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RIGID SPHERICAL HYPERSURFACES IN $\mathbf{C}^2$

Abdelaziz CHAOUECH
Université Abdelmalek Essaâdi, Maroc

Abstract

In this paper we describe explicitly one class of real-analytic hypersurfaces in $\mathbf{C}^2$ *rigid and spherical at the origin.*

1. Introduction

A real-analytic hypersurface M in $\mathbf{C}^2$ is called rigid if it is given by an equation of the form $r(w,\overline{w},z,\overline{z}) =: Imw + F(z,\overline{z}) = 0$,

where F is a real-analytic function such that: $F(0,0) = \dfrac{\partial F}{\partial z}(0,0) = 0$.

In this paper we study the real-analytic hypersurfaces M in $\mathbf{C}^2$ rigid and spherical at the origin, i.e. there exists a local biholomorphic which maps M to the euclidean unit sphere. We note that recently A. Isaev [4] has given a characterization of spherical rigid real hypersurfaces in $\mathbf{C}^n$ $(n \geq 2)$ in terms of a certain system of differential equations for a defining function of such hypersurfaces, but this does not permit to describe the spherical rigid real hypersurfaces even if in $\mathbf{C}^2$. Nowadays these hypersurfaces are not known. The only examples have been given by N. Stanton [5] (see also [6]). More recently, B. Coupet and A. Sukhov [3] have described the spherical hypersurfaces of the form: $Imw + P(z,\overline{z}) = 0$, where P is a non-identically zero subharmonic homogeneous polynomial without purely harmonic terms.

The goal of this paper is to give one description of one class of real-analytic hypersurfaces in $\mathbf{C}^2$ rigid and spherical at the origin.

2. Prelimenaries and results

Let M be a hypersurface in $\mathbf{C}^2$, strictly pseudoconvex at the origin, defined by:

$M =: \{Rew + \varphi(z,\overline{z}) = 0\}$, where φ is a real-analytic function.

Without any loss of generality, we may assume that $\dfrac{\partial^2 \varphi}{\partial z \partial \overline{z}}(0,0) = 1$.

According to a theorem of N. Stanton [5] (see theorem 1.7) there exists an holomorphic change of coordinates ψ of the form $(w, g(z))$ defined in a neighborhood V of the origin and such that $\psi(M \cap V)$ is defined by:

$Rew + |z|^2 + |z|^4 b(z,\overline{z}) = 0$,

where b is a real-analytic function.

Theorem. *Let* M *be a hypersurface in* $\mathbf{C}^2$ *defined by*
$M =: \{Rew + \varphi(z,\overline{z}) = 0\}$,

where $\varphi(z,\overline{z}) = |z|^2 + |z|^4 b(z,\overline{z})$ *and* b *being a real-analytic function in a neighbourhood of the origin.*

Suppose that $\frac{\partial b}{\partial z}(0,0) = 0$. *Then* M *is spherical at the origin if and only if* φ *is given by one of the functions:*

i) $|z|^2$, *ii)* $\frac{1}{c}\sin^{-1}\left(c\,|z|^2\right)$, *iii)* $\frac{1}{c}sh^{-1}\left(c\,|z|^2\right)$

for some $c \in \mathbf{R}^*$.

Proof. Let $F = (F_1, F_2)$ be a local biholomorphism at the origin which maps M to the euclidean unit sphere:
$\left\{(w,z) \in \mathbf{C}^2 : \rho(w,z) =: Rew + |z|^2 = 0\right\}$.

We may assume that $F_1(0,0) = F_2(0,0) = 0$ and $\frac{\partial F_1}{\partial w}(0,0) = \frac{\partial F_2}{\partial z}(0,0) = 1$.

By conjugating F with some automorphism of the euclidean unit ball of $\mathbf{C}^2$, we may also assume that $\frac{\partial F_2}{\partial w}(0,0) = 0$.

The principal idea of the proof is to determine explicitly F by solving a system of partial differential equations. To this end we consider the direct image of the translation vector field $i\frac{\partial}{\partial w}$: $F_*(i\frac{\partial}{\partial w})$ which is holomorphic tangent vector of the euclidean unit sphere, i.e. $F_*(i\frac{\partial}{\partial w}) =: A(w,z)\frac{\partial}{\partial w} + B(w,z)\frac{\partial}{\partial z}$, where A and B are two holomorphic functions in a neighborhood of the origin and such that $Re\left[A(w,z)\frac{\partial \rho}{\partial w} + B(w,z)\frac{\partial \rho}{\partial z}\right]$ is identically null on the unit sphere.

We note that the real dimension of the lie algebra of holomorphic tangent vector fields on the unit sphere is equal to 8 (see E. Cartan [1]).

We now proceed in three steps.

<u>**First step**</u>: $\frac{\partial^n F_1}{\partial z^n}(0,0) = 0, \ \forall n \geq 0$ and $\frac{\partial^n F_2}{\partial z^n}(0,0) = 0, \ \forall n \geq 2.$

We write $w = u + iv$ and $\rho(w, z) =: Rew + |z|^2$.

Let $A(v, z) =: (-\varphi(z) + iv, z)$ be a parametrization of M. The vector field defined by: $L =: \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial w} - \frac{1}{2}\frac{\partial}{\partial z}$ is tangent to M, so $L(\rho o F) \equiv 0$ on M in a neighbourhood of the origin, which implies the following identity:

$$(1) \quad \frac{\partial\varphi}{\partial z}\left[\frac{1}{2}\frac{\partial F_1}{\partial w}oA + (\overline{F}_2 oA).\frac{\partial F_2}{\partial w}oA\right]$$

$$-\frac{1}{2}\left[\frac{1}{2}\frac{\partial F_1}{\partial z}oA + (\overline{F}_2 oA).\frac{\partial F_2}{\partial z}oA\right] \equiv 0$$

near $v = 0$ and $z = 0$.

Differentiating (1) with respect to z to arbitrary order, we get:

$$(2) \qquad \frac{\partial^n F_1}{\partial z^n}(0,0) = 0, \ \ \forall n \geq 0 \qquad (See\,[2] \text{ page } 47-49).$$

We write $F_2(w, z)$ in the following form:

$$F_2(w,z) = z + K(z) + \sum_{n\geq 2} b_n w^n + \sum_{n,m\geq 1} B_{nm} z^n w^m.$$

Setting $v = 0$ and identifing the pure $\overline{z}$ terms in (1), and taking (2) into account we obtain $K(z) \equiv 0$.

Second step: F is one of the four following forms:

i) $F(w,z) = \left(\dfrac{w}{1+i\Gamma w}, \ \dfrac{ze^{-\frac{\beta_0}{2}w}}{1+i\Gamma w}\right)$

ii) $F(w,z) = \left(\dfrac{1}{\gamma}tg\gamma w, \ \dfrac{ze^{-\frac{\beta_0}{2}w}}{cos\gamma w}\right)$

iii) $F(w,z) = \left(\dfrac{i}{k_0}(e^{-ik_0 w} - 1), \ ze^{-\frac{\beta_0}{2}w}\ e^{-i\frac{k_0}{2}w}\right)$

iv) $F(w,z) = \left(\dfrac{a-1}{k}\ \dfrac{e^{kw}-1}{ae^{kw}-1}, \ (a-1)ze^{-\frac{\beta_0}{2}w}\ \dfrac{e^{\frac{k}{2}w}}{ae^{kw}-1}\right)$

where $\beta_0 = b(0,0)$, $\Gamma \in \mathbf{R}$. $\gamma \in \mathbf{C}^*$ and $\gamma^2 \in \mathbf{R}^*$, $k_0 \in \mathbf{R}^*$. $k \in \{\mathbf{R}^*, i\mathbf{R}^*\}$, $a \in \mathbf{C}^*$ and $a \neq 1$, $|a|^2 = 1$ if $k \in \mathbf{R}^*$ and $a \in \mathbf{R}^*$ if $k \in i\mathbf{R}^*$.

First, we shall prove that $F_1(w, z) = F_1(w)$, *i.e.* F_1 depends only on w.

We consider the holomorphic vector field $F_*(i\dfrac{\partial}{\partial w})$ which is defined in a neighbourhood of the origin. Since $F_*(i\dfrac{\partial}{\partial w})$ is tangent to the euclidean unit sphere: $\left\{(w,z) \in \mathbf{C}^2 : Rew + |z|^2 = 0\right\}$, it may be written as a real linear combination of the following fields:

$X_1 = i\dfrac{\partial}{\partial w}$

$X_2 = -2z\dfrac{\partial}{\partial w} + \dfrac{\partial}{\partial z}$

$X_3 = 2iz\dfrac{\partial}{\partial w} + i\dfrac{\partial}{\partial z}$

$X_4 = 2w\dfrac{\partial}{\partial w} + z\dfrac{\partial}{\partial z}$

$X_5 = iz\dfrac{\partial}{\partial z}$

$X_6 = 2izw\dfrac{\partial}{\partial w} + (2iz^2 - iw)\dfrac{\partial}{\partial z}$

$X_7 = 2zw\dfrac{\partial}{\partial w} + (2z^2 + w)\dfrac{\partial}{\partial z}$

$X_8 = -iw^2\dfrac{\partial}{\partial w} - izw\dfrac{\partial}{\partial z}$.

Then there are real numbers $\alpha_1, ..., \alpha_8$ such that:

$$F_*(i\frac{\partial}{\partial w}) = \sum_{j=1}^{8} \alpha_j X_j$$

We note $\left[F_*(i\dfrac{\partial}{\partial w})\right]_{(w,z)} =: A(w,z)\dfrac{\partial}{\partial w} + B(w,z)\dfrac{\partial}{\partial z}$.

Then $A(w,z) = i\alpha_1 + 2(-\alpha_2 + i\alpha_3)z + 2\alpha_4 w - i\alpha_8 w^2 + 2\lambda wz$
and $B(w,z) = (\alpha_2 + i\alpha_3) + \mu z + 2\lambda z^2 + \overline{\lambda} w - i\alpha_8 wz$
where $\mu = \alpha_4 + i\alpha_5$ and $\lambda = \alpha_7 + i\alpha_6$.

On the other hand we have:

$$\left[F_*(i\frac{\partial}{\partial w})\right]_{F(w,z)} = i\frac{\partial F_1}{\partial w}(w,z)\frac{\partial}{\partial w} + i\frac{\partial F_2}{\partial w}(w,z)\frac{\partial}{\partial z}.$$

Then, we obtain:

$$i\frac{\partial F_1}{\partial w}(w,z) = (AoF)(w,z) \tag{3}$$

and

(4) $$i\frac{\partial F_2}{\partial w}(w,z) = (BoF)(w,z)$$

Since $\dfrac{\partial F_1}{\partial w}(0,0) = 1$ and $\dfrac{\partial F_2}{\partial w}(0,0) = 0$, then the identities (3) and (4) become:

(5) $$i\frac{\partial F_1}{\partial w} = i + 2\alpha_4 F_1 - i\alpha_8 F_1^2 + 2\lambda F_1 F_2$$

and

(6) $$i\frac{\partial F_2}{\partial w} = \mu F_2 + 2\lambda F_2^2 + \overline{\lambda} F_1 - i\alpha_8 F_1 F_2$$

where $\mu = \alpha_4 + i\alpha_5$ and $\lambda = \alpha_7 + i\alpha_6$.

We momentarily admit that $\lambda = \dfrac{1}{2i}\dfrac{\partial b}{\partial z}(0,0)$ and $\alpha_5 = -\dfrac{\beta_0}{2}$, the proof will be given in the end of this paper.

By hypothesis $\dfrac{\partial b}{\partial z}(0,0) = 0$, then $\lambda = 0$, so the identities (5) and (6) become:

(7) $$\frac{\partial F_1}{\partial w} = 1 - 2i\alpha_4 F_1 - \alpha_8 F_1^2$$

and

(8) $$\frac{\partial F_2}{\partial w} = -i\mu F_2 - \alpha_8 F_1 F_2$$

where $\mu = \alpha_4 - i\dfrac{\beta_0}{2}$.

Since $F_2(0,0) = \dfrac{\partial F_2}{\partial w}(0,0) = 0$, from (8) we deduce by induction that

$\dfrac{\partial^n F_2}{\partial w^n}(0,0) = 0, \ \forall n \geq 0,$ *i.e.* $F_2(w,0) \equiv 0$, this implies that:

$F_1(w,z) = F_1(w)$ (See[2] page 47).

We are now in order to solve the system of differential equations (7) and (8).

Let us recall: (9) $F(0,0) = (0,0), \ \dfrac{\partial F_1}{\partial w}(0,0) = \dfrac{\partial F_2}{\partial z}(0,0) = 1$ and

$$\frac{\partial^n F_2}{\partial z^n}(0,0) = 0, \ \forall n \geq 2.$$

There are two cases to consider.

Case 1. $\alpha_4 = 0$

We suppose that $\alpha_8 = 0$. In this case (7) and (8) become:

$$\frac{\partial F_1}{\partial w} = 1 \tag{10}$$

$$\frac{\partial F_2}{\partial w} = -\frac{\beta_0}{2} F_2 \tag{11}$$

then $F_1(w,z) = w$ and $F_2(w,z) = h(z)e^{-\frac{\beta_0}{2}w}$ where h is a holomorphic function. Taking (9) into account we obtain $F_2(w,z) = ze^{-\frac{\beta_0}{2}w}$, this corresponds to the case i).

We suppose now that $\alpha_8 \neq 0$. Let $\eta \in \mathbf{C}^*$ such that: $\eta^2 = \dfrac{1}{\alpha_8}$ a particular solution of the Riccati equation (7). Then it is easy to show that:

$$F_1(w,z) = \frac{1}{\gamma} tg\gamma w, \quad \text{where } \gamma = \frac{1}{i\eta}; \;\; \gamma^2 = -\alpha_8. \tag{12}$$

Replacing$F_1(w,z)$ by its expression (12) into (8) and observing that $\dfrac{\alpha_8}{\gamma} = -\gamma$, then we obtain:

$$\frac{\partial F_2}{\partial w} = (-\frac{\beta_0}{2} + \gamma tg\gamma w)F_2. \tag{13}$$

From (13) and (9) we obtain:

$$F_2(w,z) = ze^{-\frac{\beta_0}{2}w} \frac{1}{\cos\gamma w}, \;\; \text{this corresponds to the case ii).}$$

Case 2. $\alpha_4 \neq \mathbf{0}$

We proceed analogously to the first case.

First we suppose that $\alpha_8 = 0$. From (7), (8) and (9) we obtain:

$$F_1(w,z) = \frac{i}{k_0}(e^{-ik_0 w} - 1)$$

and $F_2(w,z) = ze^{-\frac{\beta_0}{2}w} e^{-i\frac{k_0}{2}w}$

where $k_0 = 2\alpha_4$, this corresponds to the case iii).

Now, we suppose that $\alpha_8 \neq 0$. Let $\eta \in \mathbf{C}^*$ such that: $\alpha_8\eta^2 + 2i\alpha_4\eta = 1$, a particular solution of the Riccati equation (7).

We put: $k = 2(\alpha_8\eta + i\alpha_4)$.

If $k = 0$, in this case, we obtain from (7), (8) and (9):

$$F_1(w,z) = \frac{w}{1 + i\Gamma w} \quad \text{and} \quad F_2(w,z) = \frac{ze^{-\frac{\beta_0}{2}w}}{1 + i\Gamma w}$$

where $\Gamma = \alpha_4$, this corresponds to the case i).

If $k \neq 0$, then from (7) we deduce:

$$(14) \qquad F_1(w,z) = \frac{1}{\delta e^{kw} - \frac{\alpha_8}{k}} + \eta\,; \quad \delta \in \mathbf{C}^*$$

or also

$$(15) \qquad F_1(w,z) = \frac{a-1}{k}\ \frac{e^{kw}-1}{ae^{kw}-1}, \text{ where } a = \frac{\delta k}{\alpha_8}.$$

Replacing $F_1(w,z)$ by its expression (14) into (8) and observing that $\alpha_8\eta + i\alpha_4 = \dfrac{k}{2}$, then, we obtain:

$$(16) \qquad \frac{\partial F_2}{\partial w} = (-\frac{\beta_0}{2} - \frac{k}{2}\ \frac{ae^{kw}+1}{ae^{kw}-1})F_2$$

From (16) and (9) we deduce:

$$(17) \qquad F_2(w,z) = (a-1)ze^{-\frac{\beta_0 w}{2}}\frac{e^{\frac{k}{2}w}}{ae^{kw}-1}$$

Now, we shall prove that $k \in \{\mathbf{R}^*, i\mathbf{R}^*\}$, $a \in \mathbf{C}^*$ and $a \neq 1$, $|a|^2 = 1$ if $k \in \mathbf{R}^*$ and $a \in \mathbf{R}^*$ if $k \in i\mathbf{R}.^*$.

F is a local biholomorphism, so, $a \neq 1$. Since the image $F(M)$ is contained in the unit sphere:

$\left\{(w,z) \in \mathbf{C}^2 : Rew + |z|^2 = 0\right\}$ near $(0,0)$, hence $ReF_1(w) = 0$ for $Rew = 0$, then we obtain:

$$(18) \qquad Re\left(\frac{a-1}{k}\right) + \left(Re(\overline{a}\left(\frac{a-1}{k}\right)\right)e^{2vReik} -$$
$$\left(\frac{a-1}{k} + a\,\overline{\left(\frac{a-1}{k}\right)}\right)e^{ikv} -$$
$$\left(\overline{a}\left(\frac{a-1}{k}\right) + \overline{\left(\frac{a-1}{k}\right)}\right)e^{-i\overline{k}v} \equiv 0 \text{ for } v \text{ near } 0.$$

First we prove that $k \in \{\mathbf{R}^*, i\mathbf{R}^*\}$. Assume, to the contrary, that $k \in \mathbf{C}\backslash\{\mathbf{R}^*, i\mathbf{R}^*\}$, then $Re(ik) \neq 0$ and $ik \neq \overline{ik}$. From (18) we obtain:

$$Re\left(\frac{a-1}{k}\right) = 0 \text{ and } \frac{a-1}{k} + a\,\overline{\left(\frac{a-1}{k}\right)} = 0.$$

Thus, it follows that : $\dfrac{a-1}{k} + a\ \overline{\left(\dfrac{a-1}{k}\right)} - 2Re\left(\dfrac{a-1}{k}\right) = \dfrac{1}{\overline{k}}\left|a-1\right|^2 = 0$, then $a = 1$, this is a contradiction, so, $k \in \{\mathbf{R}^*, i\mathbf{R}^*\}$.

From (18) we deduce: $|a|^2 = 1$ if $k \in \mathbf{R}^*$ and $a \in \mathbf{R}^*$ if $k \in i\mathbf{R}^*$

Third step: **conclusion**

We return to the second step. Let us first prove that $\beta_0 = 0$.

According to the result of N. stanton [5] (theorem 1.7), it suffices to prove that $\varphi(z,\overline{z}) = \varphi(|z|^2)$, ($\beta_0$ is the coefficient of $|z|^4$).

There are four cases to consider. For example, we suppose that F is given by ii), i.e. $F(w,z) = \left(\dfrac{1}{\gamma} tg\gamma w,\ \dfrac{ze^{-\frac{\beta_0}{2}w}}{\cos\gamma w}\right)$, where $\gamma \in \mathbf{C}^*$ and $\gamma^2 \in \mathbf{R}^*$.

So, $\gamma^2 \in \mathbf{R}^*$ then $\gamma \in \mathbf{R}^*$ or $\gamma = i\gamma_0,\ \gamma_0 \in \mathbf{R}^*$.

Since the image $F(M)$ is contained in the unit sphere:

$\left\{(w,z) \in \mathbf{C}^2 : Rew + |z|^2 = 0\right\}$ near $(0,0)$, we have

$$h(\varphi(z,\overline{z})) = |z|^2,\ \text{where } h(x) = \begin{cases} \dfrac{1}{2\gamma}(\sin 2\gamma x)e^{-\beta_0 x} & \text{if } \gamma \in \mathbf{R}^* \\ \dfrac{1}{2\gamma_0}(sh 2\gamma_0 x)e^{-\beta_0 x} & \text{if } \gamma = i\gamma_0 \end{cases}$$

Hence $\varphi(z,\overline{z}) = h^{-1}(|z|^2)$ near 0.

So, $\beta_0 = 0$ and

$$\varphi(z,\overline{z}) = \begin{cases} \dfrac{1}{2\gamma}\sin^{-1}\left(2\gamma\,|z|^2\right) & \text{if } \gamma \in \mathbf{R}^* \\ \dfrac{1}{2\gamma_0} sh^{-1}\left(2\gamma_0\,|z|^2\right) & \text{if } \gamma = i\gamma_0 \end{cases}$$

By following the same way we obtain the other cases.

To end the proof of the theorem it remains to show that:

$$\lambda = \frac{1}{2i}\frac{\partial b}{\partial z}(0,0) \quad \text{and} \quad \alpha_5 = -\frac{\beta_0}{2}.$$

Let's return to the indentities(5) and (6). From the first step we have:

$\dfrac{\partial^n F_1}{\partial z^n}(0,0) = 0, \ \ \forall n \geq 0$, then, from (5) we deduce that:

$\dfrac{\partial^{n+1} F_1}{\partial z^n \partial w}(0,0) = 0, \ \ \forall n \geq 1$, So, in a neighbourhood fo the origin, we can write:

(19) $\quad F_1(w,z) = w + \sum_{n\geq 2} a_n w^n + w^2 \sum_{n\geq 1} A_{n_2} z^n + \sum_{q\geq 3} w^q \sum_{n\geq 1} A_{nq} z^n$

(20) $\quad F_2(w,z) = z + \sum_{n\geq 2} b_n w^n + w \sum_{n\geq 1} B_{n_1} z^n + w^2 \sum_{n\geq 1} B_{n_2} z^n + \sum_{q\geq 3}$ $w^q \sum_{n\geq 1} B_{nq} z^n$

The idea to prove $\lambda = \dfrac{1}{2i}\dfrac{\partial b}{\partial z}(0,0)$ and $\alpha_5 = -\dfrac{\beta_0}{2}$ is to observe that the terms of degree less or equal to 4 on $z, \overline{z}$ in the left hand-side of (1) are null.

First, we observe that from (5) and (19) we have:

(21) $\qquad A_{12} = -i\lambda$ and $a_2 = -i\alpha_4$

Next, from (6) and (20) we have:

(22) $\qquad b_2 = -i\dfrac{\overline{\lambda}}{2}, \ \ B_{21} = -2i\lambda$ and $B_{11} = \alpha_5 - i\alpha_4$

We are now in order to collect the terms of degree less or equal to 4 on $z, \overline{z}$ in the left hand-side of (1).

Let us write $b(z,\overline{z}) = \beta_0 + \beta_1 z + \overline{\beta}_1 \overline{z} + ...$

The terms of degree less or equal to 4 in $\dfrac{\partial \varphi}{\partial z}$ are:

$\overline{z} + 2\beta_0 \overline{z} \left|z\right|^2 + 2\overline{\beta}_1 \overline{z}^2 \left|z\right|^2 + 3\beta_1 \left|z\right|^4 .$

Since $\dfrac{\partial \varphi}{\partial z}(0,0) = 0$, it suffices to collect the terms of degree less or equal to 3 on $z, \overline{z}$ in $\left[\dfrac{1}{2}\dfrac{\partial F_1}{\partial w} oA + \left(\overline{F}_2 oA\right).\dfrac{\partial F_2}{\partial w} oA\right]$, which are:

$$\frac{1}{2} + (B_{11} - a_2)\left|z\right|^2 + (B_{21} - A_{12}) z \left|z\right|^2 - 2b_2 \overline{z} \left|z\right|^2 .$$

the terms of degree less or equal to 4 on $z, \overline{z}$ in

$$\frac{\partial \varphi}{\partial z}\left[\frac{1}{2}\frac{\partial F_1}{\partial w} oA + (\overline{F}_2 oA).\frac{\partial F_2}{\partial w} oA\right]$$

are:

$$\frac{1}{2}\overline{z} + (\beta_0 + B_{11} - a_2)\overline{z}\,|z|^2 + (\overline{\beta}_1 - 2b_2)\overline{z}^2\,|z|^2 + (B_{21} - A_{12} + \frac{3}{2}\beta_1)\,|z|^4$$

On the other hand, the terms of degree less or equal to 4 on $z, \overline{z}$ in $\frac{1}{2}\left[\frac{1}{2}\frac{\partial F_1}{\partial z}oA + \left(\overline{F}_2 oA\right).\frac{\partial F_2}{\partial z}oA\right]$ are:

$$\frac{1}{2}\overline{z} - \overline{z}\,|z|^2\,ReB_{11} - \frac{1}{2}\overline{B}_{21}\,\overline{z}^2\,|z|^2 + \frac{1}{2}\left(\overline{b}_2 - 2B_{21} + \frac{1}{2}A_{12}\right)|z|^4.$$

Finally, the terms of degree less or equal to 4 on $z, \overline{z}$ in the left hand-side of (1) are:

$$(\beta_0 + B_{11} - a_2 + ReB_{11})\overline{z}\,|z|^2 + (\overline{\beta}_1 - 2b_2 + \frac{1}{2}\overline{B}_{21})\overline{z}^2\,|z|^2$$

$+(2\mathrm{B}_{21} - \frac{5}{4}A_{12} + \frac{3}{2}\beta_1 - \frac{1}{2}\overline{b}_2)\,|z|^4.$

These terms are null, then:

(23) $$\beta_0 + B_{11} - a_2 + ReB_{11} = 0$$

and

(24) $$\overline{\beta}_1 - 2b_2 + \frac{1}{2}\overline{B}_{21} = 0$$

From (21), (22) and (23) we obtain: $\beta_0 = -2ReB_{11} = -2\alpha_5$

and

From (22) and (24) we obtain: $\lambda = \frac{\beta_1}{2i} = \frac{1}{2i}\frac{\partial b}{\partial z}(0,0)$.

This ends the proof of the theorem.

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References

[1] E. Cartan., sur la géometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes. Part I. Ann. Mat. 11, 17-90 (1932); or oeuvres complètes, vol.II, pp. 1231-1304. Paris: Gauthiers-Villars, (1953).

[2] A. Chaouech., Auto-applications holomorphes propres des domaines polynômiaux rigides de $\mathbf{C}^2$. Publicacions Matemàtiques, Vol.40, N1, pp. 41-66, (1996).

[3] B. Coupet and A. Sukhov., Rigidity of algebraic CR structures and regularity of CR mappings, Preprint.

[4] A. Isaev., Rigid spherical hypersurfaces. Complex variables 31, pp. 141-163, (1996).

[5] N. Stanton., A normal form for rigid hypersurfaces in $\mathbf{C}^2$. American Journal of Mathematics 113, pp. 877-910, (1991).

[6] N. Stanton., Infinitesimal CR automorphisms of rigid hypersurfaces in $\mathbf{C}^2$. Journal of geometric analysis, Vol. 1, N. 3, pp. 231-267, (1991).

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Abdelaziz Chaouech

Université Abdelmalek Essaâdi

Faculté des Sciences

Département de Mathématiques & Informatiques

B. P. 21 21 Tétouan

Maroc

E-MAIL : chaouech@hotmail.com