^3, y^2, z^2 are linearly independent. In fact: if $\alpha_1 y + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 +$

$\alpha_5y^2 + \alpha_6z^2 = 0$, then $\alpha_1y = -(\alpha_2x + \alpha_3x^2 + \alpha_4x^3 + \alpha_5y^2 + \alpha_6z^2)$ which implies $\alpha_1^2y^2 = \alpha_2^2x^2 + v$ with $v \in A^3 =< x^3 >_K$. Hence $\alpha_1 = \alpha_2 = 0$, and so $\{y, x, x^2, x^3, y^2, z^2\}$ is a basis of A. If $z = \beta_1y + \beta_2x + \beta_3x^2 + \beta_4x^3 + \beta_5y^2 + \beta_6z^2$, then $z^2 - (\beta_1^2y^2 + 2\beta_1\beta_2yx + \beta_2^2x^2) \in A^3$ which implies that $yx \notin < x^2, x^3, y^2 >_K$, and therefore $A^2 =< x^2, x^3, y^2, yx >_K$. We see that as $yx^2 = \alpha x^3$, then $x^2(y - \alpha x^2) = 0$. Therefore we can assume that in the basis $\{y, x, x^2, x^3, y^2, yx\}$ of A we have that $yx^2 = 0$, and moreover $y^3 = \beta x^3$, $y(yx) = \gamma x^3$, $xy^2 = \delta x^3$, $x(yx) = \varepsilon x^3$. Finally, if we define $u_1 = y$, $u_2 = x$, $u_3 = x^2$, $u_4 = x^3$, $u_5 = y^2$, $u_6 = yx$, we obtain our Proposition.

Proposition 3.5 If $\dim_K(A^2) = 3$ and $\dim_K(A^3) = 2$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A sucha that $u_1^2 = \gamma_1u_4 + \gamma_2u_5 + \gamma_3u_6$, $u_1u_2 = \delta_1u_4 + \delta_1u_4 + \delta_2u_5 + \delta_3u_6$, $u_1u_3 = \beta u_5$, $u_1u_4 = \lambda_1u_5 + \lambda_2u_6$, $u_2^2 = \varepsilon_1u_4 + \varepsilon_2u_5 + \varepsilon_3u_6$, $u_2u_4 = u_5$, $u_3^2 = u_4$, $u_3u_4 = u_6$, all other products being zero.

Proof. By Proposition 3.1, it is clear that $(A^2)^2 = 0$. We consider $x \in A$ such that $x^3 \neq 0$. We note that if $I =< x^2, x^3 >_K$ is an ideal of A, then $yx^2 \in< x^3 >_K$ for all $y \in A$, and A^3 is not a subset of I. If I is an ideal of A and $z^3 \in I$ for all $z \in A$, then the quotient algebra $\overline{A} = A/I$ is of nilindex 3 with $\overline{A}^3 = \overline{0}$, which implies $\dim_K(\overline{A}^3) \geq 5$, a contradiction. Hence if I is an ideal of A, then there is $y \in A$ such that $y^3 \notin I$, and so $A^2 =< x^2, x^3, y^3 >_K$. Since $y^2 \in A^2 =< x^2, x^3, y^3 >_K$, then $y^3 \in< yx^2 >_K \subset < x^3 >_K$, a contradiction. Therefore we conclude that $I =< x^2, x^3 >_K$ is not an ideal of A and so there exists an element $y \in A$ such that yx^2, x^3 are linearly independent. In this case it is possible to prove that y, x, x^2, yx^2, x^3 are linearly independent, and thus $A^2 =< x^2, yx^2, x^3 >_K$ and $A^3 =< yx^2, x^3 >_K$. If $yx = \beta_1x^2 + \beta_2yx^2 + \beta_3x^3$, then $yx_0 = \beta_1x^2 + \beta_3x^3 \in< x_0^2, x_0^3 >_K$ where $x_0 = x - \beta_2x^2$. Thus we can suppose that $yx = \beta_1x^2 + \beta_3x^3$, which implies $y_0x = 0$ where $y_0 = y - \beta_1x - \beta_3x^2$. Therefore we can assume that y, x, x^2, yx^2, x^3 are linearly independent with $yx = 0$. Now it is easy to find an element $z \in A$ such that $\{z, y, x, x^2, yx^2, x^3\}$ is a basis of A with $xz = \beta yx^2$. Moreover we have that $z^2 = \gamma_1x^2 + \gamma_2yx^2 + \gamma_3x^3$, $yz = \delta_1x^2 + \delta_2yx^2 + \delta_3x^3$, $y^2 = \varepsilon_1x^2 + \varepsilon_2yx^2 + \varepsilon_3x^3$, $zx^2 = \lambda_1yx^2 + \lambda_2x^3$. If we define $u_1 = z$, $u_2 = y$, $u_3 = x$, $u_4 = x^2$, $u_5 = yx^2$, $u_6 = x^3$, we obtain our Proposition.

Proposition 3.6 If $\dim_K(A^2) = 3$ and $\dim_K(A^3) = 1$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = \alpha_1u_4 + \alpha_2u_5 + \alpha_3u_6$,

$u_1u_2 = \beta_1u_4 + \beta_2u_5 + \beta_3u_6$, $u_1u_3 = \gamma u_6$, $u_1u_4 = \delta_0u_5$, $u_1u_6 = \lambda u_5$, $u_2^2 = u_6$, $u_2u_3 = \gamma_1u_4 + \gamma_2u_5 + \gamma_3u_6$, $u_2u_4 = \delta u_5$, $u_2u_6 = \varepsilon u_5$, $u_3^2 = u_4$, $u_3u_4 = u_5$, $u_3u_6 = \theta u_5$, all other products being zero.

Proof. Clearly $(A^2)^2 = 0$. We consider $x \in A$ with $x^3 \neq 0$. Then $A^3 =< x^3 >_K$ and there is $y \in A$ such that $A^2 =< x^2, x^3, y^2 >_K$. It is easy to show that y, x, x^2, x^3, y^2 are linearly independent. It is easy to find an element $z \in A$ such that $\{z, y, x, x^2, x^3, y^2\}$ is a basis of A with $zx = \gamma y^2$. If we define $u_1 = z$, $u_2 = y$, $u_3 = x$, $u_4 = x^2$, $u_5 = x^3$, $u_6 = y^2$, we obtain our Proposition.

Proposition 3.7 If $\dim_K (A^2) = 2$ and $\dim_K (A^3) = 1$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = \alpha_1u_5 + \alpha_2u_6$, $u_1u_2 = \beta_1u_5 + \beta_2u_6$, $u_1u_3 = \gamma_1u_5 + \gamma_2u_6$, $u_1u_5 = \alpha u_6$, $u_2^2 = \delta_1u_5 + \delta_2u_6$, $u_2u_3 = \varepsilon_1u_5 + \varepsilon_2u_6$, $u_2u_5 = \beta u_6$, $u_3^2 = \lambda_1u_5 + \lambda_2u_6$, $u_3u_5 = \gamma u_6$, $u_4^2 = u_5$, $u_4u_5 = u_6$, all other products being zero.

Proof. We consider $x \in A$ with $x^3 \neq 0$. Then $A^3 =< x^3 >_K$ and $A^2 =< x^2, x^3 >_K$. It is easy find elements y, z, v in A such that $\{y, z, v, x, x^2, x^3\}$ is a basis of A with $yx = zx = vx = 0$. Now we have that $y^2 = \alpha_1x^2 + \alpha_2x^3$, $yz = \beta_1x^2 + \beta_2x^3$, $yv = \gamma_1x^2 + \gamma_2x^3$, $yx^2 = \alpha x^3$, $z^2 = \delta_1x^2 + \delta_2x^3$, $zv = \varepsilon_1x^2 + \varepsilon_2x^3$, $zx^2 = \beta x^3$, $v^2 = \lambda_1x^2 + \lambda_2x^3$, $vx^2 = \gamma x^3$. Finally, if $u_1 = y$, $u_2 = z$, $u_3 = v$, $u_4 = x$, $u_5 = x^2$, $u_6 = x^3$, we obtain our Proposition.

4. JORDAN NILALGEBRAS OF NILINDEX k AND DIMENSION 6 WITH k ≥ 5

In [2], we describe Jordan nilalgebras of nilindex $n$and dimension $n+1$. In this work, we find the following results:

Proposition 4.1 If A is a Jordan nilalgebra of nilindex 5 and dimension $6, \dim_K(A^2) = 4$ and $\dim_K(A^3) = 2$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = \alpha u_2 + \gamma_2u_4 + \gamma_3u_5 + \gamma_4u_6$, $u_1u_2 = \beta_0u_5 + \gamma_0u_6$, $u_1u_3 = u_2$, $u_1u_5 = -2\beta u_6$, $2u_2^2 = \beta(\alpha - 4\beta)u_6$, $u_2u_3 = \beta u_5 + \gamma u_6$, $u_3^2 = u_4$, $u_3u_4 = u_5$, $u_3u_5 = u_6$, $u_4^2 = u_6$, all other products being zero. Moreover, if $\beta = 0$ then $\gamma_2 = \beta_0 = 0$, if $\beta \neq 0$ and $\alpha = 4\beta$ then $\gamma_2 = -4\beta^2$, $\beta_0 = -2\beta^2$, if $\beta \neq 0$ and $\alpha \neq 4\beta$ then $\alpha = -4\beta$, $\gamma_2 = -4\beta^2$ and $\beta_0 = -6\beta^2$.

Proposition 4.2 If A is a Jordan nilalgebra of nilindex 5 and dimension 6, $\dim_K(A^2) = 4$ and $\dim_K(A^3) = 3$, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1u_4 = u_2$, $u_1^2 = \lambda u_2 + \delta u_4 + \gamma u_5 + \varepsilon u_6$, $u_1u_2 = \delta u_6$, $u_3^2 = u_4$, $u_3u_4 = u_5$, $u_3u_5 = u_6$, $u_4^2 = u_6$, all other products zero.

Proposition 4.3 If A is a Jordan nilalgebra of nilindex 5 and dimension 6 and $\dim_K(A^2) = 3$, and then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$of A such that $u_1u_3 = \alpha u_5$, $u_1^2 = \beta u_5 + \gamma u_6$, $u_2u_3 = \alpha_0 u_5$, $u_2^2 = \beta_0 u_5 + \gamma_0 u_6$, $u_1u_2 = \delta u_5 + \varepsilon u_6$, $u_3^2 = u_4$, $u_3u_4 = u_5$, $u_3u_5 = u_6$, $u_4^2 = u_6$, all other products being zero.

In [1], the authors proved the following result:

Proposition 4.4 If A is a Jordan nilalgebra of nilindex 6 and dimension 6, then there exists a basis $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ of A such that $u_1^2 = \beta u_5 + \gamma u_6$, $u_1u_2 = \alpha u_5$, $u_2^2 = u_3$, $u_2u_3 = u_4$, $u_2u_4 = u_5$, $u_2u_5 = u_6$, $u_3^2 = u_5$, $u_3u_4 = u_6$, all other products zero.

Moreover in this case it is possible to find five classes of algebras which are not isomorphic (see [1], Theorem 3).

Remark Finally, it is clear that there is a unique Jordan nilalgebra of nilindex 7 and dimension 6.

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