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THE HOMOTOPY TYPE OF INVARIANT CONTROL SET

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Abstract

Let G be a noncompact semi-simple Lie group, consider S a semigroup which contains a large Lie semigroup. We computer the homotopy type $\pi_n(C)$, where C is the invariant control set of the homogeneous space G/P with $P \subset G$ a parabolic subgroup of G.

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1. Introduction

The homotopy type of semigroups was studied in [10], making use of the invariant control set C resulted from action of S in $G/AN = K$, where K is the compact subgroup of Iwasawa decomposition of G. It was showed that $\pi_n(S)$ is isomorphic to $\pi_n(K(S))$, where $K(S)$ is a certain compact and connected subgroup in K. More formally, let AN be the solvable Iwasawa component of G and form the coset space G/AN, which is diffeomorphic to the compact Iwasawa component K. The semigroup S acts on G/AN by the restriction of the G-action. Typically this action of S is not transitive. Actually, if $S \neq G$ there are proper compact subsets of G/AN invariant under S, the so-called invariant control sets. Let C be an invariant control set of S in G/AN. Considering $P(S)$ the parabolic subgroup which give the parabolic type of S, the main result of the paper [10] states that the homotopy groups of S are isomorphic to those of $C_{P(S)}$, where $C_{P(S)}$ is the invariant control set in $G/P(S)$. Hence, $C_{P(S)}$ can be continuously deformed into $K(S)$ implying that the homotopy type of this control set is the compact subgroup $K(S) \subset K$.

A natural question that arose is about the homotopy type of invariant control set C_Q in others G/Q, where Q is a parabolic subgroup of G. Here we study this question and establish the homotopy groups of C_Q by using the homotopy type of $C_{P(S)}$. In order to get this we consider the equivariant fibration $G/P \to G/Q$ and $G/P \to G/P(S)$ where P is a minimal parabolic subgroup. Take $C_P \subset G/P, C_{P(S)} \subset G/P(S)$ and $C_Q \subset G/Q$ their respective invariant control sets. As C_P can be write as product of $C_{P(S)}$ by $P(S)/P$ and $C_{P(S)}$ is contractible we have that C_P is homotopic to $P(S)/P$. Now, showing that any cycle γ in C_Q can be deformed to $P(S)/Q \cap P(S)$, its possible to prove that the homotopy type of C_Q is given in function of $K(S)$.

Concerning of structure of this paper, in the next section we establish a quick summary of the theory of the flag manifold (in sense of the G-homogeneous spaces), semi-simple Lie groups and parabolic subgroup as well as the study of the action of semigroups in flag manifolds which give origin to control sets, we also recall that the geometry of this S-control sets (more formal, the one invariant) allow us to distinguish the semigroups S in parabolic types, and moreover, allow us study the topology of S (see [8], [9] and [10]), here we unify the different definitions given in the references cited above. And in this section we recall also some results about dual flag which are important in future sections. In section 3, the study of the invariant control sets of $G/AN \approx K$ are recalled as well as its homotopy type, where $G = KAN$ is the Iwasawa decomposition. The main results are stated and proved in section 4 and 5, where are studied the homotopy type of the invariant control sets in others G/Q, with Q a parabolic subgroup.

2. Semigroups, control sets and homogeneous spaces

The purpose of this section is to get important notations and background results that will be important to future sections. As this paper makes use of the several notations and results proved in [10], in this section we reproduce some notations and definitions of [10].

2.1. Flag manifolds

Denote by G a connected noncompact Lie group with finite center and put g as its Lie algebra. The flag manifolds of G can be described by subsets of the set of simple (restricted) roots of g.

Choose an Iwasawa decomposition $g = k \oplus a \oplus n$. Let Π be the set of roots of the pair (g, a) and Π^+ [respectively Σ] be the set of positive [respectively simple] roots giving rise to the nilpotent component n, that is,

$$n = \sum_{\alpha \in \Pi^+} g_\alpha,$$

where g_α stands for the α-root space. Let m be the centralizer of a in k and put $p = m \oplus a \oplus n$ for the corresponding minimal parabolic subalgebra. By definition, the maximal flag manifold B of G is the set of subalgebras $\mathrm{Ad}\,(G)\,p$, where Ad stands for the adjoint representation of G in g. There is an identification of B with G/P where P is the normalizer of p in G. Furthermore, $P = MAN$, $A = \exp a$, $N = \exp n$ and M is the centralizer of A in $K = \exp k$ ($M = \{u \in K : uh = hu \text{ for all } h \in A\}$). Now, considering M^* the normalizer of A in K, we have that the finite group $W = M^*/M$ is the Weyl group of the pair (g, a).

For a given G there exists only a finite number of flag manifolds, where the maximal one fibres equivariantly over others. Precisely, given a subset $\Theta \subset \Sigma$, denote by p_Θ the corresponding parabolic subalgebra, namely,

$$p_\Theta = n^-(\Theta) \oplus p,$$

where $n^-(\Theta)$ is the subalgebra spanned by the root spaces $g_{-\alpha}$, $\alpha \in \langle\Theta\rangle$. Here $\langle\Theta\rangle$ is the set of positive roots generated by Θ. The set of parabolic subalgebras conjugate to p_Θ identifies with the homogenous space G/P_Θ, where P_Θ is the normalizer of p_Θ in G:

$$P_\Theta = \{g \in G : \mathrm{Ad}\,(g)\, p_\Theta = p_\Theta\}.$$

Then with the above construction we have the flag manifold $B_\Theta = G/P_\Theta$, $\Theta \subset \Sigma$. Furthermore, given two subsets $\Theta_1 \subset \Theta_2 \subset \Sigma$, the corresponding parabolic subgroups satisfy $P_{\Theta_1} \subset P_{\Theta_2}$, so that there is a canonical fibration $G/P_{\Theta_1} \to G/P_{\Theta_2}$, $gP_{\Theta_1} \mapsto gP_{\Theta_2}$. In particular, $B = B_\emptyset$ projects onto every flag manifold B_Θ. And from the structure of the parabolic subgroup P_Θ the fiber P_Θ/P of $B \to B_\Theta$ is obtained.

Let

$$a^+ = \{H \in a : \alpha\,(H) > 0 \text{ for all } \alpha \in \Sigma\}$$

be the Weyl chamber associated to Σ. We say that $X \in g$ is split-regular in case $X = \mathrm{Ad}\,(g)\,(H)$ for some $g \in G$, $H \in a^+$. Analogously, $x \in G$ is said to be split-regular in case $x = ghg^{-1}$ with $h \in A^+ = \exp a^+$, that is, $x = \exp X$, with X split-regular in g.

Let $n^- = \sum_{\alpha\in\Pi} g_{-\alpha}$ be the nilpotent subalgebra opposed to n. Put $N^- = \exp n^-$. Then in any flag manifold B_Θ, the orbit (called open Bruhat cell) $\mathrm{Ad}\,(N^-)\, p_\Theta$ is open and dense. Furthermore, if $h \in A^+$ then $\lim h^k y = p_\Theta$ for any $y \in \mathrm{Ad}\,(N^-)\, p_\Theta$. In other words, p_Θ is an attractor in B_Θ for any $h \in A^+$, having $\mathrm{Ad}\,(N^-)\, p_\Theta$ as stable manifold. The same way, $\mathrm{Ad}\,(g)\, p_\Theta$ is the attractor fixed point in B_Θ of the split-regular $g = xhx^{-1}$ such that $\mathrm{Ad}\,(xN^-x^{-1})$ is the open and dense stable manifold. In the sequel we denote the attractor fixed point of g by $\mathrm{at}\,(g)$, while the corresponding open cell, which is the stable manifold, is denoted by $\mathrm{st}\,(h)$. Although this notation does not specify the flag manifold under consideration, this will become clear from the context.

Following closely the notation of Warner [12], Section 1.2., we denote by a_Θ the annihilator of Θ in a:

$$a_\Theta = \{H \in a : \alpha\,(H) = 0 \text{ for all } \alpha \in \Theta\}.$$

Let L_Θ stand for the centralizer of a_Θ in G and put $M_\Theta\,(K) = L_\Theta \cap K$ for the centralizer of a_Θ in K. Let M_Θ^0 (identity component of M_Θ) for the connected subgroup whose Lie algebra is m_Θ, semi-simple component of the Lie algebra of L_Θ. Put $M_\Theta = M_\Theta\,(K)\, M_\Theta^0$. By Bruhat-Moore Theorem (see [12], Theorem 1.2.4.8) we have $P_\Theta = M_\Theta A_\Theta N_\Theta$ and $P_\Theta = M_\Theta\,(K)\, AN$.

In this context we can define W_Θ as the subset of W generated by the reflexions defined by the elements in Θ, in the next subsection we characterize W_Θ making use of control set.

2.2. Semigroups actions

Now we recall some facts about the general theory of semigroups as well as about the action of semigroups in flag manifolds. Consider a semigroup S with $\text{int}S \neq \emptyset$, and contained in a semi-simple Lie Group G with finite center. Take the action of S in the flag manifolds of G. It was proved in [11], Theorem 6.2, that S is not transitive in B_Θ unless $S = G$. Moreover, there exists just one closed invariant subset $C_\Theta \subset B_\Theta$ such that Sx is dense in C_Θ for all $x \in C_\Theta$. This subset is called the invariant control set of S in C_Θ (abbreviated S-i.c.s.).

We can be more formal,

Definition 2.1. *A control set for S in an homogeneous space G/H, with H a closed subgroup of the Lie group G, is a subset $D \subset G/H$ satisfying*

1. $D \subset \text{cl}(Sx)$ *for all* $x \in D$.
2. D *is maximal with this property.*
3. $\text{int}D \neq \emptyset$.

There exists a natural partial order between control sets: D_1 is smaller than D_2 in case it is possible to attain D_2 from D_1, that is if there are $x \in D_1$ and $g \in S$ such that $gx \in D_2$. With respect to this order a control set is maximal if it is S-invariant ($SD \subset D$) and it is minimal if it is S^{-1}-invariant. In case G/H is compact there is equivalence between D being maximal and S-invariant (respectively, minimal and S^{-1}-invariant).

We would like to enunciate the proposition that extend some results about control set to other ones by using fibrations.

Proposition 2.2. *Let Q be a parabolic subgroup, $P \subset Q$ a minimal parabolic subgroup and $\pi : G/P \to G/Q$ the canonical equivariant fibration. Let $E \subset G/Q$ be a control set for S, with $E^0 \neq \emptyset$. Then*

1. *There exists a control set for S $D \subset G/P$ satisfying $\pi(D^0) = E^0$, with $D^0 \neq \emptyset$.*
2. $\pi(D^{0\prime}) = E^0$ *if D' is a control set for S such that $D^{0\prime} \cap \pi^{-1}(E^0) \neq \emptyset$ and $D^{0\prime} \neq \emptyset$.*
3. *If E is invariant $\pi^{-1}(E)$ contains the invariant control set and vice-versa.*

Proof: See [11]. □

In order to finish this subsection, we have the following definition that will be useful in the others sections. Consider G a connected Lie group with Lie algebra g. We say a semigroup $S \subset G$ is a ray semigroup provided there exists a subset $U \subset g$ such that

$$S = \langle \exp\left(R^{+}U\right)\rangle.$$

In this case S is said to be generated by U and we can claim that these kind of condition is fulfilled by Lie semigroups (closure of ray semigroup).

Returning to flag manifold, since S is not transitive, $C_\Theta \neq B_\Theta$. The fact that Sx is dense in C_Θ for all $x \in C_\Theta$ implies the existence of an open subset $C_\Theta^0 \subset C_\Theta$ such that for all $x, y \in C_\Theta^0$ there exists $g \in S$ with $gx = y$. Furthermore, C_Θ^0 is dense in C_Θ. This subset C_Θ^0 is called the set of transitivity of C_Θ, and is given by $C_\Theta^0 = (\mathrm{int}S)\, x \cap (\mathrm{int}S)^{-1} x$, for all $x \in C_\Theta$, when the set of transitivity of the control set is non empty, the control set is called effective control set. In case S is a ray semigroup, it follows that $C_\Theta^0 = \mathrm{int}C_\Theta$ (see [11], Section 2).

2.3. Parabolic type

Now, we recall some facts about control sets that permit us to classify the semigroup. The classification and study about the parabolic type of the semigroup was made in [8] and [9], but making use of different definitions, in this subsection we proof the equivalence of this definitions.

Consider for a moment the maximal flag manifold $B = B_\emptyset$, from [11] we know that for each $w \in W$ there exists a control set $D(w)$ such that $x \in D(w)^0$ if and only if x is the w-fixed point for some split-regular $h \in \mathrm{int}S$. Moreover, any control set D is $D(w)$ for some $w \in W$. The assignment $w \mapsto D(w)$ permits to single out, from S, a flag manifold $B(S)$ as follows: Take a split-regular $h \in \mathrm{int}S$ and denote by $A^+ = \exp a^+$ the Weyl chamber containing h. Recall that we write W_{A^+} to emphasize the bijection of W with subsets of B. Let $1 \in W_{A^+}$ be the identity. Then the control set $D(w_0)$ is the only minimal control set in B where w_0 is the principal involution of W.

The subset $W_{A^+} = \{w \in W : D(w) = D(1)\}$ is a parabolic subgroup of W, that is, it is generated by the reflections with respect to the simple roots in a proper subset $\Theta(S) \subset \Sigma$, where $D(1)$ is the invariant control set for S. Hence we can denote W_{A^+} by $W(S)$ and put $B(S) = B_{\Theta(S)}$.

Hence, we can state the following Theorem, where the proof can be found in [11] (Theorem 4.3).

Theorem 2.3. *Consider A^+ the Weyl chamber that intercept* $\text{int}(S)$ *and let Σ be the set of simple roots defined by A^+. Then $W(S) = W_\Theta$ for some $\Theta \subset \Sigma$. And furthermore, take P_Θ the parabolic subgroup associated to Θ, $\pi : G/P \to G/P_\Theta$ the canonical projection and C the invariant control set for S in G/P. Then $C = \pi^{-1}(\pi(C))$.*

There exists a maximal Θ satisfying this property, i.e., there exists a maximal Θ such that $\pi^{-1}(C_\Theta) \subset B$ is the invariant control set B. In fact, the existence is found in [11], Theorem 4.3. In order to show the uniqueness suppose that there exist Θ_1 and Θ_2 satisfying the above property, then we have that $W_{\Theta_1} = W(S)$ and $W_{\Theta_2} = W(S)$ (see [11], Theorem 4.3). Hence, $W_{\Theta_1} = W_{\Theta_2}$. Knowing that those two subgroups are parabolics, we have that the set of reflections defined by Θ_1 and the one defined by Θ_2 are equals (because there exists a one to one correspondence between the set of reflections defined by the roots in Θ and W_Θ, see [3] Section 1.2). Hence, $\Theta_1 = \Theta_2$.

Now, for the sake of completeness we show the equivalence between the definitions of type of S, due to San Martin (see [8], [9] and [10]).

Proposition 2.4. *There exists $\Theta \in \Sigma$ such that $\pi_\Theta^{-1}(C_\Theta) \subset G/P$ is the invariant control set for S and such that C_Θ is contained in stable manifold to some split-regular element if and only if this Θ is the maximal subset such that $\pi_\Theta^{-1}(C_\Theta) \subset G/P$ is the invariant control set for S.*

Proof: Suppose that Θ is the maximal one, then by Proposition 6.8 in [11] we have that $W(S) = W_\Theta$, hence, using the Proposition 4.8 in [11], we conclude that C_Θ is contained in the above stable manifold.

Now, supposing that there exists Θ_1 containing Θ such that $\pi_{\Theta_1}^{-1}(C_{\Theta_1})$ is the invariant control set for S in the maximal flag, we get the following contradiction: C_Θ can not be in the stable manifold. In fact, if invariant control set for S, in the maximal flag, is $\pi_{\Theta_1}^{-1}(C_{\Theta_1})$ then the invariant control set for S in the B_Θ is $C_\Theta = \pi_\Theta\left(\pi_{\Theta_1}^{-1}(C_{\Theta_1})\right)$. As $\Theta \subset \Theta_1$, we have that $C_\Theta = \left(\pi_{\Theta_1}^{\Theta}\right)^{-1}(C_{\Theta_1})$, where $\pi_{\Theta_1}^{\Theta} : B_\Theta \to B_{\Theta_1}$ is the canonical projection. But one inverse image of this projection can not be contained in the stable manifold.

This show that Θ is maximal. And in the comment before this Proposition it was showed that this one maximal is unique. □

Definition 2.5. *We denote such Θ by $\Theta(S)$ and we say that it is the parabolic type of S. Remember that any proper semigroup with non-empty interior is of the parabolic type Θ for some Θ, (see [11]).*

An important property of the flag $B(S)$ is,

Proposition 2.6. *With the above notations let $C \subset B(S)$ be the invariant control set. If $\Theta \subset \Theta(S)$ and $\pi : B_\Theta \to B(S)$ is the canonical fibration then $\pi^{-1}(C)$ is the invariant control set for S in B_Θ.*

Proof: See Theorem 4.3 in [11]. □

We can yet to improve the statement of the last results.

Proposition 2.7. *Let $C \subset B(S)$ be the invariant control set. Then C is contained in the stable manifold $\mathrm{st}\,(h)$ for any split-regular $h \in \mathrm{int}(S)$. Moreover if $\Theta \subset \Theta(S)$ and $\pi : B_\Theta \to B(S)$ is the canonical fibration then $\pi^{-1}(C)$ is the invariant control set for S in B_Θ.*

Proof: In order to obtain the proof, see Proposition 4.8 and Theorem 4.3 in [11]. □

In short, we can say that the semigroups in G can be distinguished according to the geometry of their invariant control sets and when $\Theta = \Theta\,(S)$, the invariant control set $C_{\Theta(S)}$ has the following nice property: The set $R\,(S)$ of split-regular elements in $\mathrm{int}S$ is not empty. Let $h \in R\,(S)$ and let as before $\mathrm{at}\,(h)$ and $\mathrm{st}\,(h)$ stand for its attractor and stable open cell in $B_{\Theta(S)}$. Then $C_{\Theta(S)} \subset \mathrm{st}\,(h)$ and $\mathrm{at}\,(h) \in C^0_{\Theta(S)}$. As we can see in [10], this property completely describes the topology of the invariant control set in B.

2.4. Dual flag manifold

The purpose of this subsection is to set some notation concerning flag dual that will be necessary.

Let W be the Weyl group of G, we recall that $w \in W$ is an involution then $w = r_1 \cdots r_k$ is a product of reflections with respect to the simple roots pairwise ortogonals. It is called principal involution if the length $l(w) = l(r_1 \cdots r_k) = k$ is maximal, or rather, if w is the only element of W such that $w(\Sigma) = -\Sigma$. Furthermore, this principal involution is equal to $-\iota$, where ι is an involutive automorphism of the Dynkin diagram associated with Σ (see [7], Chapter 9) and we can define $\Theta^* = -\iota(\Theta)$. The flag manifold $B_{\Theta^*} = G/P_{\Theta^*}$ is called to be dual to B_Θ.

Returning to parabolic subgroup, take a split-regular $h \in \text{int}S$ and let $A^+ = \exp\mathfrak{a}^+$ be the Weyl chamber containing h, we have that $W_{A^-}(S^{-1}) = W_{A^+}(S)$, where $A^{-1} = (A^+)^{-1}$ (see Corollary 4.6 in [11] and Proposition 6.1 in [8]).

From this proposition we can prove that $B(S^{-1})$ is the dual to $B(S)$, this proof was made in [11], but in order to clarify the notations we offer here the proof. Consider $\Theta = \Theta(S)$, that is, S a semigroup of parabolic type Θ, what is equal to say that S has parabolic type of the flag manifold $B(S)$, then

Proposition 2.8. *The flag $B(S^{-1})$ is the dual to $B(S)$, and moreover, $G/P_{\Theta^*} = G/P_{\Theta(S^{-1})}$, where $\Theta = \Theta(S)$.*

Proof: Take a split-regular $h \in \text{int}S$ and assume that $h \in A^+$. If Σ is the associated set of simple roots, then $W_{A^+}(S)$ is generated by reflections with respect to the subset $\Theta(S) \subset \Sigma$. By above commentary, $W_{A^-}(S^{-1})$ is generated by the same set of reflections. However by definition of $W_{A^-}(S^{-1})$ we must look the generators of this subgroup in the subset $-\Theta \subset -\Sigma$. Hence the parabolic subalgebra associated to $W_{A^-}(S^{-1})$ is

$$p_\Theta^- = p + n^+(\Theta).$$

Then $B(S^-) = G/P_\Theta^-$. Now, $w_0(-\Theta) = \iota\Theta$ and $w_0 p_\Theta^- = p_{\pi\Theta}$, where

$$p_{\pi\Theta} = p \oplus n^-(\iota\Theta)$$

and $n^-(\pi\Theta)$ is spanned by $g_{-\alpha}$ with $\alpha \in \langle\iota\Theta\rangle$. Hence $B(S^{-1}) = B_{\iota\Theta}$, the dual of $B(S) = B_\Theta$. □

Proposition 2.9. *Let S be a semigroup of type Θ and denote by C its invariant control set in $B(S)$. Then C^* is the invariant control set in $B(S^{-1})$.*

Proof: This follows of general results about duality operator, in order to obtain more detail, see Proposition 3.5 and Corollary 3.10 in [8]. □

With this it is possible to compute the homotopy group of the S^{-1}-invariant control set in $B(S^{-1})$, what is made in future section.

And finally, we recall that a subsemigroup T of a group L is right reversible if for any finite subset $\{h_1, \ldots, h_k\} \subset L$, $k \geq 1$, one of the following conditions holds

1. $(Th_1) \cap \cdots \cap (Th_k) \neq \emptyset$.
2. There exists $h \in L$ such that $h_i \in hT$, $i = 1, \ldots, k$ (and hence $h_i T \subset hT$).

Symmetric conditions are true for left reversibility.

3. G/AN

In this section we reproduce some results and notations (see [10]), which are important in the sequel. We also compute the homotopy type of the invariant control set of G/AN. And in order to conclude this section we comment the isomorphism Theorem, which supply us the homotopy type of the semigroup $S \subset G$.

For ray semigroup its invariant control set on the flag manifold is contractible to a point as shows the following lemma proved by Mittenhuber [5], Lemma 2.11, as in [5] this result was proved to rank one groups, we give a sketch of the proof, as it was made in [10].

Lemma 3.1. *Suppose that T is a ray semigroup with* $\mathrm{int}T \neq \emptyset$. *Put* $\Theta = \Theta(T)$ *and let C_Θ be its invariant control set in G/P_Θ and C_Θ^0 its interior. Then C_Θ and C_Θ^0 are contractible.*

Proof: There exists a split-regular $h \in \mathrm{int}T$ such that its attractor, say $x \in G/P_\Theta$, belongs to C_0, and C is contained in the stable manifold $\mathrm{st}(h)$ of h. This implies that $h^k y \to x$ for all $y \in C_\Theta$. By compactness of C_Θ, for any neighborhood U of x there exists k_0 such that $h^k C_\Theta \subset U$ if $k \geq k_0$. In particular, we can choose U contractible to x, that is, there exists a continuous map $\Phi : [0,1] \times U \to U$ such that $\Phi(0,\cdot) = 1_U$, $\Phi(1,\cdot) = x$. On the other hand since T is a ray semigroup, there exists a continuous curve $g_t \in T$, $t \in [0,u]$, with $g_0 = 1$ and $g_u = h^k$. Then the map $\Phi_1 : [0,u] \times C_\Theta \to C_\Theta$, $\Phi_1(t,y) = g_t y$ contracts C_Θ into U. Joining together these maps, we get a contraction of C_Θ to x. □

In [10] was proved that the homotopy groups of the semigroup $S \subset G$ is isomorphic to the homotopy groups of the invariant control sets of S in G/AN and as consequence it was set up that the invariant control set in G/AN is diffeomorphic to Cartesian products. We recall this discussion. Assume that S is connected, put $\Theta = \Theta(S)$, and let C_Θ stand for the unique invariant control set of S in $B_\Theta = G/P_\Theta$. The set of transitivity of C_Θ is denoted by C_Θ^0. Put $C(B)$ for the unique invariant control set in the maximal flag manifold G/MAN. It is given by $C(B) = \pi_\Theta^{-1}(C_\Theta)$, where $\pi_\Theta : B \to B_\Theta$ is the canonical fibration. Also, its set of transitivity is $C(B)^0 = \pi_\Theta^{-1}(C_\Theta^0)$. These inverse images are actually diffeomorphic to Cartesian products. In fact, recall that C_Θ is contained in some open Bruhat cell, say σ, of B_Θ. Since the bundle $\pi_\Theta : B \to B_\Theta$ is trivial over σ, it follows that $\pi_\Theta^{-1}(\sigma) \approx \sigma \times F_\Theta$, where F_Θ is the fiber P_Θ/P. Therefore $C(B) \approx C_\Theta \times F_\Theta$ and $C(B)^0 \approx C_\Theta^0 \times F_\Theta$.

Now, consider the fibration

$$\pi_1 : G/AN \longrightarrow G/MAN.$$

This is a principal bundle whose fiber is the compact group $M \approx MAN/AN$. The projection π_1 is equivariant with respect to the actions of G on the homogenous spaces G/AN and G/MAN, i.e., $g \circ \pi_1 = \pi_1 \circ g$ for all $g \in G$. Also, M has a natural right action on G/AN, which commutes with the left action of G. The equivariance of π_1 implies that $\pi_1(C)$ is an S-i.c.s. in B if $C \subset G/AN$ is an invariant control set. In other words the invariant control sets of S in G/AN are contained in $\pi_1^{-1}(C(B))$. Analogously, the set of transitivity C^0 of an invariant control set is contained in $\pi_1^{-1}\left(C(B)^0\right)$.

The assumption that S is connected implies that its invariant control sets are also connected. In particular if $C \subset G/AN$ is an S-i.c.s. then it is contained in a connected component of $\pi_1^{-1}(C(B))$. Its set of transitivity C^0 is also connected and hence contained in a component of $\pi^{-1}\left(C(B)^0\right)$. Actually we have

Lemma 3.2. *S acts transitively on any connected component of $\pi_1^{-1}\left(C(B)^0\right)$.*

Proof: Consider the restriction of the principal bundle $\pi_1 : G/AN \to G/MAN$ to the open set $C(B)^0$. Its structural group is compact, and S acts on it as a semigroup of automorphisms of the bundle. Also, S acts transitively on the basis $C(B)^0$. Hence the lemma follows from Proposition 3.9 in [1], which asserts that a semigroup acting on a connected principal bundle with compact fiber is transitive provided it is transitive on the base space. □

From this lemma we get the following characterization of the invariant control sets of S in G/MAN.

Proposition 3.3. *Keep assuming that S is connected. Denote as before by C_{Θ}^{0} the set of transitivity of the S-i.c.s. in G/P_{Θ}, where $\Theta = \Theta(S)$. Put $\pi : G/AN \to G/P_{\Theta}$ for the canonical fibration. Let $C \subset G/AN$ be an S-i.c.s. Then C is a connected component of $\pi^{-1}(C_{\Theta})$ and C^{0} is a connected component of $\pi^{-1}(C_{\Theta}^{0})$. Conversely, the closure in G/AN of a connected component of $\pi^{-1}(C_{\Theta}^{0})$ is an S-i.c.s. in G/AN.*

Proof: By the choice of G/P_{Θ} it follows that $C(B)^{0} = \pi_{\Theta}^{-1}(C_{\Theta}^{0})$ where π_{Θ} is the projection $G/P \to G/P_{\Theta}$. Also there exists an open Bruhat cell $\sigma \subset G/P_{\Theta}$ such that $C_{\Theta} \subset \sigma$. The restriction of π to σ defines a trivial bundle. Since C_{Θ} is connected and contained in σ, it follows that the connected components of $\pi^{-1}(C_{\Theta})$ are contained in the connected components of $\pi^{-1}(\sigma)$. Analogously, the connected components of $\pi^{-1}(C_{\Theta}^{0})$ are contained in the components of $\pi^{-1}(\sigma)$. This together with the fact that C_{Θ}^{0} is dense in C_{Θ} implies that the closure of a component of $\pi^{-1}(C_{\Theta}^{0})$ is a component of $\pi^{-1}(C_{\Theta})$, so that the closures of two different components of $\pi^{-1}(C_{\Theta}^{0})$ are disjoint.

Now, π decomposes as

$$G/AN \to G/MAN \to G/P_{\Theta}.$$

Since the fiber of G/MAN is connected, it follows that the connected components of $\pi^{-1}(C_{\Theta}^{0})$ are the components over $C(B)^{0}$ in the fibration $G/AN \to G/MAN$. Hence, the previous lemma ensures that a connected component of $\pi^{-1}(C_{\Theta})$ is an invariant control set of S. Furthermore, this implies that any invariant control set is such a component, because their union is $\pi^{-1}(C_{\Theta})$. □

Note that the triviality of the bundle $G/AN \to G/P_{\Theta}$ over C_{Θ} means that $\pi^{-1}(C_{\Theta})$ is diffeomorphic to $C_{\Theta} \times F$ where $F = P_{\Theta}/AN$ is the fiber. Clearly, a connected component of the fiber is P_{Θ}^{0}/AN where P_{Θ}^{0} is the identity component of P_{Θ}. The other components are obtained similarly from the components of P_{Θ}. These components are diffeomorphic to each other, since they are interchanged by the right action of M as follows by Lemma 1.2.4.5 in [12], which ensures that $P_{\Theta} = MP_{\Theta}^{0}$. Therefore the previous proposition together with the fact that C_{Θ} is connected implies the

Corollary 3.4. *With the assumption that S is connected, any invariant control set $C \subset G/AN$ is diffeomorphic to $C_{\Theta} \times P_{\Theta}^{0}/AN$. Moreover, $C^{0} \approx C_{\Theta}^{0} \times P_{\Theta}^{0}/AN$.*

In order to complete the picture we recall that $P_\Theta = M_\Theta(K)AN$, where $M_\Theta(K)$ is the centralizer of a_Θ in K (see [12], Theorem 1.2.4.8). We denote by $K(\Theta)$ the identity component of $M_\Theta(K)$. Then $P_\Theta^0 = K(\Theta)AN$, so that the fiber P_Θ^0/AN is diffeomorphic to $K(\Theta)$. Therefore there is following version of the above corollary

Corollary 3.5. *With the assumption that S is connected, any invariant control set $C \subset G/AN$ is diffeomorphic to $C_\Theta \times K(\Theta)$. Moreover, $C^0 \approx C_\Theta^0 \times K(\Theta)$. Here $\Theta = \Theta(S)$.*

We regard the n-th homotopy group $\pi_n(X, x_0)$ of a space based at $x_0 \in X$ as the set of homotopy classes of pointed maps $\gamma : (S^n, s_0) \to (X, x_0)$ where S^n stands for the n sphere and s_0 is a base point in S^n, e.g., $s_0 = (1, 0, \ldots, 0)$. Hence, we have the consequence of the results above, which give us the homotopy type of the invariant control set in G/AN and of the invariant control set in $G/P_{\Theta(S)}$.

Corollary 3.6. $\pi_n(C_{\Theta(S)}) = 1$ *and* $\pi_n(C^0) = \pi_n(K(\Theta))$.

For the rest of this section we comment anything about the homotopy type of S, for better study see [10]. In the paper cited previously, it was proved that the homotopy type of S is given by homotopy type of C^0, that is, $\pi_n(S)$ is isomorphic to $\pi_n(C^0)$. In order to get this, it was considered the evaluation map $e : S \to C^0$, given by $e(g) = gx$, and it was proved that the induced homomorphisms e_* between the homotopy groups are isomorphisms.

In [10] it was proved that C^0 is diffeomorphic to $C_\Theta^0 \times F_0$ where $F_0 = P_\Theta^0/AN$ and it was showed also that, for the semigroup considered here, C_Θ^0 is contractible, so that any cycle in C^0 is homotopic to one in F_0. The surjectivity e_* is proved by showing the existence of a cross section $\sigma : F \to \mathrm{int}S$ for the evaluation map.

Proposition 3.7. *Any compact subset $Q \subset M_\Theta$ can be lifted to $(\mathrm{int}S) \cap P_\Theta$, that is, there exists $z \in S \cap A_\Theta N_\Theta$ such that $Qz \subset (\mathrm{int}S) \cap P_\Theta$.*

Proof: See [10], subsection 4.1. □

With this result it is possible to prove

Theorem 3.8. *Suppose that $\mathrm{int}S \neq \emptyset$ and put $\Theta = \Theta(S)$. Let C be one of its invariant control sets in G/AN, fix $x \in C^0$ and consider the* *evaluation map* $e : S \to C_0$, $e(g) = gx$. *Denote by* F *the fiber* $\pi^{-1}(\pi(x))$, *where* $\pi : G/AN \to G/P_\Theta$ *is the canonical projection. Then there exists a continuous cross section* $\sigma : F \to \mathrm{int}S \cap P_\Theta$, *satisfying* $e\sigma = 1_F$.

Proof: See Theorem 4.4 in [10]. □

For the proof of the injectivity of the evaluation map was assumed that S is connected and contains a Θ-large ray semigroup T, where $\Theta = \Theta(S)$. The proof was made as follows: Fix basic points $x \in C^0$ and $g_0 \in \mathrm{int}S$ such that $g_0 x = x$. If $\gamma : (S^n, s_0) \to (S, g_0)$ satisfies $e_*[\gamma] = [e \circ \gamma] = 1$, then it was proved that $[\gamma] = 1$, that is, there is a homotopy, based at g_0, carrying γ into g_0. The construction is not made directly. Instead, was built homotopies inside S carrying γ successively into smaller groups until reach $A_\Theta N_\Theta$. Using reversibility properties of $S \cap A_\Theta N_\Theta$, it followed then that there exists a (unbased) homotopy $\Phi : [0, 1] \times S^n \to S$ between γ and a constant cycle g_1. Now, from Φ we have the path, say α, given by the restriction of Φ to $[0, 1] \times \{s_0\}$. This path settles an isomorphism $\alpha_* : \pi_n(S, g_0) \to \pi_n(S, g_1)$, where $g_1 = \Phi(1, s_0)$, such that $\alpha_*[\gamma] = [g_1] = 1$ (see e.g. [4], Theorem 7.2.3). This shows the triviality of $\ker e_*$.

It was proved, in different steps, that any cycle γ in S is homotopic within S to a cycle in $S \cap A_\Theta N_\Theta$. We only put here the Lemmas that explain this steps. Its proofs can be found in [10].

Lemma 3.9. *Suppose that* S *contains a* Θ*-large ray semigroup* T *with* $\mathrm{int}T \neq \emptyset$, *where* $\Theta = \Theta(S)$. *Fix* $y \in C^0_\Theta$ *and assume without loss of generality that* P_Θ *is the isotropy at* y. *Then any cycle* $\gamma : (S^n, s_0) \to (S, g_0)$ *with* $g_0 y = y$ *is (unbased) homotopic to a cycle* $\beta : S^n \to P_\Theta$.

Lemma 3.10. *Let* $\gamma : S^n \to S \cap P^0_\Theta$ *be a cycle with* $\gamma(s_0) = g_1 \in A_\Theta N_\Theta$. *Assume that* $e_*[\gamma] = 1$. *Then* γ *is homotopic in* S *to a cycle* β *contained in* $A_\Theta N_\Theta$.

We can now have the injectivity of the evaluation map.

Theorem 3.11. *Assume that* S *is connected and contains a large ray semigroup* T *with nonempty interior. As before let* C *be an* S*-i.c.s. in* G/AN *and* C^0 *its interior. Then the homomorphism* $e_* : \pi_n(S) \to \pi_n(C^0)$ *induced by a evaluation map* $e : S \to C^0$, $e(g) = gx$, $x \in C^0$, *is injective. The same statement is true with* e *defined in* $\mathrm{int}S$ *instead of* S.

4. Homotopy type of i.c.s.

In the previous section we computed the homotopy type of the invariant control set in G/AN, then a natural question that to arise is about the homotopy type of the invariant control sets on other homogeneous spaces.

Take $\Theta \subset \Sigma$ a subset of the set of simple roots and P_Θ a parabolic subgroup associated to Θ. We consider the following equivariants fibration:

$$\pi_\Theta : G/P \to G/P_\Theta \text{ e } \pi_{\Theta(S)} : G/P \to G/P_{\Theta(S)}.$$

Let $C(B)$, $C_{\Theta(S)}$ and C_Θ be the invariant control set for S in G/P, $G/P_{\Theta(S)}$ and G/P_Θ respectively, with $C(B)^0$, $C^0_{\Theta(S)}$ and C^0_Θ the respective set of transitivity. Consider also the equivariant fibration $G/AN \to G/P$, with $C \subset G/AN$ the invariant control set and the fiber bundle $G \to G/AN$.

Remembering that $C_{\Theta(S)}$ is contractible and $C^0 \approx C^0_{\Theta(S)} \times P^0_{\Theta(S)}/AN$ (see Corollary 3.4), we can note that the homotopy type of $C(B)$ is the same that the homotopy type of $P_{\Theta(S)}/P$ (see Lemma 3.1). Analogously, it is possible to prove the following Lemma

Lemma 4.1. *$C(B)^0 \approx C^0_{\Theta(S)} \times P^0_{\Theta(S)}/P$. Moreover, $C(B)^0$ has the same homotopy type that $P^0_{\Theta(S)}/P$*

Now, the next result is important in order to obtain the main result here

Lemma 4.2. *Suppose that S contains a ray semigroup and $\Theta(S)$-large semigroup T, with nonempty interior. Fix $y \in C^0_{\Theta(S)}$ and assume, without loss of generality that $P_{\Theta(S)}$ is the isotropy in y. Then any cycle $\eta : (S^n, s_0) \to (C_\Theta, g_0 x)$, com $g_0 y = y$, is homotopic to a cycle $\beta : S^n \to P_{\Theta(S)}/P_\Theta \cap P_{\Theta(S)}$, where $x \in C_\Theta \cap (P_{\Theta(S)}/P_\Theta \cap P_{\Theta(S)})$.*

Proof: Consider $\gamma : (S^n, s_0) \to (S, g_0)$ such that $\gamma x = \eta$, observe that it is possible find such γ make using the cross section of the fiber bundle $G \to G/P_\Theta$.

Now we use some consideration of Lemma 4.10 in [10]. We take the neighborhood U of y, consider $U' := \{g\} \times U \subset \text{int}T^{-1}$ and define the cycle $\delta(z) = (g, \gamma(z)y)$ in U' and the continuous contraction of U', $\phi : I \times U' \to U'$ with $\phi(0, l) = (g, l)$ and $\phi(1, l) = (g, g)$, for all $l \in U'$. Note that $\delta^{-1}(z)\gamma(z)x$ is a cycle in $(P_{\Theta(S)}/P_\Theta \cap P_{\Theta(S)})$, because $\delta^{-1}(z)\gamma(z)$ is a cycle in $P_{\Theta(S)}$ (see Lemma 4.10 in [10]).

For $(t,z) \in I \times S^n$ we define

$$\Phi(t,z) = \phi(t, \delta(z))^{-1} \gamma(z) x.$$

Note that, Φ is a continuous mapping defined in $[0,1] \times S^n$ having its values in C_Θ^0 because $U' \subset S^{-1}$ and $\gamma(z) x \in C_\Theta^0$. Hence, we have that $\Phi(0,z) = \delta(z)^{-1} \gamma(z) x$ and $\Phi(1,z) = g^{-1}\gamma(z) x$, such that the cycle $\delta(z)^{-1} \gamma(z) x$ and $g^{-1}\gamma(z) x$ are homotopics in C_Θ^0. But, $\delta(z)^{-1} \gamma(z) x$ and $\gamma(z)x$ are homotopics in C_Θ^0, hence it is enough to take $\beta(z) := \delta(z)^{-1} \gamma(z) x$. □

Consequently, the homotopy type of C_Θ is

$$P^0_{\Theta(S)}/(P_\Theta \cap P^0_{\Theta(S)}) \approx K(\Theta(S))AN/M_\Theta(K)AN \cap K(\Theta(S))AN$$

$$\approx K(\Theta(S))/M_\Theta(K) \cap K(\Theta(S)).$$

As $M_\Theta(K) = M_\Theta \cap K$, we have that the homotopy type of C_Θ is $K(\Theta(S))/K \cap M_\Theta \cap K(\Theta(S))$. Hence,

Proposition 4.3. *With the above notations, we have that the homotopy type of the invariant control set $C_\Theta \subset G/P_\Theta$ is $K(\Theta(S))/K \cap M_\Theta$.*

Exemple 4.4. *Take $G = \mathrm{Sl}(n,R)$, a canonical choice is given by taking $\mathfrak{a}$ as the algebra of diagonal matrices with zero trace. The roots are $\alpha_{ij} = \lambda_i - \lambda_j$ where $\lambda_i(H) = a_i$ if $H = \mathrm{diag}\{a_1, \cdots, a_n\}$. A simple system of roots is given by $\Sigma = \{\alpha_{i,i+1} : i = 1,...,n-1\}$, and the Weyl group is just the group of permutations in n elements. Let $S = \mathrm{Sl}^+(n,R)$ be the semigroup of determinant one matrices having nonnegative entries. This is the compression semigroup of the positive orthant R^n_+ in R^n:*

$$R^n_+ = \{(x_1, \ldots, x_n) : x_i \geq 0\}.$$

It turns out that the parabolic type of $S = \mathrm{Sl}^+(n,R)$ is $\Theta(\mathrm{Sl}^+(n,R)) = \{\alpha_{i,i+1} : i = 2,...,n-1\}$, and the invariant control set in $G/P_{\Theta(S)} = P^{n-1}$ is the set $[R^n_+]$ of lines contained in R^n_+. In our previous notation, $C_\Theta = [R^n_+]$. The semigroup $\mathrm{Sl}^+(n,R)$ is closed but not a Lie semigroup.

As we can see in [10] it is possible to prove that $S = \mathrm{Sl}^+(n,R)$ contains a $\Theta(S)$-large ray semigroup and that S is connected. Hence the isomorphism theorem (see Section 3) holds for $\mathrm{Sl}^+(n,R)$. With the canonical choices, it is not difficult to check that P^0_Θ/AN is diffeomorphic to $\mathrm{SO}(n-1)$. It *follows that the homotopy groups of* $\mathrm{Sl}^{+}(n, R)$ *are isomorphic to the homotopy groups of* $\mathrm{SO}(n-1)$.

In this case, taking $\Theta = \{\alpha_{i,i+1} : i = 1, ..., n-2\}$ *we have that* $G/P_{\Theta} = Gr_{n-1}(n)$ *and*

$$M_{\Theta}(K) \cap K(\Theta(S)) = \begin{pmatrix} 1 & & \\ & \mathrm{SO}(n-2) & \\ & & 1 \end{pmatrix},$$

then considering $S = \mathrm{Sl}^{+}(n, R)$ *the homotopy type of the* S*-i.c.s.* C_{Θ} *is*

$$\mathrm{SO}(n-1)/\mathrm{SO}(n-2).$$

For others $\Theta \subset \Sigma$ *we have that* $M_{\Theta}(K) \simeq \mathrm{SO}(r_1) \times \cdots \times \mathrm{SO}(r_s)$ *for some positive integers* $r_1, \cdots, r_s$*, we have also that* $K(\Theta(S)) = \mathrm{SO}(n-1)$ *and the homotopy type of the* S*-i.c.s.* C_{Θ} *is*

$$\mathrm{SO}(n-1)/\mathrm{SO}(r_1) \times \cdots \times \mathrm{SO}(r_s - 1) \times 1 \approx \mathrm{SO}(r_s)/\mathrm{SO}(r_s - 1).$$

Therefore, the homotopy type of the invariant control set, for this semigroup, in any flag manifold is the same that sphere.

5. Homotopy type of S^{-1}

We have that the canonical homomorphism of G, $f(g) = g^{-1}$, considered in S, put that the homotopy type of S is the same that S^{-1}. In this section we look this equality by Theorem of isomorphism (see section 3), showing that $K(\Theta(S)) \approx K(\Theta(S^{-1}))$.

We begin recalling that if S is connected, ray semigroup, reversible and with non-empty interior then it is also to S^{-1}. Moreover, if the semigroup S has an Θ-large and ray semigroup with non-empty interior so it is true to S^{-1}. Hence, the same results got to S, in the section 4 in [10], is valid to S^{-1}.

As we saw, the minimal control set of S is a maximal (invariant) control set of S^{-1}, then we can consider the following sequence,

$$G \to G/AN \to G/P \to G/P_{\Theta^*}.$$

Take $C_*^0, C(B)_*^0$ and $C_{\Theta^*}^0$ invariant control sets for S^{-1} in G/AN, G/P and G/P_{Θ^*} respectively, where $G/P_{\Theta}^* = G/P_{\Theta(S^{-1})}$. We have that $K(\Theta^*) \approx P_{\Theta^*}^0/AN$ is a deformation retract of $\mathrm{int}S^{-1}$.

On the other hand, there exists the following decomposition,

$$p_\Theta = n^-(\Theta) \oplus p,\ p_{\Theta^*} = n^-(\Theta^*) \oplus p \text{ e } p^-_{\Theta^*} = n(\Theta^*) \oplus p^-,$$

where $p^- = m \oplus a \oplus n^-$, $p = m \oplus a \oplus n$, $n(\Theta^*) = \sum g_\alpha$, with $\alpha \in \langle \Theta^* \rangle$, $n^-(\Theta) = \sum g_\alpha$, with $\alpha \in -\langle \Theta^* \rangle$ and $n^-(\Theta^*) = \sum g_\alpha$ with $\alpha \in -\langle \Theta^* \rangle$.

It is called Cartan involution the automorphism θ of g such that $\theta^2 = 1$ and the quadratic form $B(X, \theta(X)) = \text{tr}(\text{ad}(X)\text{ad}(\theta(Y))$, $X \in g$ is negative definite. There exists one to one correspondence between the Cartan involution of g and Cartan decompositions of g (see [7], Chapter 12).

We have that $p_{\Theta^*} = -\varphi p^-_{\Theta^*}$, where φ is the Cartan involution of p_{Θ^*} (because $\varphi(n(\Theta^*)) = n^-(\Theta^*)$ and $\varphi(p^-) = p$).

Taking ζ the extension of automorphism w_0 and ψ the composition $\zeta\varphi$, we have that $p_\Theta = \psi p^-_{\Theta^*}$. Hence , we can say that p_Θ is the same subalgebra that p_{Θ^*}, except by automorphism, then we have that $K(\Theta^*) \approx K(\Theta)$, where $K(\Theta^*) \approx P^0_{\Theta^*}/AN$ and $K(\Theta) \approx P^0_\Theta/AN$. Therefore,

Proposition 5.1. *The semigroups S and S^{-1} have the same homotopy type. Furthermore, there exists $w \in AN$ such that $K(\Theta^*)w$ is a retract deformation of* $\text{int}S^{-1}$.

Remark 5.2. *If we consider the order of control set defined behind, we can obtain the homotopy group of minimal control set, since we know the homotopy group of maximal invariant control set. In fact, recalling that the minimal control set in G/P is S^{-1}-invariant control set we have that the minimal control set in G/P has the same homotopy type of the S-invariant control set (maximal control set for S).*

References

[1] *C. J. Braga Barros* e *L. A. B. San Martin*, *On the action of semigroups in fiber bundles*. Mat. Contemp. **13**, pp. 1-19, (1997).

[2] *J. Hilgert* and *K.-H. Neeb*, *Lie semigroups and their applications*. Lecture Notes in Math. **1552**. Springer-Verlag 1993.

[3] *J. E. Humphreys*, *Reflection groups and coxeter groups*. Cambridge studies in advanced mathematics **29** (1990).

[4] *C. R. F. Maunder*, *Algebraic Topology.* Van Nostrand (1970).

[5] *D. Mittenhuber*, *On maximal Lie semigroups in real hyperbolic geometry.*

[6] *L. A. B. San Martin*, *Invariant control sets on flag manifolds.* Mathematics of Control, Signals, and Systems **6**, pp. 41-61, (1993).

[7] *L. A. B. San Martin*, *Álgebras de Lie.* Editora da Unicamp (1999).

[8] *L. A. B. San Martin*, *Maximal semigroups in semi-simple Lie groups.* Trans. Amer. Math. Soc. To appear.

[9] *L. A. B. San Martin*, *Nonexistence of invariant semigroups in affine symmetric spaces.* Math. Ann. To appear.

[10] *L. A. B San Martin* and *A. J. Santana*, *The homotopy type of Lie semigroups in semi-simple Lie groups.* Monatsh. Math. To appear.

[11] *L. A. B. San Martin* and *P. A. Tonelli*, *Semigroup actions on homogeneous spaces.* Semigroup Forum **50**, pp. 59-88, (1995).

[12] *G. Warner*, *Harmonic analysis on semi-simple Lie groups I.* Springer-Verlag (1972).

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