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ASYMPTOTIC EQUILIBRIUM FOR CERTAIN TYPE OF DIFFERENTIAL EQUATIONS WITH MAXIMUM *

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Abstract

In this work we obtain asymptotic representations for the solutions of certain type of differential equations with maximum. We deduce the asymptotic equilibrium for this class of differential equations.

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1. Introduction

Differential equations with maximum arise naturally when solving practical problems, in particular, in those which appear in the study of systems with automatic regulation. A classical example is that of an electric generator. In this case, the mechanism becomes actived when the maximum voltage variation that is permitted is reached in an interval of time $I_t = [t - h, t]$, with h a positive constant. The equation which describes the actioning of this regulator has the form

$$V'(t) = -\delta V(t) + p \max_{s \in I_t} V(s) + F(t),$$

where δ and p are constants that are determined by the characteristic of the system, $V(t)$ is the voltage and $F(t)$ is the effect of the perturbation that appears associated to the change of voltage [2].

Much work on these equations has been carried out in the last three decades. We mention the work in [1-2] and [8,11].

We study differential equations with maximum of the form

$$(1.1) \quad \begin{cases} x'(t) = f(t, x(t), \max_{u \in I_t} x(u)), & \text{with } t \in I \\ x(t) = \varphi(t), & \text{with } t \in [-h, 0] \end{cases}$$

where $I = [0, b)$ and the possibility that b be infinity is not excluded. In addition, f is a real - valued continuous function defined on $I \times R \times R$.

We denote by $\|\varphi\|$ the norm

$$\|\varphi\| = \max\{|\varphi(t)| / t \in [-h, 0]\}.$$

Definition 1. A differential equation with maximum (1.1) as above has the property of asymptotic equilibrium if:

1. Every solution $x(t)$ of (1.1) with initial condition $x(t) = \varphi(t)$, for all $t \in [-h, 0]$, is defined for all $t \geq -h$ and there exists $\xi \in R$ which satisfies

$$(1.2) \quad \lim_{t \rightarrow \infty} x(t) = \xi$$

2. For all $\xi \in R$, there exists a solution $x(t)$ of (1.1), which is defined on the interval $[-h, \infty)$ and verifies (2.1).

Our main result asserts, under certain hypotheses on the function f , equation (1.1) has asymptotic equilibrium. The techniques used in the proof are based on an inequality of Gronwall - Bellman type and successive approximations. The method used is analogous to those of [3-10].

Our principal results are applied to automatic control problems described by nonlinear equations of the type

$$V'(t) = -\delta V(t) + p \max_{s \in I_t} V(s) + F(t, V(t)).$$

2. Main Results

In this paragraph we prove theorems about asymptotic behavior and boundedness of the solutions of equation (1.1). We first recall some basic inequality of Gronwall - Bellman type which we use in the study of the differential equation with maximum (1.1).

Lemma 1. *Let p, x be continuous and nonnegative functions on $[0, b)$ and $\varphi_{[-h, 0]}$ a nonnegative continuous function. If the inequality*

$$x(t) \leq x(0) + \int_0^t p(s) \max_{u \in I_t} x(u) ds$$

holds, for all $t \in [0, b)$, with $x(t) = \varphi(t)$ for all $t \in [-h, 0]$, then

$$x(t) \leq \|\varphi\| e^{\int_0^t p(s) ds}$$

for all $t \in [0, b)$.

Theorem 1. Suppose that f satisfies the following hypotheses:

[H₀] f is continuous on $I \times R \times R$, where $I = [0, \infty)$.

[H₁] There exist λ and β integrable functions on I such that, for all $(t, x, y) \in I \times R \times R$, we have

$$(2.1) \quad |f(t, x, y)| \leq \lambda(t)(|x| + |y|) + \beta(t).$$

Then, every solution $x(t)$ with $x(t) = \varphi(t)$, for all $t \in [-h, 0]$, is defined on $[-h, \infty)$ and satisfies (1.2) for some $\xi \in R$. In addition, we have $x(t) = \xi + O\left(\int_t^\infty [\lambda(s) + \beta(s)] ds\right)$

Proof. If $x(t)$ is a solution of equation (1.1) such that $x(t) = \varphi(t)$, for all t in $[-h, 0]$, and defined on a subinterval $J = [-h, T)$ of $[-h, \infty)$, then

$$(2.2) \quad x(t) = x(0) + \int_0^t f(s, x(s), \max_{u \in I_s} x(u)) ds,$$

for all t in $[0, T)$. Therefore, for all t in $[0, T)$, we have

$$\begin{aligned} |x(t)| &\leq |x(0)| + \int_0^t |\lambda(s)| \left(|x(s)| + \left| \max_{u \in I_s} x(u) \right| \right) ds + \int_0^t \beta(s) ds \\ &\leq |x(0)| + \int_0^\infty \beta(s) ds + \int_0^t 2\lambda(s) \max_{u \in I_s} |x(u)| ds. \end{aligned}$$

By lemma 1,

$$|x(t)| \leq \left[\|\varphi\| + \int_0^\infty \beta(s) ds \right] e^{\int_0^t 2\lambda(s) ds} \leq \left[\|\varphi\| + \int_0^\infty \beta(s) ds \right] e^{\int_0^\infty 2\lambda(s) ds}$$

for all t in the interval J , which shows that $x(t)$ is bounded in the interval J . Therefore, the left limit $x(T - 0)$ exists, when t tends to T . Since the initial value problem

$$y'(t) = f(t, y(t), \max_{u \in I_t} y(u)),$$

with initial condition $y(t) = x(t)$ in $[T - h, T)$ and where $y(T) = x(T - 0)$ has a local solution, we conclude that it is possible to extend

x beyond T . This shows that solution $x(t)$ of equation (1.1) is defined on $[-h, \infty)$ and there exists a constant M such that $|x(t)| \leq M$, for all $t \in [-h, \infty)$.

Since $f(s, x(s), \max_{u \in I_s} x(u))$ is integrable on I , from equation (2.2) it follows that

$$x(t) = x(0) + \int_0^{\infty} f(s, x(s), \max_{u \in I_s} x(u)) ds - \int_t^{\infty} f(s, x(s), \max_{u \in I_s} x(u)) ds.$$

Let ξ denote the real number given by

$$x(0) + \int_0^{\infty} f(s, x(s), \max_{u \in I_s} x(u)) ds, \text{ then, we have}$$

$$\begin{aligned} |x(t) - \xi| &\leq \int_t^{\infty} \lambda(s) \left(|x(s)| + \max_{u \in I_s} |x(u)| \right) ds + \int_t^{\infty} \beta(s) ds \\ &\leq 2M \int_t^{\infty} \lambda(s) + \int_t^{\infty} \beta(s) ds \end{aligned}$$

thus,

$$x(t) = \xi + O\left(\int_t^{\infty} [\lambda(s) + \beta(s)] ds\right).$$

Theorem 2. *If f satisfies conditions*

[H_0] *f is continuous on $I \times R \times R$ and $f(t, 0, 0)$ is integrable on I , where $I = [0, \infty)$.*

[H_2] *There exists a positive and integrable function μ on I such that, for all $(t, x_1, y_1), (t, x_2, y_2)$ in $I \times R \times R$, we have*

$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \mu(t) (|x_1 - x_2| + |y_1 - y_2|)$ and there exist a nonnegative real constant K such that, for all $s \in [0, \infty)$

$$\mu(s) \leq K\mu(s - h).$$

Then for all $\xi \in R$ there exists a solution $x(t)$ of (1.1) which is defined on $[-h, \infty)$ and verifies (1.2). In addition, we have

$$(2.3) \quad x(t) = \xi + \int_t^{\infty} f(s, 0, 0) ds + O\left(\int_t^{\infty} \mu(s) ds\right).$$

Proof. Let B denote the Banach space of the real bounded functions g defined on $[-h, \infty)$ with the norm defined by

$$\|g\| = \{\sup |g(t)| / t \in [-h, \infty)\}.$$

Consider the operator $T : B \rightarrow B$ defined by

$$T(x)(t) = \xi - \int_t^{\infty} f(s, x(s), \max_{u \in I_s} x(u)) ds \text{ for all } t \geq 0, \text{ and by } T(x)(t) = \xi, \\ \text{for all } t \in [-h, 0].$$

In order to prove that $T : B \rightarrow B$ has a fixed point, we must show that there exists an integer n such that T^n is a contractive operator. For this, it suffices to show that, for all x, y in B and for all positive integer n , we have

$$(2.4) \quad |(T^n y)(t) - (T^n x)(t)| \leq \frac{1}{K n!} \left(\int_{t-h}^{\infty} 2K \mu(s) ds \right)^n \|y - x\|.$$

We proceed by induction.

First, we verify that the inequality is true for $n = 1$.

$$\begin{aligned} & |(Ty)(t) - (Tx)(t)| \\ &= \left| \int_t^{\infty} \left(f(s, y(s), \max_{u \in I_s} y(u)) - f(s, x(s), \max_{u \in I_s} x(u)) \right) ds \right| \\ &\leq \int_t^{\infty} \mu(s) \left\{ |y(s) - x(s)| + \max_{u \in I_s} |y(u) - x(u)| \right\} ds \\ &\leq \int_t^{\infty} 2\mu(s) \max_{u \in I_s} |y(u) - x(u)| ds \\ &\leq \left(\int_t^{\infty} 2\mu(s) ds \right) \|y - x\| \\ &\leq \frac{1}{K 1!} \left(\int_{t-h}^{\infty} 2K \mu(s) ds \right) \|y - x\| \end{aligned}$$

Next, if the formula were true for $n - 1$, then

$$\begin{aligned}
 & |(T^n y)(t) - (T^n x)(t)| \\
 &= \left| \int_t^\infty \left(f \left(s, (T^{n-1} y)(s), \max_{u \in I_s} (T^{n-1} y)(u) \right) - \right. \right. \\
 &\quad \left. \left. f \left(s, (T^{n-1} x)(s), \max_{u \in I_s} (T^{n-1} x)(u) \right) \right) ds \right| \\
 &\leq \int_t^\infty \mu(s) \{ |(T^{n-1} y)(s) - (T^{n-1} x)(s)| + \\
 &\quad \max_{u \in I_s} (T^{n-1} y)(u) - (T^{n-1} x)(u) \} ds \\
 &\leq \int_t^\infty 2\mu(s) \max_{u \in I_s} |(T^{n-1} y)(u) - (T^{n-1} x)(u)| ds \\
 &\leq \int_t^\infty 2\mu(s) \frac{1}{K(n-1)!} \left(\int_t^\infty 2K\mu(\tau) d\tau \right)^{n-1} \|y - x\| ds \\
 &\leq \int_t^\infty 2K\mu(s-h) \frac{1}{K(n-1)!} \left(\int_t^\infty 2K\mu(\tau) d\tau \right)^{n-1} \|y - x\| ds \\
 &\leq \frac{1}{K n!} \left(\int_t^\infty 2K\mu(s) ds \right)^n \|y - x\|.
 \end{aligned}$$

Since $\mu(t)$ is integrable on I , we can find a positive integer n such that the operator T^n is a contraction. Therefore, there exists a fixed point x of T , and then $x(t) = \xi - \int_t^\infty f(s, x(s), \max_{u \in I_s} x(u)) ds$ for all $t \geq 0$, and $x(t) = \xi$, for all $t \in [-h, 0]$, i. e. $x(t)$ is a solution of equation (1.1) such that $x(t) = \varphi(t)$, for all t in $[-h, 0]$.

Since $\left| x(t) - \xi - \int_t^\infty f(s, 0, 0) ds \right| \leq K \int_t^\infty \mu(s) ds$ we can see that (2.3) is verified.

Example 1 : Consider the differential equation with maximum

$$\begin{cases} x'(t) = \frac{2}{(1+t)^2} \max_{u \in I_t} x(u) \\ x(t) = e^{-2}, \text{ for } t \in [-h, 0] \end{cases}$$

We have that $f(t, x, y) = \frac{2}{(1+t)^2} y$ is a real-valued, continuous, integrable function on $I \times R \times R$ and

$$|f(t, x, y)| = \left| \frac{2}{(1+t)^2} \cdot y \right| \leq \frac{2}{(1+t)^2} (|x| + |y|).$$

The function $\lambda(t) = \frac{2}{(1+t)^2}$ is continuous and integrable on I . Thus f satisfies conditions H_0 and H_1 and by Theorems 1 and 2, every solution $x(t)$ is defined on $[-h, \infty)$ and verifies

$$(2.5) \quad x(t) = \xi + O\left(\int_t^\infty \lambda(s) ds\right) = \xi + O\left(\frac{2}{1+t}\right)$$

and reciprocally for any $\xi \in R$ there exists a solution x defined on $[-h, \infty]$ and satisfying asymptotic formula (2.5).

In this example, the function $x(t) = \begin{cases} \exp\left(-\frac{2}{1+t}\right), & \text{for } t \in I \\ \exp(-2), & \text{for } t \in [-h, 0] \end{cases}$ is an explicit solution of the differential equation which in effect satisfies formula (2.5).

Example 2. In the differential equation with maximum

$$\begin{cases} x'(t) = \frac{2+\sin t}{1+t^2}x(t) + \frac{2}{(1+t)^2} \max_{u \in I_t} x(u) \\ x(t) = \sigma \end{cases}$$

the function $f(t, x, y) = \frac{2 + \sin t}{1 + t^2}x + \frac{2}{(1 + t)^2}y$ is continuous for $t > -1$, $x \in R$, $y \in R$.

Moreover,

$$\begin{aligned} |f(t, x_1, y_1) - f(t, x_2, y_2)| &= \mu_1(t) |x_1 - x_2| + \mu_2(t) |y_1 - y_2| \\ &\leq \mu(t) (|x_1 - x_2| + |y_1 - y_2|) \end{aligned}$$

with $\mu(t) = \max\{\mu_1(t), \mu_2(t)\}$, where $\mu_1(t) = \frac{|2 + \sin t|}{1 + t^2}$ and

$$\mu_2(t) = \frac{2}{(1 + t)^2}.$$

Since f satisfies conditions H_0 , H_1 and H_2 we have that for every solution $x(t)$ of the given equation, there exists $\xi \in R$ such that $x(t) = \xi + O\left(\frac{\pi}{2} - \arctan t\right)$.

Conversely, for each $\xi \in R$, there exist a solution $x(t)$ defined on

the interval $[-h, \infty)$ such that

$$x(t) = \xi + O\left(\int_t^\infty \lambda(s) ds\right) = \xi + O\left(\frac{\pi}{2} - \arctan t\right).$$

Example 3 : (Application to semilinear electrogenerators) : Consider the differential equation with maximum

$$\begin{cases} x'(t) = a(t)x(t) + g(t, x(t), \max_{s \in I_t} x(s)), & \text{for } t \in [0, \infty) \\ x(t) = e^{\int_0^t a(s) ds} \xi, & \text{for } t \in [-h, 0] \end{cases}$$

where a is a continuous function on I and g is a continuous function on $I \times R \times R$, such that $f(t, x, y) = e^{-\int_0^t a(s) ds} g(t, e^{\int_0^t a(s) ds} x, e^{\int_0^t a(s) ds} y)$ verifies conditions H_0 and H_2 .

Substitution $x(t) = e^{\int_0^t a(s) ds} v(t)$ in this equation gives

$$v'(t) = e^{-\int_0^t a(s) ds} g(t, e^{\int_0^t a(s) ds} v(t), \max_{s \in I_t} e^{\int_0^s a(\tau) d\tau} v(s))$$

By Theorem 2 , there exists a solution v of the differential equation

$$\begin{cases} v'(t) = f(t, v(t), \max_{s \in I_t} v(s)) \\ v(t) = \xi, \quad t \in [-h, 0] \end{cases} \quad \text{defined in } [-h, \infty) \text{ such that}$$

$$v(t) = \xi + \int_t^\infty f(s, 0, 0) ds + O\left(\int_t^\infty \mu(s) ds\right) \text{ then}$$

$$x(t) = e^{\int_0^t a(s) ds} \left\{ \xi + \int_t^\infty e^{-\int_0^s a(\tau) d\tau} g(s, 0, 0) ds + O\left(\int_t^\infty \mu(s) ds\right) \right\}$$

If function a is integrable in the interval I then the solutions of the equation are stables. If $\int_0^t a(s) ds$ tends to $-\infty$ as t approaches infinity then, the solutions of the system are asymptotically stables.

We can apply this result to an automatic control problem described by a nonlinear equation of the type

$$\begin{cases} V'(t) = -\delta V(t) + p(t) \max_{s \in I_t} V(s) + F(t), & \text{if } t \in I \\ V(t) = e^{-\delta t} \xi, & \text{if } t \in [-h, 0] \end{cases}$$

where δ is a positive constant, p, F are continuous and integrable functions on I and there exists a nonnegative real constant K , such that, for all $s \in [0, \infty)$, $p(s) \leq Kp(s-h)$. We can see there exists a solution

$V(t)$ of this equation such that the following asymptotic representation is verified $V(t) = e^{-\delta t} \left\{ \xi + \int_t^\infty e^{-\delta s} F(s) ds + O\left(\int_t^\infty p(s) ds\right) \right\}$

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