Proyecciones
Vol. 21, N^o 1, pp. 65-95, May 2002.
Universidad Católica del Norte
Antofagasta - Chile

REGULARITY OF SOLUTIONS OF PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY *

EDUARDO HERNÁNDEZ
Universidade de Sao Paulo - Brasil

Abstract

We prove the existence of regular solutions for a class of quasi-linear partial neutral functional differential equations with unbounded delay that can be described as the abstract retarded functional differential equation $\frac{d}{dt}(x(t) + F(t, x_t)) = Ax(t) + G(t, x_t)$*, where* A *is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on a Banach space* X *and* F*,* G *are appropriated functions.*

Keywords: *Retarded functional differential equations, abstract Cauchy problem, semigroup of bounded linear operators, regularity of solutions.*

---

*Research Partially Supported by FAPESP-SP-Brazil, grant # 13394-5.

1. Introduction

The purpose of this paper is to establish some results of regularity, in a sense to be specified later, for solutions of a class of quasi-linear neutral functional differential equations with unbounded delay that can be described in the form

$$\text{(1.1)} \qquad \begin{cases} \frac{d}{dt}(x(t) + F(t, x_t)) & = Ax(t) + G(t, x_t), \qquad t > \sigma, \\ x_\sigma & = \varphi \in \mathcal{B}, \end{cases}$$

where A is the infinitesimal generator of an uniformly bounded analytic semigroup of bounded linear operators, $(T(t))_{t\geq 0}$, on a Banach space X, the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, $\Omega \subset \mathcal{B}$ is open, $0 \leq \sigma < a$ and $F, G : [\sigma, a] \times \Omega \to X$ are appropriate continuous functions.

Neutral differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for neutral functional differential equations is the Hale & lunel book [3] and the references therein. The work in partial neutral functional differential equations with unbounded delay was initiated by Hernández & Henríquez in [4, 5]. In these papers, Hernández & Henríquez proved the existence of mild, strong and periodic solutions for the neutral equation (1.1). In general, the results were obtained using the semigroup theory and the Sadovskii fixed point theorem (see [11]).

The results obtained in this paper are the continuation of papers [4], [5] on the existence of mild, strong and periodical solutions for the neutral system (1.1) and generalization of the results reported by Henriquez in [6].

Throughout this paper, X will be a Banach space provided with norm $\| \cdot \|$ and $A : D(A) \to X$ will be the infinitesimal generator of an uniformly bounded analytic semigroup, $T = (T(t))_{t\geq 0}$, of linear operators on X. For the theory of strongly continuous semigroup, we refer to Pazy [10] and Krein [9]. We will point out here some notations and properties that will be used in this work. It is well know that there exist constants $\tilde{M}$ and $w \in \mathbb{R}$ such that

$$\|T(t)\| \leq \tilde{M}e^{wt}, \qquad t \geq 0.$$

If T is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\| x \|_\alpha = \| (-A)^\alpha x \|$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\| \cdot \|_\alpha$, then the following properties are well known ([10], pp. 74):

Lemma 1. *If the previous conditions hold:*

1. *Let $0 < \alpha \leq 1$. Then X_α is a Banach space.*
2. *If $0 < \beta \leq \alpha$ then $X_\alpha \to X_\beta$ is continuous.*
3. *For every constant $a > 0$, there exists $C_a > 0$ such that*

$$\|(-A)^\alpha T(t)\| \leq \frac{C_a}{t^\alpha}, \quad 0 < t \leq a.$$

4. *For every $a > 0$ there exists a positive constant C'_a such that*

$$\left\|(T(t) - I)(-A)^{-\alpha}\right\| \leq C'_a\, t^\alpha, \quad 0 < t \leq a.$$

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$, introduced by Hale and Kato [2]. To establish the axioms of the space $\mathcal{B}$ we follows the terminology used in Hino-Murakami-Naito [8], and thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X, endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(**A**) If $x : (-\infty, \sigma + \omega) \to X$, $\omega > 0$, is continuous on $[\sigma, \sigma + \omega)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + \omega)$ the following conditions hold:

i) x_t is in $\mathcal{B}$.

ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.

iii) $\| x_t \|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t-\sigma) \| x_\sigma \|_{\mathcal{B}}$.

Where $H > 0$ is a constant; $K, M : [0,\infty) \to [0,\infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

$(\mathbf{A-1})$ For the function $x(\cdot)$ in $(\mathbf{A})$, x_t is a $\mathcal{B}$-.valued continuous function on $[\sigma, \sigma + w)$.

$(\mathbf{B})$ The space $\mathcal{B}$ is complete.

For the literature on phase space, we refer the reader to [8]. While noting here that from the axiom $(\mathbf{A-1})$, it follows, that the operator function $W(\cdot)$ dfined by

$$(1.2)[W(t)\varphi](\theta) := \begin{cases} T(t+\theta)\varphi(0) & \text{for} \quad -t \leq \theta \leq 0, \\ \varphi(t+\theta) & \text{for} \quad -\infty < \theta < -t, \end{cases}$$

is an strongly continuous semigroup of bounded linear operators on $\mathcal{B}$. In this paper, A_w with domain $D(A_w)$ will be the infinitesimal generator of $W(\cdot)$.

To obtain some of our results we will require additional properties for the phase space $\mathcal{B}$, in particular we consider the following axiom (see [6], pp. 526 for details) ;

$(\mathbf{C_3})$ Let $\rho > 0$. Let $x : (-\infty, \sigma + \rho] \to X$ be a continuous function such that $x_\sigma \equiv 0$and the right derivative, denoted $x'(\sigma^+)$, exists. If the function ψ defined by $\psi(\theta) = 0$ for $\theta < 0$ and $\psi(0) := x'(\sigma^+)$ belongs to $\mathcal{B}$ then $\left\| \left(\frac{1}{h}\right) x_{\sigma+h} - \psi \right\|_{\mathcal{B}} \to 0$ as $h \to 0^+$.

On the other hand, for a linear map $P : D(P) \subset X \to X$ and $\varphi \in \mathcal{B}$ such that $\varphi(\theta) \in D(P)$ for every $\theta \leq 0$, we denote by $P\varphi : (-\infty, 0] \to X$, defined by $P\varphi = P(\varphi(\theta))$. For any $0 < \alpha \leq 1$ we use the notation $\mathcal{B}_\alpha$ for the vector space

$$\mathcal{B}_\alpha = \{(-A)^{-\alpha}\varphi : \varphi \in \mathcal{B}\}.$$

It is easily to prove that $\mathcal{B}_\alpha$, endowed with the seminorm defined by

$$\| \psi \|_{\mathcal{B}_\alpha} := \| (-A)^\alpha \psi \|_{\mathcal{B}} ,$$

is a phase space of functions with values in X_α.

The paper is organized as follows. In section 2, we define the different concepts used in this work and establish the existence of N-classical and classical solutions for the initial value problem (1.1) . Our results are based on the properties of analytic semigroups and the ideas contained in Pazy [10] and Henríquez [6].

Throughout this work we assume that X is an abstract Banach space. The terminology and notations are those generally used in operator theory. In particular, if X and Y are Banach spaces, we indicate by $\mathcal{L}\,(X:Y)$ the Banach space of the bounded linear operators of X in Y and we abbreviate this notation to $\mathcal{L}\,(X)$ when ever $X = Y$. In addition $B_r(x:X)$ will denote the closed ball in the space X with center at x and radius r.

For some bounded function $\xi : [\sigma, a] \to X$ and $\sigma \leq s < t \leq a$ we employ the notation

$$(1.3) \qquad \|\xi(\cdot)\|_{[s,t]} = \sup\{\| \xi(\theta) \|: \theta \in [s,t]\}$$

and we will write simply ξ_t for $\|\xi(\cdot)\|_{[\sigma,t]}$ when no confusion arises.

If $x \in X$, we will use the notation $\mathcal{X}_x$ for the function $\mathcal{X}_x : (-\infty, 0] \to X$ where $\mathcal{X}_x = 0$ for $\theta < 0$ and $\mathcal{X}_x(0) = x$.

Finally, a function $f : I \subset \mathbb{R} \to X$ is α-Hölder continuous, $0 < \alpha \leq 1$, if there exists a constant $L > 0$ such that

$$\|f(s) - f(t)\| \leq L \mid t - s \mid^\alpha , \quad s, t \in I.$$

We represent by $C^{0,\alpha}(I;X)$ the space of α-Hölder continuous function from I into X. Similarly, $C^{k,\alpha}(I;X)$ consist of those functions from I into X, that are k-times continuously differentiable and whose k^{th}-derivative is α-Hölder continuous.

2. Regularity of Mild Solutions

In this section we will study the regularity of mild solutions of the abstract Cauchy problem (1.1). Henceforth we will assume that A is the infinitesimal generator of a uniformly bounded analytic semigroup, $T = (T(t))_{t\geq 0}$, on X, that $\Omega \subset \mathcal{B}$ is open and that $F, G : [\sigma, a]\times\Omega \to X$ are continuous functions. Further, to avoid unnecessary notation, we suppose that $0 \in \rho(A)$ and that $\|T(t)\| \leq \tilde{M}$, for some constant $\tilde{M} \geq 1$ and every $t \geq 0$. Our regularity results are based on those of regularity of mild solutions for the abstract Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t), \\ x(0) = x_0. \end{cases} \tag{2.1}$$

By analogy with the abstract Cauchy problem (2.1) we adopt the following definitions:

Definition 1. *We will say that a function* $x : (-\infty, \sigma+b) \to X$, $\sigma + b \leq a$, *is a mild solution of the abstract Cauchy problem (1.1) if:* $x_\sigma = \varphi$; *the restriction of* $x(\cdot)$ *to the interval* $[\sigma, \sigma+b)$ *is continuous; for each* $\sigma \leq t < \sigma+b$ *the function* $AT(t-s)F(s, x_s)$, $s \in [\sigma, t)$, *is integrable and*

$$\begin{aligned} &x(t) = T(t-\sigma)(\varphi(0) + F(\sigma, \varphi)) - F(t, x_t) - \\ &\textstyle\int_\sigma^t AT(t-s)F(s, x_s)ds + \int_\sigma^t T(t-s)G(s, x_s)ds, \end{aligned} \tag{2.2}$$

for every $t \in [\sigma, \sigma + b)$.

The existence and uniqueness of mild solution of system (1.1) was established in [5] as consequence of the contraction principle. More precisely:

Theorem 1. *Let* $\varphi \in \Omega$ *and assume that the following conditions hold:*

a) *There exist* $\beta \in (0, 1)$ *and* $L \geq 0$ *such that the function* F *is* X_β*-valued and satisfies the Lipschitz condition*

$$\begin{aligned} &\left\|(-A)^\beta F(t, \psi_1) - (-A)^\beta F(s, \psi_2)\right\| \leq \\ &L\,(\mid t-s\mid + \ \|\psi_1 - \psi_2\|_{\mathcal{B}}) \end{aligned} \tag{2.3}$$

for every $\sigma \leq s, t \leq a$, $\psi_1, \psi_2 \in \Omega$ *and*

$$K(0)L \left\| (-A)^{-\beta} \right\| < 1. \tag{2.4}$$

b) The function G *is continuous and there exist* $N > 0$ *such that*

$$\| G(t, \psi_1) - G(s, \psi_2) \| \leq N(| t - s | + \| \psi_1 - \psi_2 \|_{\mathcal{B}}) \tag{2.5}$$

for every $\sigma \leq s, t \leq a$ *and* $\psi_1, \psi_2 \in \Omega$.

Then there exists a unique mild solution $x(\cdot, \varphi)$ *of the abstract Cauchy problem (1.1) defined on* $(-\infty, \sigma + r)$, *for some* $0 < r < a - \sigma$. *Furthermore, if* $\Omega = \mathcal{B}$ *then* r *can be chosen independent of* φ.

Considering the concepts of mild and classical solutions adopted by Henriquez in [6], we introduce the followings definitions.

Definition 2. *We will say that a function* $x : (-\infty, \sigma + b) \to X$, $b > 0$, *is a classical solution of the abstract Cauchy problem (1.1) if:* $x_\sigma = \varphi$; $x(\cdot) \in C([\sigma, \sigma + b); X) \bigcap C^1((\sigma, \sigma + b); X)$; $x(t) \in D(A)$ *for every* $t \in (\sigma, \sigma + b)$; *for each* $\sigma \leq t < \sigma + b$ *the function* $AT(t - s)F(s, x_s)$, $s \in [\sigma, t)$, *is integrable and* $x(\cdot)$ *satisfies equation (1.1) on* $[\sigma, \sigma + b)$.

Definition 3. *We will say that a function* $x : (-\infty, \sigma + b) \to X$, $b > 0$, *is an N-classical solution of the abstract Cauchy problem (1.1) if:* $x_\sigma = \varphi$, $x(\cdot) \in C([\sigma, \sigma + b); X)$; $x(t) \in D(A)$ *for every* $t \in (\sigma, \sigma + b)$; $\dfrac{d}{dt}(x(t) + F(t, x_t))$ *is continuous on* $(\sigma, \sigma + b)$; *for each* $\sigma \leq t < \sigma + b$ *the function* $AT(t - s)F(s, x_s)$, $s \in [\sigma, t)$, *is integrable and* $x(\cdot)$ *satisfies equation (1.1) on* $[\sigma, \sigma + b)$.

In relation with the previous definitions, we consider the following result.

Proposition 1. *The following properties hold.*

(a) *If $x(\cdot):(-\infty,\sigma+b)\to X$ is a classical or N-classical solution of (1.1) then $x(\cdot)$ is a mild solution.*

(b) *If $x(\cdot):(-\infty,\sigma+b)\to X$ is a N-classical solution of (1.1) and $\dfrac{d}{dt}F(t,x_t)$ is continuous on $(\sigma,\sigma+b)$, then $x(\cdot)$ is a classical solution.*

(c) *Every classical solution is a N-classical solution.*

Proof: We only prove (**a**). Using that $(T(t))_{t\geq 0}$ is analytic; for $t,s\in[\sigma,\sigma+b)$ with $t>s$, we find that

$$\begin{aligned}\tfrac{d}{ds}[T(t-s)(x(s)+F(s,x_s))] &= -AT(t-s)(x(s)+F(s,x_s))\\ &\quad +T(t-s)\tfrac{d}{ds}(x(s)+F(s,x_s))\\ &= -AT(t-s)F(s,x_s+T(t-s)G(s,x_s),\end{aligned}$$

which in turn implies that

$$\begin{aligned}&x(t)+F(t,x_t)=T(t-\sigma)(\varphi(0)+F(\sigma,\varphi))\\ &-\textstyle\int_\sigma^t AT(t-s)F(s,x_s)ds+\int_\sigma^t T(t-s)G(s,x_s)ds,\end{aligned}$$

since $s\to AT(t-s)F(s,x_s)$ is integrable on $[\sigma,t)$. Thus $x(\cdot)$ is a mild solution. The proof is complete

To prove our first regularity Theorem, we need previously some technical results. Next, we study the regularity of mild solutions of the abstract Cauchy problem

$$\left\{\begin{aligned}\dot{x}(t) =&\ \ Ax(t)+(-A)^{1-\beta}g(t),\\ x(0) =&\ \ 0,\end{aligned}\right. \tag{2.6}$$

where $g(\cdot)\in C([0,a];X_{1-\beta})\bigcap C^{0,\vartheta}([0,a];X)$; $\beta,\vartheta\in(0,1)$ and $\beta+\vartheta>1$. To this end, for a mild solution, $u(\cdot)$, of (2.6) we introduce the decomposition

$$\begin{aligned}u\,(t) &= \textstyle\int_0^t(-A)^{1-\beta}T(t-s)(g(s)-g(t))ds\\ &\quad +\textstyle\int_0^t(-A)^{1-\beta}T(t-s)g(t)ds\\ &= u_1(t)+u_2(t).\end{aligned}$$

The proofs of the following three results follow from the proofs of Theorems 4.3.2, 4.3.5 and Lemma 4.3.4 in Pazy [10]. However there are some differences that require special attention and we include the principal ideas of the proofs for completeness.

Lemma 2. *Let ϑ, $\beta \in (0,1)$ with $\beta + \vartheta > 1$ and $g(\cdot) \in C([0,a]; X_{1-\beta}) \bigcap C^{0,\vartheta}([0,a]; X)$. If $u(\cdot)$ is a mild solution of (2.6) then $u(\cdot) \in C([0,a]; X_1) \bigcap C^1([0,a]; X)$.*

Proof. Clearly $Au_2(t) \in C([0,a];X)$, since $Au_2(t) = T(-A)^{1-\beta} g(t) - (-A)^{1-\beta} g(t)$. In order to study the function u_1, for $\epsilon > 0$ we define the function $u_{1,\epsilon} \in C([0,a];X)$ by

$$u_{1,\epsilon}(t) := \begin{array}{ll} \int_0^{t-\epsilon} (-A)^{1-\beta}\, T(t-s)\,(g(s) - g(t))\, ds, & \text{for} \quad t \in [\epsilon, a], \\ 0 & \text{for} \quad t \in [0,\epsilon), \end{array}$$

(2.7)

From Lemma 1.1.5 in [1], $u_{1,\epsilon}(t) \in D(A)$ for every $t \in [0,a]$ and

$$Au_{1,\epsilon}(t) := \begin{cases} -\int_0^{t-\epsilon} (-A)^{2-\beta} T(t-s)(g(s) - g(t))ds\,, & t \in [\epsilon, a], \\ 0, & t \in [0,\epsilon). \end{cases}$$

(2.8)

Moreover, from the Lebesgue dominated convergence theorem, the estimate

$$\begin{array}{l} \left\|(-A)^{2-\beta} T(t-s)(g(s) - g(t))\right\| ds \\ \leq \frac{C_{2-\beta}}{(t-s)^{2-\beta-\beta}},\ t > s, \end{array} \tag{2.9}$$

and the assumption $\beta + \vartheta > 1$, follow that $u_1(t) \in D(A)$ for every $t \in [0,a]$ and that

$$\begin{array}{ll} -\int_0^t (-A)^{2-\beta} T(t-s)(g(s) - g(t))ds, & \text{for } t > 0, \\ 0, & \text{for } t = 0, \end{array} \tag{2.10}$$

since A is a closed operator. The continuity of Au_1 on $[0,a]$ follows from the Lebesgue dominated convergence theorem and inequality (2.9).

The property for $u'(\cdot)$ is proved in usual form. The proof is complete.

Lemma 3. *Under the assumptions of Lemma 2,*
$Au_1 \in C^{0,\nu}([0,a];X)$ *for* $\beta + \vartheta \geq 1 + \nu$.

Proof. At first we observe that for $t > s$ and $\xi \in (0,1)$,

$$\left\|(-A)^{2-\xi}T(t) - (-A)^{2-\xi}T(s)\right\| \leq \int_s^t \left\|(-A)^{3-\xi}T(\tau)\right\| d\tau$$

$$\leq C_{3-\xi} \int_s^t \frac{d\tau}{\tau^{3-\xi}}$$

$$\leq \frac{C_{3-\xi} t^{1-\xi}(t-s)}{t^{2-\xi} s^{2-\xi}},$$

and hence

$$\left\|(-A)^{2-\xi}\, T(t) - (-A)^{2-\xi}\, T(s)\right\| \leq \frac{C_{3-\xi}(t-s)}{t^{1-\xi} s^{2-\xi}}. \tag{2.11}$$

Let $\delta > 0$ and $t \in (\delta, a]$. Using (2.10), for $h > 0$ we get

$$\begin{aligned}
&\|Au_1(t+h) - Au_1(t)\| \\
&\leq \left\| \int_0^t (-A)^{2-\beta}(T(t+h-s) - T(t-s))(g(s) - g(t)) \right\| ds \\
&+ \int_0^t \left\|(-A)^{2-\beta} T(t+h-s)(g(t) - g(t+h))\right\| ds \\
&+ \int_t^{t+h} \left\|(-A)^{2-\beta} T(t+h-s)(g(s) - g(t+h))\right\| ds \\
&= I_1(t,h) + I_2(t,h) + I_3(t,h).
\end{aligned} \tag{2.12}$$

Next we estimate each $I_i(t,h)$ separately. From inequality (2.11) we find that

$$\begin{aligned} I_1(t,h) &\leq \int_0^t \Big\|(-A)^{2-\beta}(T(t+h-s)-T(t-s))(g(s)-g(t))\Big\|\,ds \\ &\leq \int_0^{t-h} \frac{C_{3-\beta}h ds}{(t+h-s)^{1-\beta}(t-s)^{2-\beta-\vartheta}} \\ &\quad + \int_{t-h}^{t} \frac{C_{3-\beta}h ds}{(t+h-s)^{1-\beta}(t-s)^{2-\beta-\vartheta}} \\ &\leq C_{3-\beta}h^{\beta+\vartheta-1}\int_0^{t-h} \frac{ds}{(t+h-s)^{1-\beta}} \\ &\quad + C_{3-\beta}h^{\beta}\int_{t-h}^{t} \frac{ds}{(t-s)^{2-\beta-\vartheta}} \\ &\leq c_1 h^{\beta+\vartheta-1} + c_2 h^{2\beta+\vartheta-1}, \end{aligned} \tag{2.13}$$

consequently

$$I_1(t,h) \leq c_3 h^{\beta+\vartheta-1}. \tag{2.14}$$

For the second term we see that

$$\begin{aligned} I_1(t,h) &\leq \int_0^t \Big\|(-A)^{2-\beta}\,T(t+h-s)(g(t+h)-g(t))\Big\|\,ds \\ &\leq c_4 h^{\vartheta}\int_0^t \frac{ds}{(t+h-s)^{2-\beta}} \\ &\leq c_5 \frac{h^{\vartheta}}{h^{1-\beta}}\ , \end{aligned}$$

and hence

$$I_2(t,h) \leq c_5 h^{\beta+\vartheta-1}. \tag{2.15}$$

Similarly, for the third term we find that

$$I_3(t,h) \leq c_6 h^{\beta+\vartheta-1}. \tag{2.16}$$

The assertion is now consequence of (2.12), (2.14), (2.15) and (2.16). The proof is complete.

Proposition 2. *Assume that the assumptions of Lemma 2 hold. If $u(\cdot)$ is the mild solution of (2.6), then the following properties are verified.*

(a) *If* $g \in C^{0,\mu}([0,a]; X_{1-\beta})$ *then*
$u \in C^{0,\varrho}([\delta,a]; X_1) \bigcap C^{1,\varrho}([\delta,a]; X)$ *for every* $\delta > 0$ *and every* $0 < \varrho \leq \min\{\mu, \beta + \vartheta - 1\}$.

(b) *If* $g \in C^{0,\mu}([0,a]; X_{1-\beta})$ *and* $g(0) = 0$, *then*
$u \in C^{0,\varrho}([0,a]; X_1) \bigcap C^{1,\varrho}([0,a]; X)$ *for every* $\varrho \leq \min\{\mu, \beta + \vartheta - 1\}$.

(c) *If* $g \in C([0,a]; X_{1-\beta+\nu})$ *then*
$u \in C^{0,\varrho}([0,a]; X_1) \bigcap C^{1}([0,a]; X)$ *for every* $\varrho < \nu$.

Proof. From Lemma 2, $Au \in C([0,a]; X)$ and

$$\begin{aligned} Au(t) &= -\int_0^t (-A)^{2-\beta} T(t-s)(g(s) - g(t))ds \\ &\quad -\int_0^t (-A)^{2-\beta} T(t-s) g(t) ds \\ &= -\int_0^t (-A)^{2-\beta} T(t-s)(g(s) - g(t))ds \\ &\quad -(-A)^{1-\beta} T(t) g(t) + A^{1-\beta} g(t) \\ &:= Au_1(t) + v(t) + w(t). \end{aligned}$$

Next we discuss (**a**), (**b**), (**c**) separately.

(a) Let $\delta > 0$. From the assumptions and Lemma 3, we know that $w \in C^{0,\mu}([0,a]; X)$ and that $Au_1 \in C^{0,\beta+\vartheta-1}([0,a]; X)$. On the other hand, since

$$\begin{aligned} \|v(t+h) - v(t)\| &\leq \left\|(T(t+h) - T(t))(-A)^{1-\beta} g(t+h)\right\| \\ &\quad + \left\|(-A)^{1-\beta} T(t)(g(t+h) - g(t))\right\| \end{aligned}$$

the followings inequalities hold

$$\|v\,(t+h) - v\,(t)\| \leq \begin{cases} \frac{c_1 h}{\delta} + \frac{c_2 h^{\vartheta}}{\delta^{1-\beta}} \\ \frac{c_3 h}{\delta} + c_4 h^{\mu} \ , \\ \frac{c_5 h^{v}}{\delta^{1-\beta+\vartheta}} + \frac{c_6 h^{\vartheta}}{\delta^{1-\beta}} \ , \ \nu < \beta \\ \frac{c_7 h^{\nu}}{\delta^{1-\beta+\vartheta}} + c_8 h^{\mu} \ , \ \nu < \beta \end{cases}$$

and hence

$$\|v(t+h)-v(t)\| \le c_3 \frac{h^{\max\{\vartheta,\mu\}}}{\delta}$$

Consequently, $u \in C^{0,\varrho}([\delta,a];X) \bigcap C^{1,\varrho}([\delta,a];X)$ for $\varrho \le \min\{\mu, \beta+\vartheta-1\}$.

(b) As in the previous case, $w \in C^{0,\mu}([0,a];X)$ and $Au_1 \in C^{\beta+\vartheta-1}([0,a];X)$. Moreover, for $h>0$ we find that

$$\begin{aligned}
\|v(t+h)-v(t)\| &\le \left\|T(t+h)((-A)^{1-\beta}g(t+h)-(-A)^{1-\beta}g(t))\right\| \\
&\quad + \int_t^{t+h} \left\|AT(s)[(-A)^{1-\beta}g(t)-(-A)^{1-\beta}g(0)]\right\| ds \\
&\le \left\|T(t+h)((-A)^{1-\beta}g(t+h)-(-A)^{1-\beta}g(t))\right\| \\
&\quad \int_t^{t+h} c_1 s^{\mu-1} ds \\
&\le \begin{cases} \dfrac{c_1 h^{\vartheta}}{h^{1-\beta}} + c_2 h^{\mu}, & \text{or} \\ c_3 h^{\mu}, \end{cases} \\
&\le c_4 h^{\mu},
\end{aligned}$$

thus $u \in C^{0,\varrho}([0,a];X_1) \bigcap C^{1,\varrho}([0,a];X)$ for $\varrho \le \min\{\beta+\vartheta-1, \mu\}$.

(c) From Lemma 2, we know that

$$(2.17) \quad Au(t) = -\int_0^t (-A)^{2-\beta} T(t-s) g(s) ds\,, \qquad t \ge 0.$$

Under this remark, for $t \in [0,a]$, $h>0$ and $\varrho < \nu$ we get

$$\begin{aligned}
&\|Au(t+h)-Au(t)\| \\
\le\ & \int_0^t \left\|(T(h)-I)(-A)^{1-\nu}T(t-s)(-A)^{1-\beta+\nu}g(s)\right\| ds \\
&+ \int_t^{t+h} \left\|(-A)^{1-\nu}T(t+h-s)(-A)^{1-\beta+\nu}g(s)\right\| ds \\
\le\ & C_{\xi} h^{\varsigma} \left\|(-A)^{1-\beta+\nu}g\right\|_a \int_0^t \tfrac{ds}{(t-s)^{\varrho+1-\nu}} \\
&+ C_{1-\nu} \left\|(-A)^{1-\beta+\nu}g\right\|_a \int_t^{t+h} \tfrac{ds}{(t+h-s)^{1-\nu}} ds \\
\le\ & c_1 h^{\varrho} + c_2 h^{\nu}.
\end{aligned}$$

Thus, $u \in C^{0,\varrho}([0,a];X_1) \bigcap C^1([0,a];X)$ for $\varrho < \nu$. This complete the proof.

The following result can be proved using the steps in the proof of Proposition 2. We will omit the proof.

Corollary 1. *Let ϑ, $\beta, \eta \in (0,1)$ with $\beta + \vartheta > 1$;*
$f(\cdot) \in C^{0,\eta}([0,a];X)$ and $g(\cdot) \in C^{0,\vartheta}([0,a];X) \bigcap C([0,a];X_{1-\beta})$. If $u(\cdot)$ is the mild solution of

$$(2.18)\quad \begin{cases} x'(t) &= Ax(t) + A^{1-\beta}g(t) + f(t), \quad t \in [0,a], \\ x(0) &= 0, \end{cases}$$

then the following properties are verified:

(a) $u \in C([0,a];X_1) \bigcap C^1([0,a],X)$,

(b) *If $g \in C^{0,\mu}([0,a];X_{1-\beta})$ then $u \in C^{0,\varrho}([\delta,a];X_1) \bigcap C^{1,\varrho}([\delta,a];X)$ for every $0 < \varrho \leq \min\{\mu, \beta + \vartheta - 1, \eta\}$,*

(c) *If $g \in C^{0,\mu}([0,a];X_{1-\beta})$ and $g(0) = f(0) = 0$ then $u \in C^{0,\varrho}([0,a];X_1) \bigcap C^{1,\varrho}([0,a];X)$ for each $0 < \varrho \leq \min\{\mu, \beta + \vartheta - 1, \eta\}$,*

(d) *If $g \in C([0,a];X_{1-\beta+\nu})$ and $f \in C([0,a];X_\mu)$ then $u \in C^{0,\varrho}([0,a];X_1) \bigcap C^1([0,a];X)$ for every $0 < \varrho < \min\{\mu, \nu\}$.*

In the rest of this paper, we always assume that the functions F, G verifies the hypothesis in Theorem 1.1. Moreover, to simplify our notations, we only consider the case $\sigma = \mathbf{0}$.

Now we establish a first result about the existence of regular solutions; specifically we prove existence of N-classical solutions.

Theorem 2. *Assume that there exist constants $0 < \alpha < \beta < 1$; $0 < \gamma_1, \gamma_2 \leq 1$ and an open subset $\Omega_\alpha \subset \mathcal{B}_\alpha$ such that $F : [0,a] \times \Omega_\alpha \to X_1$ and $G : [0,a] \times \Omega_\alpha \to X$ are continuous functions and that the following conditions hold:*

(a) *The function F is X_β-valued and there exist positive constants L_i, L_i', $i = 1, 2$, such that*

$$\begin{aligned}
&\left\|(-A)^{\beta} F(t,\psi_1) - (-A)^{\beta} F(s,\psi_2)\right\| \\
&\leq L_1 \mid t-s \mid^{\gamma_1} + L_1' \left\|\psi_1 - \psi_2\right\|_{\mathcal{B}_\alpha} , \\
&\left\|G(t,\psi_1) - G(s,\psi_2)\right\| \\
&\leq L_2 \mid t-s \mid^{\gamma_2} + L_2' \left\|\psi_1 - \psi_2\right\|_{\mathcal{B}_\alpha} ,
\end{aligned}$$

for every $0 \leq s, t \leq a$, $\psi_1, \psi_2 \in \Omega_\alpha$.

(b) $K(0)L_1' \left\|(-A)^{\alpha-\beta}\right\| < 1.$

(c) *$\varphi \in \Omega_\alpha$ and there exists $0 < \xi \leq 1$ such that the function $W(\cdot)\varphi$ is ξ-Hölder on $[0, a]$.*

If $\beta + Min\{\beta - \alpha, \xi, \gamma_1, \gamma_2\} > 1$, then there exists a unique N-classical solution $x(\cdot, \varphi)$ of the abstract Cauchy problem (1.1) defined on $(-\infty, b)$, for some $0 < b < a$.

Proof: From our assumptions on the operator A, see lemma (1), we fix positive constants C_α and $C_{\alpha+1-\beta}$ such that for all $t \in (0, T]$

$$\|(-A)^{\alpha} T(t)\| \leq C_\alpha t^{-\alpha}$$

and

$$\left\|(-A)^{1-\beta+\alpha} T(t)\right\| \leq C_{\alpha+1-\beta} t^{-(1-\beta+\alpha)}.$$

Let $b_1 > 0$ and $\delta > 0$ such that $V = \{(s, \psi) : 0 \leq s \leq b_1, \|\psi - \varphi\|_{\mathcal{B}_\alpha} < \delta\} \subset [0, a] \times \Omega_\alpha$ and $\mu = K_{b_1} \parallel (-A)^{\alpha-\beta} \parallel L_1' < 1$. Now we choose $0 < b < b_1$ such that

$$\sup_{0\in[0,b]} \left\|W(\theta)(-A)^{\alpha}(\varphi) - (-A)^{\alpha}(\varphi)\right\|_{\beta} < \frac{(1-\mu)}{4}\delta \ , \tag{2.19}$$

$$\begin{aligned}
&K_b \sup_{0\in[0,b]} \left\|T(\theta)(-A)^{\alpha}\ \varphi(0) - (-A)^{\alpha}\ \varphi(0)\right\| \\
&\qquad < \tfrac{(1-\mu)}{8}\delta,
\end{aligned} \tag{2.20}$$

$$K_b \sup_{0\in[0,b]} \|T(\theta)(-A)^{\alpha}\ f(0,\varphi) - (-A)^{\alpha}\ f(0,\varphi)\| < \tfrac{(1-\mu)}{4}\delta, \tag{2.21}$$

$$K_b L_1 \left\|(-A)^{\alpha-\beta}\right\| b^{\gamma_1} < \frac{(1-\mu)}{8}\delta, \tag{2.22}$$

$$\frac{b^{\beta-\alpha}}{\beta-\alpha}\ K_b\ C_{\alpha+1-\beta}\{L_1 b^{\gamma_1} + L_1\prime\delta + \|\ \left\|(-A)^{\beta}F(0,\varphi)\right\| < \frac{(1-\mu)}{4}\delta, \tag{2.23}$$

$$K_b\frac{b^{1-\alpha}}{1-\alpha}\ C_{\alpha}\{L_2 b^{\gamma_2} + L_2\prime\delta + \|\ G(0,\varphi)\ \|\} < \frac{(1-\mu)}{4}\delta, \tag{2.24}$$

$$K_b\{C_{\alpha+1-\beta}\ L_1\prime\ \frac{b^{\beta-\alpha}}{\beta-\alpha} + C_{\alpha}\ L_2'\frac{b^{1-\alpha}}{1-\alpha}\} < \frac{(1-\mu)}{2} \tag{2.25}$$

In the space $Y = C([0,b] : X)$ provided with the topology of uniform convergence, we define;

$$A(\varphi,\alpha,b) = \{u \in Y :\ u(0) = (-A)^{\alpha}\varphi(0),\ \ \|\ (-A)^{\alpha}\varphi - \tilde{u}_t\ \|_{\mathcal{B}} \leq \delta, \forall\ t\in[0,b]\ \},$$

where $\tilde{u}$ is the extension of u to $(-\infty,b]$ with $\tilde{u}_0 = (-A)^{\alpha}\varphi$. From (2.19), it follows that $A(\varphi,\alpha,b)$ is a nonempty, convex and closed subset of Y. On $A(\varphi,\alpha,b)$ we define the operator Φ by the expression

$$\begin{aligned}\Phi(u)(t) = T(t)((-A)^{\alpha}(\varphi(0) + F(0,\varphi)) - (-A)^{\alpha}F(t,(-A)^{-\alpha}\tilde{u}_t)\\ + \textstyle\int_0^t(-A)^{\alpha+1-\beta}T(t-s)(-A)^{\beta}F(s,(-A)^{-\alpha}\tilde{u}_s)ds\\ + \textstyle\int_0^t(-A)^{\alpha}T(t-s)G(s,(-A)^{-\alpha}\tilde{u}_s)ds\end{aligned}$$

In order to use the contraction mapping principle, we now show that the range of Φ is included in $A(\varphi,\alpha,b)$. To this end we introduce the functions $y_{\alpha}, z^i : (-\infty,b] \to X$, $i = 1,2,3$, where $y_{\alpha}(t) = T(t)(-A)^{\alpha}\varphi$ for $t \geq 0$, $(y_{\alpha})_0 = (-A)^{\alpha}\varphi$ and

$$\begin{aligned}z^1(t) &= T(t)(-A)^{\alpha}\ F(0,\varphi) - (-A)^{\alpha}\ F(t,(-A)^{-\alpha}\tilde{u}_t),\\ z^2\,(t) &= \textstyle\int_0^t(-A)^{\alpha+1-\beta}\ T(t-s)(-A)^{\beta}\ F(s,(-A)^{-\alpha}\tilde{u}_s)ds,\\ z^3\,(t) &= \textstyle\int_0^t(-A)^{\alpha}\ T(t-s)G(s,(-A)^{-\alpha}\tilde{u}_s)ds,\end{aligned}$$

for $t > 0$, $z^i_0 = 0$ for $i \in \{1,2,3\}$. Clearly

$$\Phi(u)(t) = y_{\alpha}(t) + z^1(t) + z^2(t) + z^3(t) \tag{2.26}$$

on $[0,b]$ and $\widetilde{\Phi(u)}_t = (y_\alpha)_t + z_t^1 + z_t^2 + z_t^3$. With the previous notations, for $t \in [0,b]$ we have,

$$\begin{aligned}(2.27)\quad &\left\|\widetilde{\Phi(u)}_t - (-A)^\alpha\varphi\right\|_{\mathcal{B}} \\ &\leq \|(y_\alpha)_t - (-A)^\alpha\varphi\|_{\mathcal{B}} + \|z_t^1\|_{\mathcal{B}} + \|z_t^2\|_{\mathcal{B}} + \|z_t^3\|_{\mathcal{B}}\,.\end{aligned}$$

Using axiom (**A**) concerning the phase space, we estimate each term on the right hand side of (2.27) separately. Directly from the choice of b we get the estimate

$$\begin{aligned}(2.28)\quad \|\ (y_\alpha)_t - (-A)^\alpha\varphi\ \|_{\mathcal{B}} &\leq\ \|W(t)(-A)^\alpha\varphi - (-A)^\alpha\varphi\|_{\mathcal{B}} \\ &\leq \frac{(1-\mu)}{4}\delta.\end{aligned}$$

On the other hand, for $t \in [0,b]$

$$(2.29)\qquad \left\|z_t^1\right\|_{\mathcal{B}} \leq K_b \sup_{s\in\ [0,t]} \left\|z^1(s)\right\|$$

and for

$$\begin{aligned}&\|z^1(s)\| \leq \|T(s)(-A)^\alpha\ F(0,\varphi) - (-A)^\alpha\ F(0,\varphi)\| \\ &\quad + \|(-A)^\alpha F(0,\varphi) - (-A)^\alpha F(s,(-A)^{-\alpha}\tilde{u}_s)\| \\ &\leq \frac{(1-\mu)}{8K_b}\delta + L_1\left\|(-A)^{\alpha-\beta}\right\| b^{\gamma_1} \\ &\quad + L_1'\left\|(-A)^{\alpha-\beta}\right\| \|(-A)^\alpha\ \ \varphi - \widetilde{u}_s\|_{\mathcal{B}}\end{aligned}$$

hence

$$(2.30)\qquad \|\ z^1(s)\ \| \leq \frac{2(1-\mu)}{8K_b}\delta + L_1'\left\|(-A)^{\alpha-\beta}\right\|\delta,$$

and substituting (2.30) into (2.29)

$$(2.31)\qquad \|\ z_t^1\ \|_{\mathcal{B}} \leq \frac{(1-\mu)}{4}\delta + \mu\delta.$$

Now, for the function $z^2(\cdot)$ we have that

$$(2.32)\qquad \|\ z_t^2\ \|_{\mathcal{B}} \leq K_b \sup_{s\in[0,t]} \left\|z^2(s)\right\|$$

and for $s \in [0,t]$

$$
\begin{aligned}
&\|z^2(s)\| \\
&\leq \int_0^s \left\|(-A)^{\alpha+1-\beta}T(s-\theta)((-A)^{\beta}F(\theta,(-A)^{-\alpha}\tilde{u}_{\theta}) \right.\\
&\quad \left. -(-A)^{\beta}F(0,\varphi))\right\| d\theta \\
&\quad + \int_0^s \left\|(-A)^{\alpha+1-\beta}T(s-\theta)(-A)^{\beta}F(0,\varphi))\right\| d\theta \\
&\leq \int_0^s \frac{C_{\alpha+1-\beta}}{(s-\theta)^{\alpha+1-\beta}} \ \{L_1\theta^{\gamma 1}\,\|\widetilde{u}_{\theta}-(-A)^{\alpha}\varphi\|_{\mathcal{B}}\}\}d\theta \\
&\quad + \frac{b^{\beta-\alpha}}{\beta-\alpha}C_{\alpha+1-\beta}\ \left\|(-A)^{\beta}F(0,\varphi)\right\| \\
&\leq (L_1 b^{\gamma_1}+L_1\prime\delta)\frac{b^{\gamma-\alpha}}{\beta-\alpha}C_{\alpha+1-\beta} \\
&\quad + \frac{b^{\beta-\alpha}}{\beta-\alpha}C_{\alpha+1-\beta}\ \left\|(-A)^{\beta}F(0,\varphi)\right\|.
\end{aligned}
$$

Employing the last inequality in (2.32) we obtain that;

$$
\begin{aligned}
&\| z_t^2 \| \\
(2.33)\quad &\leq \tfrac{b^{\beta-\alpha}}{\beta-\alpha}K_b C_{\alpha+1-\beta}\left\{L_1 b^{\gamma_1}+L_1'\delta+\left\|(-A)^{\beta}F(0,\varphi)\right\|\right\} \\
&\leq \tfrac{(1-\mu)}{4}\delta.
\end{aligned}
$$

Similarly for z^3,

$$
\| z_t^3 \|_{\mathcal{B}} \leq K_b\frac{b^{1-\alpha}}{1-\alpha}C_{\alpha}\left\{L_2 b^{\gamma_2}+L_2'\delta+\| G(0,\varphi) \|\right\} \leq \frac{(1-\mu)}{4}\delta.
$$
(2.34)

Combining (2.27), (2.28), (2.31), (2.33) and (2.34), we conclude that $\Phi(u) \in A(\varphi,\alpha,b)$.

Now we prove that Φ is a contraction. Let $u,v \in A(\varphi,\alpha,b)$, so that

$$
\begin{aligned}
&\|\Phi(u)(t)-\Phi(v)(t)\| \\
&\leq \|(-A)^{\alpha}F(t,(-A)^{-\alpha}\tilde{u}_t)-(-A)^{\alpha}F(t,(-A)^{-\alpha}\tilde{v}_t)\| \\
&\quad + \int_0^t\left\|(-A)^{\alpha+1-\beta}\,T(t-s)\left((-A)^{\beta}\,F\left(s,(-A)^{-\alpha}\ \widetilde{u}_s\right)\right.\right.
\end{aligned}
$$

$$-(-A)^{-\beta}\, F\left(s,(-A)^{-\alpha}\ \tilde{v}_s\right)\right)\Big\|\, ds$$

$$+\int_0^t \|(-A)^{\alpha}\, T(t-s)(G(s,(-A)^{-\alpha}\,\tilde{u}_s)-G(s,(-A)^{-\alpha}\,\tilde{v}_s))\|\, ds$$

$$\leq \left\|(-A)^{\alpha-\beta}\right\| L_1'\, \|\tilde{u}_t-\tilde{v}_t\|_{\mathcal{B}}$$

$$+\int_0^t \left(\frac{C_{\alpha+1-\mathcal{B}}\, L_1'}{(t-s)^{\alpha+1-\mathcal{B}}}+\frac{C_\alpha L_2'}{(t-s)^{\alpha}}\right)\|\tilde{u}_s-\tilde{v}_s\|_{\mathcal{B}}\ ds,$$

thus

$$\begin{aligned} &\| \Phi(u)-\Phi(v)\|_b \leq K_b\{\| (-A)^{\alpha-\beta}\| L_1' \\ &+C_{\alpha+1-\beta}L_1'\tfrac{b^{\beta-\alpha}}{\beta-\alpha}+C_\alpha L_2'\tfrac{b^{1-\alpha}}{1-\alpha}\}\ \| u-v\|_{\mathcal{B}}\,. \end{aligned} \tag{2.35}$$

From (2.25), (2.35) and the contraction mapping principle, we can conclude that Φ has a unique fixed point $x(\cdot)$ in $A(\varphi,\alpha,b)$.

From the assumptions on F and G, it follows that the functions $t\to G(t,(-A)^{-\alpha}\tilde{x}_t)$ and $t\to(-A)^{\beta}F(t,(-A)^{-\alpha}\tilde{x}_t)$ are continuous and bounded on $[0,b]$. In the following N will be a positive constant such that

$$\begin{gathered}\left\|(-A)^{\beta}\ F(t,(-A)^{-\alpha}\ \tilde{x}_t)\right\|\leq N,\\ \|G(t,(-A)^{-\alpha}\ \tilde{x}_t)\|\leq N,\end{gathered}$$

for all $t\in\ [0,b]$. Next we will prove that the functions $t\to(-A)^{\beta}F(t,(-A)^{-\alpha}\tilde{x}_t)$ and $t\to G(t,(-A)^{-\alpha}\tilde{x}_t)$ are Hölder continuous on $[0,b]$. From Lemma 1 and the condition $\beta+Min\{\beta-\alpha,\xi,\gamma_1,\gamma_2\}>1$, we fix $0<\vartheta<\min\{\beta-\alpha,\xi,\gamma_1,\gamma_2\}$ with $\vartheta+\beta>1$ and $\tilde{C}>0$ such that for all $0<s<t<b_1$ and $0<h<1$

$$\|(T(h)-I)(-A)^{\alpha}T(t-s)\|\leq\tilde{C}h^{\vartheta}(t-s)^{-(\vartheta+\alpha)}, \tag{2.36}$$

$$\begin{aligned}&\left\|(T(h)-I)(-A)^{\alpha+1-\beta}T(t-s)\right\|\\ &\leq\tilde{C}h^{\vartheta}(t-s)^{-(\alpha+1-\beta+\vartheta)}\,.\end{aligned} \tag{2.37}$$

For $t\in[0,b)$ and $0<h<1$ with $t+h<b$ we get

$$
\begin{aligned}
&\|x(t+h)-x(t)\| \\
&\leq C_1 h^{\xi} + C_2 h^{1-\alpha} \\
&+ \|(-A)^{\alpha} F(t+h,(-A)^{-\alpha}\tilde{x}_{t+h}) - (-A)^{\alpha}F(t,(-A)^{-\alpha}\tilde{x}_t)\| \\
&+ \int_0^t \left\|(-A)^{\alpha+1-\beta}\,(T(h)-I)T(t-s)(-A)^{\beta}\,F(s,(-A)^{-\alpha}\,\tilde{x}_s)\right\| ds \\
&+ \int_t^{t+h} \left\|(-A)^{\alpha+1-\beta}\,T(t+h-s)(-A)^{\beta}\,F(s,(-A)^{-\alpha}\,\tilde{x}_s)\right\| ds \\
&+ \int_t^{t+h} \|(-A)^{\alpha}\,\,(T(h)-I)\,\,T\,(t-s)\,G\,(s,(-A)^{\alpha}\,\,\widetilde{x}_s)\|\, ds \\
&+ \int_t^{t+h}\,\,\|(-A)^{\alpha}\,T(t+h-s)G(s,(-A)^{-\alpha}\tilde{x}_s)\|\, ds.
\end{aligned} \tag{2.38}
$$

We estimate each term on the right hand side of the last inequality separately. For the third term we have

$$
\begin{aligned}
I_3 &\leq \left\|(-A)^{\alpha-\beta}\right\| \left\|(-A)^{\beta}\,F(t+h,(-A)^{-\alpha}\,\tilde{x}_{t+h})\right. \\
&\quad \left. -(-A)^{\beta}F(t,(-A)^{-\alpha}\tilde{x}_t)\right\| \\
&\leq \left\|(-A)^{\alpha-\beta}\right\| \{L_1\,h^{\gamma_1} + L_1'\,\|\widetilde{x}_{t+h}-\widetilde{x}_t\|_{\mathcal{B}}\} \\
&\leq \left\|(-A)^{\alpha-\beta}\right\| L_1\,h^{\gamma_1} + \,\|\,(-A)^{\alpha-\beta}\,\|\,L_1' M_b\,\|\,\,\|\widetilde{x}_h-(-A)^{\alpha}\varphi\|_{\mathcal{B}} \\
&\quad +(-A)^{\alpha-\beta}\,\|\,L_1'\,K_b\,\sup_{\theta\in[0,t]}\|\,x(\theta+h)-x(\theta)
\end{aligned}
$$

equivalently,

$$
\begin{aligned}
I_3 &\leq C_3\,h^{\gamma_1} + \|\,(-A)^{\alpha-\beta}\,\|\,L_1'\,M_b\,\|\,\|\,\widetilde{x}_h-(-A)^{\alpha}\varphi\,\|_{\mathcal{B}} \\
&+ \left\|(-A)^{\alpha-\beta}\right\| L_1'\,K_b \sup\nolimits_{\theta\in[0,t]}\|x(\theta+h)-x(\theta)\|\,,
\end{aligned} \tag{2.39}
$$

for some constant C_3, independent of t and h.

With respect to the fourth term, we get

$$
\begin{aligned}
I_4 &= \int_0^t \left\|(T(h)-I)(-A)^{\alpha+1-\beta}T(t-s)(-A)^{\beta}\right. \\
&\quad \left. F(s,(-A)^{-\alpha}x_s)\right\| ds \\
&\leq \int_0^t \frac{h^{\vartheta}\,N\widetilde{C}}{(t-s)^{\alpha+1-\beta+\vartheta}} ds \\
&\leq \frac{t^{\beta-\alpha-\vartheta}}{\beta-\alpha-\vartheta}\widetilde{C}\,Nh^{\vartheta},
\end{aligned}
$$

which can be abbreviated as

$$I_4 \leq C_4 h^{\vartheta}, \tag{2.40}$$

where C_4 is a constant independent of t and h.

For I_5 we find that

$$\begin{aligned} I_5 &\leq \int_t^{t+h} \left\| (-A)^{\alpha+1-\beta}\, T(t+h-s)(-A)^{\beta}\, F(s,(-A)^{-\alpha}x_s) \right\| ds \\ &\leq C_{\alpha+1-\beta} N\, \tfrac{h^{\beta-\alpha}}{\beta-\alpha} \leq C_5 h^{\vartheta} \end{aligned}$$

then

$$I_5 \leq C_5 h^{\vartheta} \tag{2.41}$$

where C_5 is a constant independent of $t \in [0,b)$ and $0 < h < 1$.

In a similar manner we can prove that

$$I_6 \leq C_6 h^{\vartheta} \quad \text{and} \quad I_7 \leq C_7 h^{\vartheta}, \tag{2.42}$$

where C_6 and C_7 are positive constants independents of $t \in [0,b)$ and $0 < h < 1$.

Using the estimates (2.39)-(2.42), there exists a constant $\tilde{C}_1 > 0$, independent of $t \in [0,b)$ such that for $0 < h < 1$ with $0 < t+h < b$,

$$\|x(t+h) - x(t)\| \leq \tilde{C}_1 h^{\vartheta} + \left\| (-A)^{\alpha-\beta} \right\| L_1{\prime} M_b \| \tilde{x}_h - (-A)^{-\alpha}\varphi \|_{\mathcal{B}}$$

$$+ \left\| (-A)^{\alpha-\beta} \right\| L_1' \, K_b \sup_{\theta \in [0,t]} \|x(\theta+h) - x(\theta)\|$$

consequently

$$\begin{aligned} \| x(\theta+h) - x(\theta) \|_{[0,t]} \leq \tfrac{\tilde{C}_1 h^{\vartheta}}{1-\mu} + \tfrac{M_b L_1'}{1-\mu} \| (-A)^{\alpha-\beta} \| \\ \| \tilde{x}_h - (-A)^{-\alpha}\varphi \|_{\mathcal{B}} \end{aligned} \tag{2.43}$$

since $0 < \mu = K_{b_1} \| (-A)^{\alpha-\beta} \| L_1' < 1$. Moreover, using the definition of y_α and the decomposition of $x(t) = \Phi(x)(t)$ as indicated in (2.26), it is possible to prove that

$$\| \tilde{x}_h - (-A)^\alpha \varphi \|_{\mathcal{B}} \leq \|W(h)(-A)^\alpha \varphi - (-A)^\alpha \varphi\|_{\mathcal{B}} + \textstyle\sum_{i=1}^{3} \|z_h^i\|_{\mathcal{B}}$$

$$\leq C_8 h^\xi + \| z_h^1 \|_{\mathcal{B}} + C_9 h^\vartheta. \tag{2.44}$$

Now from axiom (**A**)

$$\left\|z_h^1\right\|_{\mathcal{B}} \leq K_h \sup_{s\in[0,h]} \left\|T(s)(-A)^\alpha F(0,\varphi) - (-A)^\alpha F(s,(-A)^{-\alpha}\tilde{x}_s)\right\| \tag{2.45}$$

and for $0 \leq s \leq h$

$$\begin{aligned} &\| T(s)(-A)^\alpha F(0,\varphi) - (-A)^\alpha F(s,(-A)^{-\alpha}\tilde{x}_s) \| \\ (2.46) \leq\ & \| T(s)(-A)^\alpha F(0,\varphi) - (-A)^\alpha F(0,\varphi) \| \\ &+ L_1 s^{\gamma_1} \| (-A)^{\alpha-\beta} \| \\ &+ L_1' \| (-A)^{\alpha-\beta} \| \| (-A)^\alpha \varphi - \tilde{x}_s \|_{\mathcal{B}} \\ \leq\ & C_{10}(h^{1-\alpha} + h^{\gamma_1}) + L_1' \| (-A)^{\alpha-\beta} \| \| (-A)^\alpha \varphi - \tilde{x}_s \|_{\mathcal{B}} . \end{aligned}$$

Employing this last inequality in (2.45), it follows that:

$$\begin{aligned} &\|z_h^1\|_{\mathcal{B}} \leq C_{10} K_b (h^{1-\alpha} + h^{\gamma_1}) \\ &+ K_b L_1' \left\|-A^{\alpha-\beta}\right\| \|-A^\alpha \varphi - \tilde{x}_\tau\|_{\tau\in[0,h]} . \end{aligned} \tag{2.47}$$

From (2.47), (2.44), the choice of b and the fact that $\vartheta < Min\{\beta-\alpha, \xi, \gamma_1, \gamma_2\}$; for $0 < h < 1$ we find that

$$\begin{aligned} &\|\tilde{x}_\tau - (-A)^\alpha \varphi\|_{\tau\in[0,h]} \leq C_{11} h^\vartheta \\ &+ K_b L_1' \left\|(-A)^{\alpha-\beta}\right\| \|(-A)^\alpha \varphi - \tilde{x}_\tau\|_{\tau\in[0,h]} \end{aligned}$$

thus

$$\|x_h - (-A)^\alpha \varphi\|_{\mathcal{B}} \leq \frac{C_{11}}{1-\mu} h^\vartheta. \tag{2.48}$$

The inequalities (2.48) and (2.43) show that there exists $C_{12} > 0$, independent of $\theta \in [0,b)$ and $h > 0$, such that

$$\|x(\theta + h) - x(\theta)\| \leq C_{12} h^\vartheta \tag{2.49}$$

where $\vartheta < Min\{\beta-\alpha, \xi, \gamma_1, \gamma_2\}$ and $\vartheta + \beta > 1$.

From (2.48), (2.49) and axiom (**A**) follows that the functions $(-A)^{\beta}F(t, (-A)^{-\alpha}\tilde{x}_t)$ and $G(t, (-A)^{-\alpha}\tilde{x}_t)$ are ϑ-Hölder with $\vartheta + \beta > 1$.

We know from Corollary (1), that the abstract Cauchy problem

$$(2.50) \qquad \begin{aligned} w'(t) &= Aw(t) + (-A)^{1-\beta}((-A)^{\beta}F(t, (-A)^{-\alpha}\tilde{x}_t)) + G(t, (-A)^{-\alpha}\tilde{x}_t), \\ x(0) &= \varphi(0) + F(0, \varphi), \end{aligned}$$

has a unique classical solution $y \in C((0, b]; X_1)$ which is given by

$$(2.51) \qquad \begin{aligned} y(t) = T(t)(\varphi(0) + F(0, \varphi)) &+ \textstyle\int_0^t (-A)^{1-\beta}T(t-s)(-A)^{\beta} F(s, (-A)^{-\alpha}\tilde{x}_s)ds \\ &+ \textstyle\int_{\sigma}^t T(t-s)G(s, (-A)^{-\alpha}\tilde{x}_s)ds. \end{aligned}$$

Operating on (2.51) with $(-A)^{\alpha}$ and using the ideas in the proof of Lemma 2, follows that $(-A)^{\alpha}y(t) = x(t) + (-A)^{\alpha}F(t, (-A)^{-\alpha}\tilde{x}_t)$, which in turn implies that $z = (-A)^{-\alpha}x \in C((0, b]; X_1)$ since $t \to F(t, x_t)$ is continuous with values in X_1. Let $\tilde{z} : (-\infty, b] \to X$ a extension of z such that $\tilde{z}_0 = \varphi$. Clearly, $\tilde{z}$ is a N-classical solution of the neutral problem (1.1). The proof is complete.

Now we turn our attention to the problem of existence of classical solutions. In the rest of this paper, for a function $j : [0, a] \times \mathcal{B} \to X$ and $h \in \mathbb{R}$ we use the notation $\partial_h j$ for the function

$$\partial_h j(t) := \frac{j(t+h, \psi) - j(t, \psi)}{h}.$$

Moreover, if j is differentiable we will employ the following decomposition

$$(2.52) \qquad \begin{aligned} & j(t+s, \psi+\psi_1) - j(t, \psi) \\ &= (D_1 j(t, \psi), D_2 j(t, \psi))(s, \psi_1) \\ &\quad + \|(s, \psi_1)\| \, R(j, t, \psi, s, \psi_1) \end{aligned}$$

where

$$R(j,t,\psi,s,\psi_1) \to 0 \ \text{ as } \ (s,\psi_1) \to 0. \tag{2.53}$$

To prove Theorem 3 below, we will employ the following property.

Lemma 4. *Let X, Y be Banach spaces, $\Omega \subset X$ open, $K \subset \Omega$ compact and $f : \Omega \subset X \to Y$ be a continuously differentiable function. Then, for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\|f(x) - f(y) - (D_x f)(x-y)\| \leq \epsilon \|x-y\|,$$

for every $x, y \in K$ such that $\|x-y\| < \delta$.

The next result establishes the existence of classical solutions for the neutral system (1.1), making use of usual regularity assumptions for the functions $(-A)^\beta F$ and G.

Theorem 3. *Let assumptions in Theorem 1 be satisfied. Assume that $\varphi \in D(A_W)$, that $(-A)^\beta F$ and G are continuously differentiable on $[0,a] \times \Omega$, that F is continuous with values in X_1 and that $D((-A)^\beta F)(0,\varphi) \equiv 0$. If $\mathcal{X}_{G(0,\varphi)} \equiv 0$ or $\mathcal{X}_{G(0,\varphi)} \in \mathcal{B}$ and $\mathcal{B}$ satisfies axiom $\mathbf{C_3}$, then there exists a unique classical solution of the system (1.1) defined on $[0,b]$ for some $0 < b < a$.*

Proof. Let $u := u(\cdot,\varphi)$ the mild solution of (1.1). In the following we assume that $u(\cdot)$ is defined on $(-\infty, 2b]$ where $0 < 2b < a$ and

$$\begin{aligned}\mu = K_{2b}\Big[&\| D_2F(s,u_s) \| + \tfrac{b^\beta C_{1-\beta}}{\beta} \| D_2(-A)^\beta F(s,u_s) \| \\ &+ b\tilde{M} \|D_2G(s,u_s)\|_{2b}\Big] < 1.\end{aligned} \tag{2.54}$$

Let $z(\cdot)$ be the solution of the integral equation

$$\begin{aligned} z(t) &= T(t)A\varphi(0) + p(t) - D_2F(t,u_t)(z_t) \\ &+ \int_0^t (-A)^{1-\beta}T(t-s)[D_2(-A)^\beta F(s,u_s)](z_s)ds \\ &+ \int_0^t T(t-s)D_2G(s,u_s)(z_s)ds, \qquad t \geq 0, \end{aligned} \tag{2.55}$$

with initial condition

$$z_0 = A_W(\varphi) + \mathcal{X}_{G(0,\varphi)}, \tag{2.56}$$

and where

$$\begin{aligned} p(t) &= -D_1F(t,u_t) + \int_0^t (-A)^{1-\beta}T(t-s)D_1(-A)^\beta F(s,u_s)ds \\ &+ \int_0^t T(t-s)D_1G(s,u_s)ds + T(t)G(0,\varphi). \end{aligned}$$

The existence and uniqueness of local solution to the integral equation (2.55)-(2.56) is clear and we omit the proof. In what follows we assume that $z(\cdot) \in C([0,b] : X)$. We affirm that $u'(\cdot) = z(\cdot)$ on $[0,b]$. In order to prove the assertion, for $t \in [0,b]$ and $0 < h < 1$ sufficiently small we have

$$\begin{aligned} &\|\partial_h \, u\,(t) - z\,(t)\| \\ &\leq \left\| T(t) \left[\frac{T(h)-I}{h}\varphi(0) - A\varphi(0) \right] \right\| \\ &+ \left\| T(t) \left(\frac{T(h)-I}{h} \right) F(0,\varphi) \right. \\ &\left. + \frac{1}{h} \textstyle\int_0^h (-A)^{1-\beta}T(t+h-s)(-A)^\beta F(s,u_s) \right\| ds \\ &+ \left\| \frac{-F(t+h,u_{t+h}) + F(t,u_t)}{h} + D_1F(t,u_t) + D_2F(t,u_t)(z_t) \right\| \\ &+ \textstyle\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \Big\| \partial_h (-A)^\beta F(s,u_s) - D_1(-A)^\beta F(s,u_s) \\ &- D_2(-A)^\beta F(s,u_s)(z_s \Big\| \, ds \end{aligned}$$

$$+\left\|\frac{1}{h}\ \int_0^h T(t+h-s)G(s,u_s)ds - T(t)G(0,\varphi)\right\|$$
$$+\int_0^t\ \tilde{M}\left\|\partial_h G(s,u_s) - D_1G(s,u_s) - D_2G(s,u_s)(z_s)\right\| ds$$

Next we use the notations $I_i(t,h)$, $i = 1, ...6$, for the terms of the right hand side of the last inequality.

It is clear that

$$I_i(t,h) \to 0, \text{ as } h \to 0, \ i \in \{1,2,5\} \tag{2.57}$$

uniformly for $t \in [0,b]$. On the other hand for the third term

$$\begin{aligned} \| I_3(t,h) \| &= \| -\partial_h F(t,u_t) + D_1F(t,u_t) + D_2F(t,u_t)(z_t) \| \\ &\leq \| D_2F(t,u_t)(\frac{u_{t+h}-u_t}{h} - z_t) \| \\ &\quad + \frac{\| (h, u_{t+h}-u_t) \|}{h} \| R(F,t,u_t,h,u_{t+h}-u_t) \| . \end{aligned}$$

Since the function $t \to x_t$ is Lipschitz continuous on $[0,b]$, see Proposition (3.1) in [5], follows from Lemma 4 we get

$$\frac{\| (h, u_{t+h}-u_t)) \|}{h} \| R(F,t,u_t,h,u_{t+h}-u_t) \| \to 0 \tag{2.58}$$
$$\text{as} \qquad h \to 0, \tag{2.59}$$

uniformly for $t \in [0,b]$. Consequently, we can rewrite the last inequality in the form

$$\|I_3(t,h)\| \,\| \leq \xi_3(t,h) + \| \,\|D_2F(t,u_t)\| \left\|\frac{u_{t+h}-u_t}{h} - z_t\right\|_{\mathcal{B}} \tag{2.60}$$

where $\xi_3(t,h) \to 0$ if $h \to 0$ uniformly for $t \in [0,b]$.

Using similar arguments, we have for I_4 that

$$\|I_4\,(t,h)\| \leq \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \left\|D_2(-A)^\beta F(s,u_s)\right\| \left\|\frac{u_{s+h}-u_s}{h} - z_s\right\|_{\mathcal{B}} ds$$
$$+\int_0^t \frac{\|(h,u_{s+h}-u_s)\|}{h} \left\|R((-A)^\beta F, s, u_s, h, u_{s+h}-u_s)\right\| ds$$

and so, from (2.58) we get

$$\|I_4(t,h)\| \leq \xi_4(t,h) + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \left\|D_2(-A)^\beta F(s,u_s)\right\| \left\|(\frac{u_{s+h}-u_s}{h} - z_s)\right\|_{\mathcal{B}} ds \tag{2.61}$$

where $\xi_4(t,h) \to 0$ as $h \to 0$ uniformly for $t \in [0,b]$.

Analogously, for I_6 we see that

$$\|I_6(t,h)\| \leq \xi_6(t,h) + \int_0^t \tilde{M} \, \|D_2 G(s,u_s)\| \left\|(\frac{u_{s+h}-u_s}{h} - z_s)\right\|_{\mathcal{B}} ds \tag{2.62}$$

where $\xi_6(t,h) \to 0$ as $h \to 0$ uniformly for $t \in [0,b]$.

Combining (2.57), (2.60), (2.61), (2.62) and the first inequality we get

$$\begin{aligned}
&\|\partial_h u(t) - z(t)\| \\
&\leq \xi_7(t,h) + \|D_2 F(t,u_t)\| \left\|\left(\frac{u_{t+h}-u_t}{h} - z_t\right)\right\|_{\mathcal{B}} \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \ \left\|D_2(-A)^\beta \ F(s,u_s)\right\| \ \left\|\left(\frac{u_{s+h}-u_s}{h} - z_s\right)\right\|_{\mathcal{B}} \ ds \\
&+ \int_0^t \tilde{M} \ \|D_2 G(s,u_s)\| \left\|\left(\frac{u_{s+h}-u_s}{h} - z_s\right)\right\|_{\mathcal{B}} \ ds,
\end{aligned}$$

where $\xi_7(t,h) \to 0$ as $h \to 0$ uniformly for $t \in [0,b]$. Using axiom (**A**) and (2.54) we infer that,

$$\|\partial_h u(\cdot) - z(\cdot)\|_{[0,t]} \leq \frac{1}{1-\mu} \ \max_{s\in[0,t]} \ \xi_7(s,h) + \frac{M_{2b}\mu}{(1-\mu)K_{2b}} + \left\|\frac{u_h-\varphi}{h} - z_0\right\|_{\mathcal{B}}.$$

Clearly $u'(\cdot) = z(\cdot)$ if $h^{-1}(u_h - \varphi) - z_0 \to 0$ as $h \to 0$. Next we will prove this convergence. For $h > 0$ we consider the decomposition

$$\begin{aligned}
&\left(\frac{u_h-\varphi}{h} - A_W(\varphi) - \mathcal{X}_{G(0,\varphi)}\right\|_{\mathcal{B}} \\
&\leq \left\|\frac{W(h)\varphi-\varphi}{h} - A_W(\varphi)\right\|_{\mathcal{B}} + \left\|\frac{z_h^1+z_h^2}{h}\right\|_{\mathcal{B}} \\
&\quad + \left\|\frac{z_h^3}{h} - \mathcal{X}_{G(0,\varphi)}\right\|_{\mathcal{B}}
\end{aligned} \tag{2.63}$$

where $z^i = 0$ on $(-\infty, 0]$ and

$$\begin{aligned} z^1(\theta) &= T(\theta)F(0,\varphi) - F(\theta,u_\theta) \\ z^2(\theta) &= \int_0^\theta (-A)^{1-\beta}T(\theta-s)(-A)^\beta F(s,u_s)ds \\ z^3(\theta) &= \int_0^\theta T(\theta-s)G(s,u_s)ds, \end{aligned}$$

for $\theta \in [0,b]$. Let $I_i(h)$, $i = 1,2,3$, be the terms of the right hand side of (2.63). Clearly

$$I_1(h) \to 0 \quad \text{as} \quad h \to 0, \tag{2.64}$$

since $\varphi \in D(A_W)$. On the other hand, in both $\mathcal{X}_{G(0,\varphi)} = 0$ or $\mathcal{X}_{G(0,\varphi)} \neq 0$, the hypothesis imply that

$$I_3(h) \to 0 \quad \text{as} \quad h \to 0. \tag{2.65}$$

For the second term we get

$$\begin{aligned} &I_2(h) \\ &\leq K_b \frac{1}{h} \| (T(\theta) - I)F(0,\varphi) \\ &\quad + \textstyle\int_0^\theta (-A)^{1-\beta}T(\theta-s)(-A)^\beta F(s,u_s)ds \Big\|_{\theta\in[0,h]} \\ &\quad + K_b \frac{1}{h} \| F(0,\varphi) - F(\theta,u_\theta) \|_{\theta\in[0,h]} . \end{aligned} \tag{2.66}$$

Moreover for $\theta \in [0,h]$,

$$\begin{aligned} &\frac{1}{h} \Big\| (T(\theta) - I) F(0,\varphi) + \textstyle\int_0^\theta (-A)^{1-\beta}T(\theta-s)(-A)^\beta F(s,u_s)ds \Big\| \\ &\leq \frac{1}{h} \int_0^\theta \Big\| (-A)^{1-\beta}T(\theta-s)\{(-A)^\beta F(s,u_s) - (-A)^\beta F(0,\varphi)\} \Big\| ds \\ &\leq \frac{1}{h} \textstyle\int_0^\theta \displaystyle\frac{C_{1-\beta}L}{(\theta-s)^{1-\beta}} (s + \| \varphi - u_s \|_{\mathcal{B}}) ds \\ &\leq \frac{1}{h} C_{1-\beta} L (h + Ch) \frac{h^\beta}{\beta} \end{aligned}$$

where we use the Lipschitz continuity of $s \to u_s$, see Proposition (3.1) in [5]. Consequently

$$\frac{1}{h}\left\|(T(\theta)-I)F(0,\varphi)+\textstyle\int_0^{\theta}(-A)^{1-\beta}T(\theta-s)(-A)^{\beta}F(s,u_s)ds\right\|_h \to 0, \quad \text{as } h \to 0. \tag{2.67}$$

Similarly, we can to prove that

$$\left\|\frac{F(0,\varphi)-F(\theta,u_\theta)}{h}\right\|_{[0,h]} \to 0 \tag{2.68}$$

as $h \to 0$, since $DF(0,\varphi) \equiv 0$. Using (2.67) and (2.68) in (2.66), we conclude that

$$I_2(h) \to 0 \qquad h \to 0. \tag{2.69}$$

Now, the convergence of $h^{-1}(u_h - \varphi)$ to z_0 follows from (2.64), (2.65), (2.69) and (2.63).

We know from Corollary 1, that the unique mild solution, $y(\cdot)$, of the Cauchy problem

$$\begin{aligned} &w'(t) = Aw(t) + (-A)^{1-\beta}((-A)^{\beta}F(t,u_t)) + G(t,u_t), \quad t \in (0,b), \\ &w(0) = \varphi(0) + F(0,\varphi), \end{aligned}$$

is a classical solution. Consequently, $y' = \frac{d}{dt}(u(t) + F(t,u_t))$ is continuous on $[0,b]$ and $u(t) \in D(A)$ for every $t \in [0,b]$, since $\varphi(0) \in D(A)$ and $F([0,a] \times \Omega) \subset D(A)$. Thus $u(\cdot)$ is a classical solution of (1.1). The proof is complete.

Acknowledgement: The author wishes to thank to the referees for their comments and suggestions.

References

[1] Goldstein, Jerome A. Semigroups of linear operators and applications. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, (1985).

[2] Hale, Jack K.; Kato, Junji.(1978) Phase space for retarded equations with infinite delay. *Funkcial. Ekvac.* **21**, No. 1, pp. 11–41.

[3] Hale, Jack K.; Verduyn Lunel, Sjoerd M. Introduction to functional-differential equations. Applied Mathematical Sciences, 99. Springer-Verlag, New York, (1993).

[4] Hernández, Eduardo; Henríquez, Hernán R. Existence of periodic solutions of partial neutral functional-differential equations with unbounded delay. J. Math. Anal. Appl. 221, No. 2, PP. 499–522, (1998).

[5] Hernández, Eduardo; Henríquez, Hernán R. Existence results for partial neutral functional-differential equations with unbounded delay. J. Math. Anal. Appl. 221 (1998), no. 2, 452–475.

[6] Henríquez, Hernán R. Regularity of solutions of abstract retarded functional-differential equations with unbounded delay. Nonlinear Anal. 28, No. 3, pp. 513–531, (1997).

[7] Henríquez, Hernán R. Periodic solutions of quasi-linear partial functional-differential equations with unbounded delay. Funkcial. Ekvac. 37, No. 2, pp. 329–343, (1994).

[8] Hino, Yoshiyuki; Murakami, Satoru; Yoshizawa, Taro. Existence of almost periodic solutions of some functional-differential equations with infinite delay in a Banach space. Tôhoku Math. J. (2) 49, No. 1, pp. 133–147, (1997).

[9] Krein, S. G. Linear differential equations in Banach space. Translated from the Russian by J. M. Danskin. Translations of Mathematical Monographs, Vol. 29. American Mathematical Society, Providence, R. I., (1971).

[10] Pazy, A. Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. Springer-Verlag, New York-Berlin, (1983).

[11] B. N. Sadovskii. On a fixed point principle. *Funct. Anal. Appl.* **1**, pp. 74-76, (1967).

[12] Travis, C. C.; Webb, G. F. Existence and stability for partial functional differential equations. Trans. Amer. Math. Soc. 200, pp. 395–418, (1974).

Received : December 2001..

Eduardo Hernández

Instituto de Ciências Matemáticas e de computação

Universidade de São Paulo

Campus de São Carlos

Caixa Postal 668, 13560-970, São Carlos, SP

Brazil

e-mail: lalohm@icmc.sc.usp.br.