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DIBARIC ALGEBRAS

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Abstract

Here we give basic properties of dibaric algebras which are motivated by genetic models. Dibaric algebras are not associative and they have a non trivial homomorphism onto the sex differentiation algebra. We define first join of dibaric algebras next indecomposable dibaric algebras. Finally, we prove the uniqueness of the decomposition of a dibaric algebra, with semiprincipal idempotent, as the join of indecomposable dibaric algebras.

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1. Introduction

The study of dibaric algebras has as motivation the algebras coming from genetic models in bisexual populations with sex linked genetic inheritance. First, Etherington [3], introduced the idea of treating the male and female components of a population separately and next Holgate [4] formalized this concept with the introduction of the sex differentiation algebra and dibaric algebras. Following the modern notation of Wörz-Busekros [7], we introduce Holgate's definitions bellow. See also the survey [6] for more information. Here F will be a field of characteristic different from two.

Let § be a bi-dimensional commutative F-algebra generated by the elements $\{\mathrm{m}, \mathsf{f}\}$, and with multiplication table $\mathrm{m}^2 = 0$, $\mathrm{m}\mathsf{f} = \mathsf{f}\mathrm{m} = (\mathrm{m} + \mathsf{f})/2$, and $\mathsf{f}^2 = 0$. This algebra § is called *sex differentiation algebra*. Now, an algebra $\mathcal{A}$ will be called *dibaric* if it admits a homomorphism onto the sex differentiation algebra.

Recall that an F-algebra is called baric if it admits a homomorphism onto the field F. Since $\S^2 = \langle \mathrm{m} + \mathsf{f} \rangle_F$ is an ideal of § isomorphic to F we get that $\S^2$ is a baric algebra and hence we obtain the following well known result

Lemma 1.1. : If an algebra $\mathcal{A}$ is dibaric, then $\mathcal{A}^2$ is baric.

Example 1.1. : Let $(, \omega)$ be a baric $\mathbf{R}$-algebra, that is, $\mathcal{B}$ is an $\mathbf{R}$-algebra and $\omega : \mathcal{B} \longrightarrow \mathbf{R}$ is a homomorphism different from zero. Consider $T : \mathcal{B} \longrightarrow \mathcal{B}$ a linear mapping satisfying $\omega \circ T = \omega$. Thus T leaves the ideal $\ker(\omega)$ invariant. Now, we introduce the vector space $\mathcal{A} := \mathcal{B} \otimes \mathcal{B} \oplus \mathcal{B}$, where $\oplus$ denotes the direct sum and $\otimes$ denotes the tensor product of vector spaces. We identify the elements $x \otimes y \oplus 0 \in \mathcal{A}$ with $x \otimes y \in \mathcal{B} \otimes \mathcal{B}$ and the elements $0 \oplus z \in \mathcal{A}$ with $z \in \mathcal{B}$. In this space we introduce a commutative multiplication by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = 0, \; z_1 z_2 = 0$$
$$(x \otimes y) z = \tfrac{1}{2} (xy \otimes T(z) \oplus \omega(z) xy) .$$

The algebra $\mathcal{A}$ is the *sex linked duplicate* of the algebra $\mathcal{B}$ with respect to the linear mapping T (see [7] for more information). Obviously, $\mathcal{A}$

is a dibaric algebra with weight $\gamma : \mathcal{A} \longrightarrow \mathcal{S}$ defined by $\gamma(x \otimes y \oplus z) := \omega(xy)h + \omega(z)m$.

Example 1.2. : Let $A = A_h \oplus A_m$ be the 5-dimensional commutative $\mathbf{R}$-algebra with $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ as basis of A_h, with $\mathbf{b}_1$ and $\mathbf{b}_2$ as basis of A_m and with multiplication table as follows: $A_h^2 = 0, A_m^2 = 0$ and (for $i, j = 1, 2$)

$$\mathbf{a}_i\mathbf{b}_j = \frac{1}{2}\left(\delta_{ij}\mathbf{a}_i + (1 - \delta_{ij})\mathbf{a}_3 + \mathbf{b}_i\right), \quad \mathbf{a}_3\mathbf{b}_j = \frac{1}{2}\left(\mathbf{a}_1\mathbf{b}_j + \mathbf{a}_2\mathbf{b}_j\right)$$

where δ_{ij} is equal to 1 if $i = j$ and is equal to 0 in another case. The algebra A is dibaric with weight $\gamma : A \longrightarrow S$ defined by $\gamma(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + y_1\mathbf{b}_1 + y_2\mathbf{b}_2) := (x_1 + x_2 + x_3)h + (y_1 + y_2)m$. This algebra is called the *zygotic algebra* for sex *linked inheritance* for two alleles with *simple Mendelian segregation rates*. We claim, without proof, the following relevant fact: every element $x \in A$ with $\gamma(x) = h + m$ satisfies the plenary train equation [8 $\mathrm{x}^{[5]} - 6x^{[4]} - 3x^{[3]} + x^{[2]} = 0] where the plenary powers are defined inductively by$ $\mathrm{x}^{[1]} = x$ and $x^{[k+1]} = x^{[k]}x^{[k]}$ for $k \geq 1$. Therefore, if $x \in A$ represents a state of a population ($\gamma(x) = h + m$), then its trajectory $\{x^{[k]}\}_{k=1}^{\infty}$ converge and $x^{[\infty]} = \lim_{kl\infty} x^{[k]}$ is equal to the idempotent $(8x^{[4]} + 2x^{[3]} - x^{[2]})/9$. We notice that an explicit form of $x^{[\infty]}$, in terms of the corresponding gametic algebra, was given by Lyubich in [5] (see also [7, 8, 9] for more information). Finally, we claim that $8x^{[4]} - 6x^{[3]} - 3x^{[2]} + x^{[1]} \in ann(A) = \mathbf{R}\langle \mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3 \rangle$ for all $x \in A$ with $\gamma(x) = \mathsf{f} + m$.

2. Dibaric Weight Homomorphisms

In the following $\mathcal{A}$ will be an algebra (not necessarily commutative or associative) over the field F. A function $\gamma : \mathcal{A} \longrightarrow \S$, where $\S$ is the sex differentiation algebra defined above, is called dibaric weight homomorphism if γ is an onto homomorphism of algebras. So, if a, b are elements in $\mathcal{A}$ such that $\gamma(a) = \mathrm{m}$ and $\gamma(b) = \mathsf{f}$, then we have the following decomposition

$$\mathcal{A} = Fa \oplus Fb \oplus \ker(\gamma), \tag{2.1}$$

where $\ker(\gamma) := \{x \in \mathcal{A} : \gamma(x) = 0\}$ is an ideal of $\mathcal{A}$ of codimension two.

Notice that for every dibaric weight homomorphism γ and every automorphism $f : \S \longrightarrow \S$, the mapping $f \circ \gamma$ is a dibaric weight homomorphism and $\ker(\gamma) = \ker(f \circ \gamma)$. We say that two dibaric weight homomorphisms γ and γ' are equivalent if there exists an automorphism $f : \S \longrightarrow \S$ such that $\gamma' = f \circ \gamma$.

Lemma 2.1. : The sex differentiation algebra has only two automorphisms, the identity and the involution $* : \S \longrightarrow \S$ given by $*(m) = \mathsf{f},\ *(\mathsf{f}) = m$.

Proof. Let $f : \S l \S$ be an onto homomorphism. Then $0 = f(\mathrm{m}^2) = f(\mathrm{m})^2$, and analogously, $0 = f(\mathsf{f})^2$ and hence either $f(\mathrm{m}) \in F\mathrm{m}$, $f(\mathsf{f}) \in F\mathsf{f}$ or $f(\mathrm{m}) \in F\mathsf{f}$, $f(\mathsf{f}) \in F\mathrm{m}$. Next using that $f(\mathrm{m})f(\mathsf{f}) = f(mh) = f((\mathrm{m}+\mathsf{f})/2) = (f(\mathrm{m}) + f(\mathsf{f}))/2$ we get the result. ∎

From this result, it follows that each equivalence class defined above has exactly two weight. So, if γ and γ' are two different and equivalent dibaric weight homomorphisms, then $\gamma' = * \circ \gamma$. We denote by $\mathscr{E}$ the set of these equivalence classes, that is, an element of $\mathscr{E}$ is $\{\gamma, \gamma^*\}$, where γ is a dibaric weight homomorphism and $\gamma^* := * \circ \gamma$.

Theorem 2.1. : The application $\{\gamma, \gamma^*\} \longmapsto \ker(\gamma)$ is a bijection between the set $\mathscr{E}$ of equivalence class of dibaric weight homomorphisms of $\mathcal{A}$, and the set of ideals I of $\mathcal{A}$ of codimension two, such that $\mathcal{A}/I \cong \S$.

Corollary 2.1. : Dibaric weight homomorphisms with same kernel are equivalent.

Lemma 2.2. : Different dibaric weight homomorphisms of an algebra $\mathcal{A}$ are linearly independent.

Proof. Let $\gamma_1, \gamma_2, \cdots, \gamma_m$ be different dibaric weight homomorphisms of $\mathcal{A}$ and consider scalars $\alpha_1, \cdots, \alpha_m$ in F such that

$$\alpha_1\gamma_1(z) + \alpha_2\gamma_2(z) + \cdots + \alpha_m\gamma_m(z) = 0 \tag{2.2}$$

for all $z \in \mathcal{A}$. We will prove that $\alpha_1 = 0 = \alpha_2 = \cdots = \alpha_m$ using induction over the number m of different dibaric weight homomorphisms of $\mathcal{A}$. The case $m = 1$ is trivial. Let $m > 1$. Then by hypothesis of induction, the lemma is true for $m - 1$ weights.

Notice that if there exists and index i such that $\alpha_i = 0$, then by hypothesis of induction, we obtain that $\alpha_j = 0$ for $j = 1, 2, ..., m$, and the result follows.

First, we suppose that all weight homomorphisms have same kernel. Under this assumption, we obtain from Corollary 2.1 that $m = 2$ and $\gamma_2 = \gamma_1^*$. Now, let $z \in \mathcal{A}$ such that $\gamma_1(z) = \mathrm{m}$. Then $0 = \alpha_1\gamma_1(z) + \alpha_2\gamma_1^*(z) = \alpha_1\mathrm{m} + \alpha_2\mathsf{f}$, and hence it follows that $\alpha_1 = 0 = \alpha_2$.

Finally, we suppose that there exist homomorphisms with different kernels. We can assume that $\ker(\gamma_1) \neq \ker(\gamma_2)$. Under this condition, consider $x \in \mathcal{A}$ such that $\gamma_1(x) \neq 0$ and $\gamma_2(x) = 0$. Since $\mathrm{im}\,(\gamma_1) = \mathcal{S}$, there exists $y \in \mathcal{A}$ such that $\mathrm{m} + \mathsf{f} = \gamma_1(x)\gamma_1(y) = \gamma_1(xy)$. Multiplying the equation (2.2) by $\gamma_1(xy)$, we obtain

$$\alpha_1\gamma_1(xy)\gamma_1(z) + \alpha_2\gamma_1(xy)\gamma_2(z) + \cdots + \alpha_m\gamma_1(xy)\gamma_m(z) = 0, \tag{2.3}$$

and replacing $z \to (xy)z$ in equation (2.2) we get

$$\alpha_1\gamma_1(xy)\gamma_1(z) + \alpha_2\gamma_2(xy)\gamma_2(z) + \cdots + \alpha_m\gamma_m(xy)\gamma_m(z) = 0. \tag{2.4}$$

for all $z \in \mathcal{A}$. Next subtracting the equation (2.4) from equation (2.3), we get

$$\alpha_2\left(\gamma_1(xy)\right)\gamma_2(z) + \cdots + \alpha_m\left(\gamma_1(xy) - \gamma_m(xy)\right)\gamma_m(z) = 0. \tag{2.5}$$

Notice that $\gamma_2(xy) = 0$.Since $\gamma_k(xy) \in \gamma_k(\mathcal{A}^2) = \S^2 = \langle \mathrm{m} + \mathsf{f} \rangle_F$ for $k = 1, \ldots, m$, there exist scalars β_k such that $\gamma_k(xy) = \beta_k(\mathrm{m} + \mathsf{f})$. So, the equation (2.5) can be written as follows

$$(\mathrm{m} + \mathsf{f})\left(\alpha_2\gamma_2(z) + \alpha_3(1 - \beta_3)\gamma_3(z) + \cdots + \alpha_m(1 - \beta_m)\gamma_m(z)\right) = 0,$$

and therefore

$$\gamma(z) := \alpha_2\gamma_2(z) + \alpha_3(1-\beta_3)\gamma_3(z) + \cdots + \alpha_m(1-\beta_m)\gamma_m(z) \in F(\mathrm{m}-\mathsf{f}).$$

Thus, $\gamma(\mathcal{A}^2) \in F(\mathrm{m}+\mathsf{f}) \cap F(\mathrm{m}-\mathsf{f}) = (0)$. Now, if $\gamma(z) = \lambda_z(\mathrm{m}-\mathsf{f})$ then $0 = \gamma(z^2) = \gamma(z)^2 = -\lambda_z^2(\mathrm{m}+\mathsf{f})$. This implies that $\lambda_z = 0$ and hence $\gamma(z) = 0$. Using the hypothesis of induction on $\gamma(z) = 0$, we have $\alpha_2 = 0$. So, $\alpha_j = 0$ for all j. ∎

From the above result, it follows that the number of different dibaric weight homomorphisms of an algebra $\mathcal{A}$ is at most n, where n is the dimension of $\mathcal{A}$. We will show that this bound can be improved. For an algebra $\mathcal{A}$ we define inductively

$$\mathcal{A}^{[1]} = \mathcal{A}, \quad \mathcal{A}^{[i]} = \mathcal{A}^{[i-1]}\mathcal{A}^{[i-1]}, \quad i > 1.$$

So, if $\mathcal{A}$ has finite dimension, there exists a natural number r, such that $\mathcal{A}^{[r+1]} = \mathcal{A}^{[r]}$. Under this condition, we can show that the number of different dibaric weight homomorphisms of $\mathcal{A}$ is at most $2 \cdot \dim(\mathcal{A}^{[r]})$. Notice that for a dibaric algebra $\mathcal{A}^2 \neq \mathcal{A}$.

According to Lemma 1.1, if $\mathcal{A}$ is a dibaric algebra with γ as dibaric weight homomorphism, then $\mathcal{A}^2$ is baric and $\hat{\gamma} : \mathcal{A}^2 \longrightarrow \S^2$, the restriction of $\gamma : \mathcal{A} \longrightarrow \S$ is a baric weight homomorphism for $\mathcal{A}^2$. From, now on we identify $\S^2$ with the field F.

Theorem 2.2. : The application $\{\gamma, \gamma^*\} \longmapsto \hat{\gamma}$ is an injection between the set $\mathcal{E}$ of equivalence classes of dibaric weight homomorphism of $\mathcal{A}$ and the set of baric weight homomorphisms of $\mathcal{A}^2$.

Proof. First, we note that the elements of $\S^2$ are invariant by the involution $*$ and hence $pq = *(pq) = *(p) * (q)$ for all $p, q \in \S$. From this fact, we obtain that the application is well defined, that is $\hat{\gamma} = \hat{\gamma^*}$.

Next, we will show that the application is injective. Let τ, γ be two dibaric weight homomorphisms, such that $\hat{\tau} = \hat{\gamma}$. We have to show that these two homomorphisms are equivalent but according to Corollary 2.1, it suffices to show that they have the same kernels. So, let $a \in \ker(\gamma)$. Since $\ker(\gamma)$ is an ideal of $\mathcal{A}$, we have that $a\mathcal{A} \subseteq \ker(\gamma) \cap \mathcal{A}^2$ and using the hypothesis, we have $a\mathcal{A} \subseteq \ker(\tau)$. Then, it follows that $a \in \ker(\tau)$, since in other case we have an element $b \in \mathcal{A}$, such that $\tau(ab) \neq 0$ and this is a contradiction. So, we showed that $\ker(\gamma) \subset \ker(\tau)$ and therefore $\ker(\gamma) = \ker(\tau)$. This implies that the two homomorphisms are equivalent. ∎

In an analogous way we can prove the following lemma

Lemma 2.3. : The application $\omega \longmapsto \hat{\omega}$ is an injection between the set of baric weight homomorphisms of a baric algebra $\mathcal{B}$ and the set of baric homomorphisms of $\mathcal{B}^2$.

Proof. Let $\omega, \tau : B \longrightarrow F$ be two baric weight homomorphisms of $\mathcal{B}$ such that $\omega(x) = \tau(x)$ for all $x \in \mathcal{B}^2$. We already know that $\omega = \tau$ if and only if $\ker(\omega) = \ker(\tau)$. If $x \in \ker(\omega)$, then $x^2 \in ker(\omega) \cap \mathcal{A}^2 = \ker(\tau) \cap \mathcal{A}^2$ and hence $0 = \tau(x^2) = \tau(x)^2$. This forces $\tau(x) = 0$. Thus, we have proved that $\ker(\omega) \subset \ker(\tau)$ that is $\ker(\omega) = \ker(\tau)$ and hence by Lemma 3.3.1 of [5] we have that $\omega = \tau$. ∎

According to [5] the number of baric weight homomorphisms of a baric algebra $\mathcal{B}$ is at most its dimension. Using this fact, Lemma 2.3 and Theorem 2.2 we have the following result:

Corollary 2.2. : Let $\mathcal{A}$ be a dibaric algebra of dimension n and r a natural number such that $\mathcal{A}^{[r+1]} = \mathcal{A}^{[r]}$. Under these conditions, the number of different dibaric weight homomorphisms of $\mathcal{A}$ is at must $2 \cdot \dim(\mathcal{A}^{[r]})$.

Lemma 2.4. : Let $\mathcal{A}$ be a dibaric algebra with γ as dibaric weight homomorphism. If there exists a monomial $p(x) \in F[x]$, $p(x) \neq 0$, such that $p(a) = 0$, for all $a \in \ker(\gamma)$, then the only dibaric weight homomorphisms of $\mathcal{A}$ are γ and γ^*.

Proof. Let $\tau : \mathcal{A} \longrightarrow \S$ be a dibaric weight homomorphism. If $\tau(a) \neq 0$, then there exists $b \in \mathcal{A}$ such that $\mathrm{m} + \mathsf{f} = \tau(a)\tau(b) = \tau(ab)$. Then $\tau(p(ab)) = p(\tau(ab)) = p(\mathrm{m} + \mathsf{f}) = \mathrm{m}+\mathsf{f}$ and hence $ab \notin \ker(\gamma)$. This forces that $a \notin \ker(\gamma)$. Consequently, $\ker(\gamma) \subset \ker(\tau)$ and $\ker(\gamma) = \ker(\tau)$. Now the result follows form Theorem 2.1. ∎

Example 2.1. **:** An important example for biological applications is the evolution algebra $\mathcal{A}_V$ described in $[5, Cap.I]$. Consider two positive integer n and ν and real scalars $p_{ij,k}^{(m)}$ and $p_{ij,l}^{(h)}$ satisfying

$$p_{ij,k}^{(m)} \geq 0, \qquad p_{ij,l}^{(h)} \geq 0, \qquad \sum_{k=1}^{n} p_{ij,k}^{(m)} = 1, \qquad \sum_{l=1}^{\nu} p_{ij,l}^{(h)} = 1, \tag{2.6}$$

for $1 \leq i \leq n,\ 1 \leq j \leq \nu$. Now we define in the space $\mathbf{R}^n \times \mathbf{R}^\nu$ a commutative product as follows

$$e_i e_k = 0, \quad e_i \bar{e}_j = \frac{1}{2}\left(\sum_{k=1}^{n} p_{ij,k}^{(m)} e_k + \sum_{l=1}^{\nu} p_{ij,l}^{(f)} \bar{e}_l\right), \quad \bar{e}_j \bar{e}_l = 0$$

where we identify $e_i \equiv (e_i, 0)$, $\bar{e}_j \equiv (0, \bar{e}_j)$ such that $(e_i)_{i=1}^n$ is a canonical basis of $\mathbf{R}^n$ and $(\bar{e}_j)_{j=1}^\nu$ is a canonical basis of $\mathbf{R}^\nu$. In this way, we obtain a commutative algebra $\mathcal{A}_V$. The following result is well known

Lemma 2.5. **:** The mapping $s : \mathcal{A}_V \longrightarrow \S$ given by $s(z) = (\sum_{i=1}^n x_i)m + (\sum_{j=1}^\nu y_j)\mathsf{f}$ where $z = (x, y) \in \mathcal{A}_V$ is a dibaric weight homomorphism.

Lemma 2.6. **:** The weight homomorphism $s : \mathcal{A}_V \longrightarrow \S$ is characterized, up to equivalence, as the only positive dibaric weight homomorphism in the sense that the image of $\Omega = \{(x, y) \in \mathcal{A}_V : x_i, y_j \geq 0, \sum_i x_i = 1, \sum_j y_j = 1\}$ is contained in the set $\{\alpha m + \beta \mathsf{f} \mid \alpha, \beta \geq 0, \alpha + \beta > 0\}$.

Proof. Let $\gamma : \mathcal{A}_V \longrightarrow \S$, be a positive dibaric weight homomorphism. For $1 \leq i \leq n$ and $1 \leq j \leq \nu$ we have that $\gamma(e_i), \gamma(\bar{e}_j) \in \S$, so

$$\gamma(e_i) = \alpha_i \mathrm{m} + \beta_i \mathsf{f}, \qquad \gamma(\bar{e}_j) = \bar{\alpha}_j \mathrm{m} + \bar{\beta}_j \mathsf{f},$$

where $\alpha_i, \beta_i, \bar{\alpha}_j, \bar{\beta}_j \in \mathbf{R}$. Then, because $(e_i)^2 = 0$, we get $0 = \gamma(e_i^2) = \gamma(e_i)^2 = (\alpha_i \mathrm{m} + \beta_i \mathsf{f})^2 = \alpha_i \beta_i (\mathrm{m} + \mathsf{f})$ and analogously, using that $(\bar{e}_j)^2 = 0$, we obtain that $0 = \bar{\alpha}_j \bar{\beta}_j (\mathrm{m} + \mathsf{f})$. On the other hand, the elements $2 e_i \bar{e}_j$ and $e_i + \bar{e}_j$ belong to Ω and their images are

$\gamma(2e_i\bar{e}_j) = 2\gamma(e_i)\gamma(\bar{e}_j) = 2(\alpha_i\mathrm{m}+\beta_j\mathsf{f})(\bar{\alpha}_j\mathrm{m}+\bar{\beta}_j\mathsf{f}) = (\alpha_i\bar{\beta}_j+\beta_i\bar{\alpha}_j)(\mathrm{m}+\mathsf{f})$ and $\gamma(e_i+\bar{e}_j) = (\alpha_i+\bar{\alpha}_j)\mathrm{m}+(\beta_i+\bar{\beta}_j)\mathsf{f}$. Therefore, we have the following relations,

$$\alpha_i\beta_i = 0, \quad \alpha_i+\bar{\alpha}_j > 0, \quad \alpha_i\bar{\beta}_j+\beta_i\bar{\alpha}_j > 0, \quad \beta_i+\bar{\beta}_j > 0, \quad \bar{\alpha}_j\bar{\beta}_j = 0.$$

In particular $\alpha_1\beta_1 = 0$ and hence either $\alpha_1 \neq 0$ and $\beta_1 = 0$ or $\alpha_1 = 0$ and $\beta_1 \neq 0$. We will consider the two cases separately. In the first case, we will prove that $\gamma = s$ and in the second case that $\gamma = * \circ s$.

First, we suppose that $\alpha_1 \neq 0$ and $\beta_1 = 0$. Then, for each j, the equation $\alpha_1\bar{\beta}_j + \beta_1\bar{\alpha}_j > 0$ implies that $\bar{\beta}_j \neq 0$. Therefore, $\bar{\alpha}_j = 0$. Now, because $\alpha_1+\bar{\alpha}_j, \beta_1+\bar{\beta}_j > 0$ we obtain that $\alpha_1 > 0$ and $\bar{\beta}_j > 0$. In particular, $\bar{\beta}_1 > 0$ and $\bar{\alpha}_1 = 0$. So, from inequality $\alpha_i\bar{\beta}_1+\beta_i\bar{\alpha}_1 > 0$ we have that $\alpha_i > 0$ and hence $\beta_i = 0$. Thus, we have proved that $\gamma(e_i) = \alpha_i\mathrm{m}$ and $\gamma(\bar{e}_j) = \bar{\beta}_j\mathsf{f}$. Then,

$$\begin{aligned}\alpha_i\bar{\beta}_j(m+\mathsf{f}) = 2\gamma(e_i)\gamma(\bar{e}_j) = 2\gamma(e_i\bar{e}_j) &= \gamma\left(\textstyle\sum_{k=1}^n p_{ij,k}^{(m)}e_k + \sum_{l=1}^{\nu} p_{ij,l}^{(f)}\bar{e}_l\right)\\ &= \textstyle\sum_{k=1}^n p_{ij,k}^{(m)}\alpha_k m + \sum_{l=1}^{\nu} p_{ij,l}^{(f)}\bar{\beta}_l\mathsf{f}.\end{aligned}$$

So, we obtain the following equalities,

$$\alpha_i\bar{\beta}_j = \sum_{k=1}^n p_{ij,k}^{(m)}\alpha_k, \qquad \alpha_i\bar{\beta}_j = \sum_{l=1}^n p_{ij,l}^{(f)}\bar{\beta}_l, \tag{2.7}$$

for $1 \leq i \leq n$ and $1 \leq j \leq \nu$. Now considering the scalars

$$\alpha_{\max} = \max(\alpha_i)_{i=1}^n, \quad \alpha_{\min} = \min(\alpha_i)_{i=1}^n,$$

$$\bar{\beta}_{\max} = \max(\bar{\beta}_j)_{j=1}^{\nu}, \quad \bar{\beta}_{\min} = \min(\bar{\beta}_j)_{j=1}^{\nu},$$

and using (2.7), we obtain

$$\alpha_{\min} = \sum_{k=1}^n p_{ij,k}^{(m)}\alpha_{\min} \leq \sum_{k=1}^n p_{ij,k}^{(m)}\alpha_k = \alpha_i\bar{\beta}_j \leq \sum_{k=1}^n p_{ij,k}^{(m)}\alpha_{\max} = \alpha_{\max}.$$

and also

$$\bar{\beta}_{\min} = \sum_{k=1}^n p_{ij,k}^{(f)}\bar{\beta}_{\min} \leq \sum_{l=1}^n p_{ij,l}^{(f)}\bar{\beta}_l = \alpha_i\bar{\beta}_j \leq \sum_{l=1}^n p_{ij,l}^{(f)}\bar{\beta}_{\max} = \bar{\beta}_{\max}.$$

In particular, $\alpha_{\min} \leq \alpha_{\min}\bar{\beta}_j$ and $\alpha_{\max}\bar{\beta}_j \leq \alpha_{\max}$, and since all scalars are positive, it follows that $1 \leq \bar{\beta}_j \leq 1$, for all j. This implies that $\bar{\beta}_j = 1$. Analogously $\bar{\beta}_{\min} \leq \alpha_i\bar{\beta}_{\min}$ and $\alpha_i\bar{\beta}_{\max} \leq \bar{\beta}_{\max}$ and then $\alpha_i = 1$, for all i. So, $\gamma(e_i) = \mathrm{m}$ and $\gamma(\bar{e}_j) = \mathsf{f}$. Therefore $\gamma = s$.

Finally, we consider the second case, that is, $\beta_1 \neq 0$ and $\alpha_1 = 0$. Analogously, we have that $\gamma(e_i) = \beta_i\mathsf{f}$ and $\gamma(\bar{e}_j) = \bar{\alpha}_j\mathrm{m}$. Repeating the calculations above with the scalars α_i and $\bar{\beta}_j$ we get that these are all equal to 1. Thus, $\gamma = * \circ s$.

So, s and $s^* = * \circ s$ are the only positive dibaric weight homomorphisms in this algebra. ∎

3. Dibaric Algebras

An ordered pair (A, γ), where A is an algebra and $\gamma : A \longrightarrow \S$ is a dibaric weight homomorphism is called dibaric algebra. Under these conditions, the homomorphism is called weight function and the affine subspace $H := \{x \in \mathcal{A} \mid \gamma(x) = \mathrm{m} + \mathsf{f}\}$ of codimension 2, is called unit subspace. For each $x \in \mathcal{A}$ with $x^2 \notin \ker(\gamma)$, we have that $x^2/\gamma(x^2) \in H$. We denote the kernel of γ, by N, that is,

$$\mathrm{N} = \{x \in \mathcal{A} \mid \gamma(x) = 0\}.$$

If B is any set contained in $\mathcal{A}$, we will denote by N_B the set $\mathrm{N} \cap \mathrm{B}$, that is,

$$\mathrm{N}_B = \{x \in \mathrm{B} \mid \gamma(x) = 0\}.$$

Let $(\mathcal{A}, \gamma)$ be a dibaric algebra. We say that a subalgebra $\mathcal{A}_1$ of $\mathcal{A}$ is a dibaric subalgebra of $\mathcal{A}$ if $\mathcal{A}_1 \cap \ker(\gamma)$ is an ideal of $\mathcal{A}_1$ of codimension 2, or equivalently, $\gamma_1 \equiv \gamma_{|\mathcal{A}_1}$ is a dibaric weight homomorphism for $\mathcal{A}_1$. This subalgebra is denoted by $(\mathcal{A}_1, \gamma_1) \subset (\mathcal{A}, \gamma)$. A dibaric algebra $\mathcal{A}$ is not trivial if N is different from zero, that is, $\mathcal{A}$ is not isomorphic to §.

Also, a subalgebra $\mathcal{A}_1$ of $\mathcal{A}$ is called baric subalgebra if $\gamma(\mathcal{A}_1) = \langle \mathrm{m} + \mathsf{f} \rangle_F$.

An ideal I is called dibaric ideal if $I \subseteq \ker(\gamma)$, that is, $\gamma_{|I} = \{0\}$. Naturally a dibaric ideal cannot be a dibaric subalgebra. We say that a dibaric ideal I is maximal if $I \neq \mathrm{N}$ and the only dibaric ideals of $\mathcal{A}$

that contain I are I and N. Notice that the biggest dibaric ideal of a dibaric algebra $(\mathcal{A}, \gamma)$ is N.

The annulator, $ann\mathcal{A} := \{x \in A \mid xA = (0)\}$ is a dibaric ideal of $\mathcal{A}$. Also, any subspace of $ann\mathcal{A}$ is a dibaric ideal. The ideal $\mathcal{A}^2$ of $\mathcal{A}$, is not dibaric, but, according to Lemma 1.1, we have that $\mathcal{A}^2$ is a baric subalgebra.

For any dibaric ideal I, we have that the quotient $\mathcal{A}/I$ is a dibaric algebra. It is called dibaric quotient and is denoted by $(\mathcal{A}, \gamma)/I$. The quotient algebra $(\mathcal{A}, \gamma)/\mathcal{N}$ is isomorphic to the sex differentiation algebra §.

Given two dibaric algebras $(\mathcal{A}_1, \gamma_1)$ and $(\mathcal{A}_2, \gamma_2)$, a dibaric homomorphism of dibaric algebras $f : (\mathcal{A}_1, \gamma_1) l (\mathcal{A}_2, \gamma_2)$ is a homomorphism of algebras $f : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ such that $\gamma_2 \circ f = \gamma_1$. For example, the embedding of a dibaric subalgebra and the quotient application are dibaric homomorphisms. Clearly, the composition of dibaric homomorphisms is dibaric. The inverse of a dibaric isomorphism is a dibaric isomorphism, because if $\gamma_2 \circ f = \gamma_1$, this imply that $\gamma_2 = \gamma_1 \circ f^{-1}$. We write $(\mathcal{A}_1, \gamma_1) \cong (\mathcal{A}_2, \gamma_2)$ for isomorphic dibaric algebras, that is, there is a dibaric isomorphism $f : (\mathcal{A}_1, \gamma_1) \longrightarrow (\mathcal{A}_2, \gamma_2)$.

Every dibaric algebra $(\mathcal{A}, \gamma)$ is not associative because $(\mathcal{A}, \gamma)/\ker(\gamma) \cong$ § and § is not associative. In particular, the subalgebra of $End\,(\mathcal{A})$ spanned by the left and right multiplication by elements of $\mathcal{A}$, that is $L_a(x) = ax$ and $R_a(x) = xa$ for all $a, x \in \mathcal{A}$ is not dibaric because it is associative. This associative algebra is called the multiplication algebra of $\mathcal{A}$ and is denoted by $\mathcal{M}\,(A)$.

Lemma 3.1. : Let $f : (\mathcal{A}_1, \gamma_1) \longrightarrow (\mathcal{A}_2, \gamma_2)$ be a dibaric homomorphism. Then $\Im(f)$ is a dibaric subalgebra of $\mathcal{A}_2$ and $\ker(f)$ is a dibaric ideal of $\mathcal{A}_1$. The bijection induced by f is a dibaric isomorphism, that is $(\mathcal{A}_1, \gamma_1)/\ker(f) \cong im(f)$.

Lemma 3.2. : A dibaric homomorphism $f : (\mathcal{A}_1, \gamma_1) \longrightarrow (\mathcal{A}_2, \gamma_2)$ is an isomorphism if and only if $\hat{f} \equiv f_{|\mathcal{N}_1} : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ is an isomorphism, where $\mathcal{N}_i = \ker(\gamma_i)$, for $i = 1, 2$.

Proof. Let $f : (\mathcal{A}_1, \gamma_1) \longrightarrow (\mathcal{A}_2, \gamma_2)$ be a dibaric homomorphism such that $\hat{f} : \mathcal{N}_1 l \mathcal{N}_2$ is an isomorphism. Consider $u, v \in \mathcal{A}_1$ satisfying $\gamma_1(u) = \mathrm{m}$ and $\gamma_1(v) = \mathsf{f}$. If $a, b \in \mathcal{A}_1$ then there exist elements $x, y \in \mathcal{N}_1$ and scalars $\alpha_1, \alpha_2, \beta_1, \beta_2 \in F$ uniquely determined such that $a = \alpha_1 u + \alpha_2 v + x$ and $b = \beta_1 u + \beta_2 v + y$. Now we assume that $f(a) = f(b)$. This give us that $f(a - b) = 0$, and since f is a dibaric homomorphism, we have that $a - b \in \mathcal{N}_1$. This implies that $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Then, because $f(a) = f(b)$, we obtain that $f(x) = f(y)$. Now by hypothesis $\hat{f}$ is an isomorphism and hence $x = y$. Consequently, $a = b$. The reverse is trivial. ∎

An idempotent element e in a dibaric algebra $(\mathcal{A}, \gamma)$, is called semiprincipal if $e = u + v$, where $\gamma(u) = \mathrm{m}$, $\gamma(v) = \mathsf{f}$, and $u^2 = 0$, $v^2 = 0$ and $uv = vu = (u + v)/2$.

Let $(\mathcal{A}, \gamma)$ be a dibaric algebra and $e = u + v$ a semiprincipal idempotent element in $\mathcal{A}$. Then, we have the decomposition $\mathcal{A} = Fu \oplus Fv \oplus \mathcal{N}$ where $Fu \oplus Fv$ is a dibaric subalgebra isomorphic to §.

There exists a natural form to get algebras with semiprincipal idempotent elements. If N is an arbitrary algebra over F and $\lambda_1, \lambda_2, \rho_1, \rho_2 : N \longrightarrow N$ are linear applications, we consider $\mathcal{A} = \S \oplus N$, with the multiplication $(\alpha \mathrm{m} + \beta \mathsf{f}, x_1)(\mu \mathrm{m} + \eta \mathsf{f}, x_2)$ defined by

$$\left(\frac{(\alpha\eta + \beta\mu)}{2}(\mathrm{m} + \mathsf{f}), x_1 x_2 + \alpha\lambda_1(x_2) + \beta\rho_1(x_2) + \mu\lambda_2(x_1) + \eta\rho_2(x_1) \right)$$

and weight function by $\gamma(\alpha \mathrm{m} + \beta \mathsf{f}, x) := \alpha \mathrm{m} + \beta \mathsf{f}$, where $\alpha, \beta, \mu, \eta \in F$ and $x_1, x_2, x \in N$. We have that γ is different from zero and the element $(\mathrm{m} + \mathsf{f}, 0)$ is a semiprincipal idempotent element of $\mathcal{A}$. This algebra is denoted by $[\lambda_1, \lambda_2, \rho_1, \rho_2, N]$.

Conversely, a dibaric algebra $(\mathcal{A}, \gamma)$ with semiprincipal idempotent $e = u + v$, is isomorphic to $[\lambda_1, \lambda_2, \rho_1, \rho_2, \mathcal{N}]$, where $\mathcal{N} = \ker(\gamma)$,

$$\lambda_1 = L_{u|\mathcal{N}}, \quad \lambda_2 = L_{v|\mathcal{N}}, \quad \rho_1 = R_{u|\mathcal{N}}, \quad \rho_2 = R_{v|\mathcal{N}}.$$

The applications $L_{a|\mathcal{N}}$, $R_{a|\mathcal{N}}$ denote the restriction of the left and right multiplications by the element a in N, that is, $L_a(x) = ax$, $R_a(x) = xa$, for every $x \in \mathrm{N}$.

If (A, γ) is a dibaric algebra with semiprincipal idempotent element $e = u + v$, and I is a dibaric ideal of $\mathcal{A}$, then $Fu \oplus Fv \oplus I$ is a dibaric subalgebra of $\mathcal{A}$. Naturally if I is maximal, it follows that this subalgebra is maximal. Conversely, if I is a dibaric ideal, it follows that the subalgebra defined above is maximal.

4. The Main Theorem

For two dibaric algebras, $(\mathcal{A}_1, \gamma_1)$ and $(\mathcal{A}_2, \gamma_2)$, we have the external product $\mathcal{A}_1 \times \mathcal{A}_2$ with the multiplication given by $(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2)$. This algebra is not necessarily dibaric, but the subspace

$$\mathcal{A}_1 \vee \mathcal{A}_2 := \{(x_1, x_2) \in \mathcal{A}_1 \times \mathcal{A}_2 \mid \gamma_1(x_1) = \gamma_2(x_2)\}$$

is a dibaric algebra with dibaric weight homomorphism given by

$$\gamma_1 \vee \gamma_2(x_1, x_2) := \gamma_1(x_1) = \gamma_2(x_2).$$

We will call this algebra $(\mathcal{A}_1 \vee \mathcal{A}_2, \gamma_1 \vee \gamma_2)$ by join of $\mathcal{A}_1$ and $\mathcal{A}_2$.

The join for baric algebras with idempotent of weight 1 was defined for Roberto Costa and H. Guzzo J. in [1]. Here, we extend this definition for dibaric algebras.

There exists a natural identification of N_1, the dibarideal of $\mathcal{A}_1$ with the ideal of $\mathcal{A}_1 \vee \mathcal{A}_2$ given by the set $\{(x, 0) \mid x \in \mathrm{N}_1\}$. Analogously, we identify N_2, the dibarideal of $\mathcal{A}_2$, with the ideal of $\mathcal{A}_1 \vee \mathcal{A}_2$ given by the set $\{(0, x) \mid x \in \mathrm{N}_2\}$. Take $u_1, v_1 \in \mathcal{A}_1$ such that $\gamma_1(u_1) = \mathrm{m}$, $\gamma_1(v_1) = \mathsf{f}$, and $u_2, v_2 \in \mathcal{A}_2$ such that $\gamma_2(u_2) = \mathrm{m}$, $\gamma_2(v_2) = \mathsf{f}$. We have that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are in $\mathcal{A}_1 \vee \mathcal{A}_2$ with $\gamma_1 \vee \gamma_2(u) = \mathrm{m}$ and $\gamma_1 \vee \gamma_2(v) = \mathsf{f}$. Therefore, we write

$$\mathcal{A}_1 \vee \mathcal{A}_2 = Fu \oplus Fv \oplus \mathrm{N}_1 \oplus \mathrm{N}_2,$$

where $\mathrm{N}_1 \oplus \mathrm{N}_2$ is the dibarideal of $\mathcal{A}_1 \vee \mathcal{A}_2$.

Lemma 4.1. : The join of dibaric algebras satisfies the following properties :

(a) $(\S \vee \mathcal{A}, Id_{|\S} \vee \gamma) \cong (\mathcal{A}, \gamma)$; where $Id_{|\S}$ is the identity in $\S$;

(b) $(\S \vee \mathcal{A}, * \vee \gamma) \cong (\mathcal{A}, \gamma)$;

(c) $(\mathcal{A}_1 \vee \mathcal{A}_2, \gamma_1 \vee \gamma_2) \cong (\mathcal{A}_2 \vee \mathcal{A}_1, \gamma_2 \vee \gamma_1)$;

(d) $((\mathcal{A}_1 \vee \mathcal{A}_2) \vee \mathcal{A}_3, (\gamma_1 \vee \gamma_2) \vee \gamma_3) \cong (\mathcal{A}_1 \vee (\mathcal{A}_2 \vee \mathcal{A}_3), \gamma_1 \vee (\gamma_2 \vee \gamma_3))$.

In view of (d) we can define the join $(\bigvee_{i\in I}\mathcal{A}_i,\ \bigvee_{i\in I}\gamma_i)$ of an arbitrary family $\{(\mathcal{A}_i,\gamma_i)\}_{i\in I}$ of dibaric algebras, where $\bigvee_{i\in I}\mathcal{A}_i$ is the subalgebra of $\times_{i\in I}\mathcal{A}_i$, given by

$$\bigvee_{i\in I}\mathcal{A}_i := ((x_i)_{i\in I} \mid x_i\in\mathcal{A}_i,\ \gamma_i(x_i)=\gamma_j(x_j),\ \forall\ i,j\in I)\,,$$

and the dibaric weight homomorphism is given by $(\bigvee_{i\in I}\gamma_i)((x_i)_{i\in I}) := \gamma_i(x_i)$, where i is a fixed and arbitrary index of I. Notice that for a family $\mathcal{A}_1,\ldots,\mathcal{A}_r$ of r dibaric algebras with dimension of $\mathcal{A}_i$ equal to n_i, we have that

$$\dim\bigvee_{i=1}^{r}\mathcal{A}_i = 2(1-r)+\sum_{i=1}^{r}n_i.$$

A dibaric algebra $(\mathcal{A},\gamma)$ is decomposable if there exist non-trivial dibaric algebras $(\mathcal{A}_1,\gamma_1)$ and $(\mathcal{A}_2,\gamma_2)$ such that $(\mathcal{A},\gamma)\cong(\mathcal{A}_1,\gamma_1)\vee(\mathcal{A}_2,\gamma_2)$. In another case, we say that $\mathcal{A}$ is indecomposable.

Lemma 4.2. A dibaric algebra (A,γ) is decomposable if and only if N is decomposable as $M(\mathcal{A})$ module.

Proof. Let $(\mathcal{A},\gamma)$ be a decomposable dibaric algebra. Then, there exist two non-trivial dibaric algebras $(\mathcal{A}_1,\gamma_1)$ and $(\mathcal{A}_2,\gamma_2)$ and a dibaric isomorphism $f\ :\ (\mathcal{A},\gamma)\ \longrightarrow\ (\mathcal{A}_1\vee\mathcal{A}_2,\gamma_1\vee\gamma_2)$. Since the dibarideal of $\mathcal{A}_1\vee\mathcal{A}_2$ is written as a direct sum of the ideals $\mathrm{N}_1\equiv\{(x,0),x\in\mathrm{N}_1\}$ and $\mathrm{N}_2\equiv\{(0,x),x\in\mathrm{N}_2\}$ and f is a dibaric isomorphism, it follows that the dibarideal N of $\mathcal{A}$ is written as the direct sum of the non-trivial ideals $f^{-1}(\mathrm{N}_1)$ and $f^{-1}(\mathrm{N}_2)$. Therefore N is decomposable.

Conversely, let $(\mathcal{A},\gamma)$ be a dibaric algebra such that $\mathcal{N}$ is decomposable as M modulo. Then, there exist $\mathrm{N}_1,\mathrm{N}_2$, two proper M submodules of N such that $\mathrm{N}=\mathrm{N}_1\oplus\mathrm{N}_2$. So, we can write $\mathcal{A}=Fu\oplus Fv\oplus\mathrm{N}_1\oplus\mathrm{N}_2$ where u,v satisfy $\gamma(u)=\mathrm{m}$ and $\gamma(v)=\mathrm{h}$. Notice that the subspaces N_1 and N_2 are ideals of $\mathcal{A}$. Then, an element $x\in\mathcal{A}$ is uniquely written as a sum $x=\alpha u+\beta v+x_1+x_2$, where $\alpha,\beta\in F$ and $x_1\in\mathrm{N}_1$, $x_2\in\mathrm{N}_2$. This decomposition give us the means to define the projections $\pi_i : \mathcal{A} \longrightarrow \mathrm{N}_i$, $\pi_i(x) = x_i$, for $i = 1, 2$. The mappings π_i, $(i = 1, 2)$ satisfy the following properties:

$$\pi_i(ux) = u\pi_i(x), \ \ \pi_i(xu) = \pi_i(x)u, \ \ \pi_i(vx) = v\pi_i(x), \ \ \pi_i(xv) = \pi_i(x)v,$$

and $\pi_i(xy) = \pi_i(x)\pi_i(y)$ for all $x, y \in \mathrm{N}$. Now, we define over the vector space $\mathcal{A}_i = Fu_i \oplus Fv_i \oplus \mathrm{N}_i$, $(i = 1, 2)$ a product " $\cdot$ " such that, restricted to N_i, it coincides with the multiplication of N_i as a subalgebra of $\mathcal{A}$ that is $x_i \cdot y_i = x_i y_i$ for $x_i, y_i \in \mathrm{N}_i$ and

$$\begin{array}{rclrcl} u_i \cdot v_i & = & \frac{(u_i+v_i)}{2} + \pi_i(uv), & v_i \cdot u_i & = & \frac{(u_i+v_i)}{2} + \pi_i(vu), \\ u_i \cdot u_i & = & \pi_i(u^2), & v_i \cdot v_i & = & \pi_i(v^2), \\ x_i \cdot u_i & = & x_i u, & u_i \cdot x_i & = & u x_i, \\ x_i \cdot v_i & = & x_i v, & v_i \cdot x_i & = & v x_i. \end{array}$$

where $x_i, y_i \in \mathrm{N}_i$. The algebra $(\mathcal{A}_i, \cdot)$ has dibaric homomorphism given by $\gamma_i(\alpha u_i + \beta v_i + x_i) = \alpha \mathrm{m} + \beta \mathsf{f}$, for all $\alpha, \beta \in F$ and $x_i \in \mathrm{N}_i$. So, $(\mathcal{A}_1, \gamma_1)$ and $(\mathcal{A}_2, \gamma_2)$ are dibaric algebras and its join $(\mathcal{A}_1, \gamma_1) \vee (\mathcal{A}_2, \gamma_2)$ is isomorphic to $(\mathcal{A}, \gamma)$. To see that this last assertion is true, we consider the mapping $f : \mathcal{A} \longrightarrow \mathcal{A}_1 \vee \mathcal{A}_2$, given by

$$f(\alpha u + \beta v + x) = (\alpha u_1 + \beta v_1 + \pi_1(x), \alpha u_2 + \beta v_2 + \pi_2(x)).$$

Simple computations show that f is a dibaric isomorphism. ∎

The above result can be generalized in the following sense: if a dibaric algebra $(\mathcal{A}, \gamma)$ is written as join of a family of dibaric algebras $\{(\mathcal{A}_i, \gamma_i)\}_{i=i}^n$, that is $(\mathcal{A}, \gamma) = (\bigvee_{i=1}^n \mathcal{A}_i, \bigvee_{i=1}^n \gamma_i)$, then we identify $\mathrm{N}_j \equiv \{(0, \ldots, x_j, \ldots, 0) \in \bigvee_{i=1}^n \mathcal{A}_i, x_j \in \mathrm{N}_j\}$, where $\mathrm{N}_j = \ker(\gamma_j)$ and we have that the dibaric ideal N of $\mathcal{A}$ is written as a direct sum of the M submodules as follows $\mathrm{N} = \mathrm{N}_1 \oplus \cdots \oplus \mathrm{N}_n$.

Conversely, if we have that the dibaric ideal N of an algebra $(\mathcal{A}, \gamma)$ is written as a direct sum of M submodules of N, that is $\mathrm{N} = \bigoplus_{i=1\mathrm{N}i}^n$, where N_i is M submodule, then for each index i we define an algebra over the vector space $\mathcal{A}_i = Fu_i \oplus Fv_i \oplus \mathrm{N}_i$ with a product as in the above lemma where $\pi_i(\alpha u_i + \beta v_i + x) = x_i$ whenever $x = \sum_{j=1}^n x_j$ with $x_j \in \mathrm{N}_j$. This algebra has weight homomorphisms given by $\gamma_i(\alpha u_i + \beta v_i + \pi_i(x)) = \alpha \mathrm{m} + \beta \mathsf{f}$. So, $(\mathcal{A}_i, \gamma_i)$ is a dibaric algebras and $f : (\mathcal{A}, \gamma) l (\bigvee_{i=1}^n \mathcal{A}_i, \bigvee_{i=1}^n \gamma_i)$, defined by

$$f(\alpha u + \beta v + x) = (\alpha u_1 + \beta v_1 + \pi_1(x), \ldots, \alpha u_n + \beta v_n + \pi_n(x)),$$

where $\alpha, \beta \in F, x \in \mathrm{N}$, is a dibaric homomorphism.

Corollary 4.1. : If a dibaric algebra $(\mathcal{A}, \gamma)$ is written as a join of a finite family of dibaric algebras, that is, $(\mathcal{A}, \gamma) \cong (\bigvee_{i=1}^{n} A_i, \bigvee_{i=1}^{n} \gamma_i)$, then $N \cong N_1 \times \cdots \times N_n$, where $N_i := \ker(\gamma_i)$. Conversely, if the dibaric ideal N of an arbitrary dibaric algebra $(\mathcal{A}, \gamma)$, is written as a direct sum of ideals $I_1, \ldots, I_n$, then there exist dibaric algebras $\{(\mathcal{A}_i, \gamma_i)\}_{i=1}^{n}$, with $\ker(\gamma_i) \cong I_i$, such that $(\mathcal{A}, \gamma) \cong (\bigvee_{i=1}^{n} \mathcal{A}_i, \bigvee_{i=1}^{n} \gamma_i)$.

We say that a dibaric algebra $(\mathcal{A}, \gamma)$ satisfies the ascendent chain condition (a.c.c.) if $\mathrm{N} = \ker(\gamma)$ satisfies (a.c.c.) as M module, where M is the multiplication algebra of $\mathcal{A}$. Analogously, we say that a dibaric algebra $(\mathcal{A}, \gamma)$ satisfy the descendent chain condition (d.c.c.) if N satisfy (d.c.c.) as $\mathrm{M}(\mathcal{A})$ module.

Lemma 4.3. : Let $(\mathcal{A}_1 \vee \mathcal{A}_2, \gamma_1 \vee \gamma_2)$ be the join of two dibaric algebras $(\mathcal{A}_1, \gamma_1)$ and $(\mathcal{A}_2, \gamma_2)$. Then (e_1, e_2) is a semiprincipal idempotent in $\mathcal{A}_1 \vee \mathcal{A}_2$ if and only if e_1, e_2 are semiprincipal idempotent elements in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Therefore, it follows that $\mathcal{A}_1 \vee \mathcal{A}_2$ has a semiprincipal idempotent if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ have a semiprincipal idempotent.

Proof. $(\Rightarrow)$ Let $e := (e_1, e_2) = (u_1, u_2) + (v_1, v_2)$ be a semiprincipal idempotent element in $\mathcal{A}_1 \vee \mathcal{A}_2$, with $u_1, v_1 \in \mathcal{A}_1$ and $u_2, v_2 \in \mathcal{A}_2$. Under these conditions, we have that $(u_1, u_2)^2 = (0, 0)$. So, it follows that $(u_1^2, u_2^2) = (0, 0)$, therefore $u_1^2 = 0$ and $u_2^2 = 0$. Analogously, $(v_1, v_2)^2 = (0, 0)$, and so $v_1^2 = 0$ and $v_2^2 = 0$. On the other hand, $(u_1, u_2)(v_1, v_2) = (u_1 v_1, u_2 v_2) = ((u_1, u_2) + (v_1, v_2))/2$. Then, $u_1 v_1 = (u_1 + v_1)/2$ and $u_2 v_2 = (u_2 + v_2)/2$. Finally, we have $(\gamma_1 \vee \gamma_2)(u_1, u_2) = \gamma_1(u_1) = \gamma_2(u_2) = \mathrm{m}$ and $(\gamma_1 \vee \gamma_2)(v_1, v_2) = \gamma_1(v_1) = \gamma_2(v_2) = \mathsf{f}$. Hence, $e_1 = u_1 + v_1$ and $e_2 = u_2 + v_2$ are semiprincipal idempotent elements in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.

$(\Leftarrow)$ If $e_1 = u_1 + v_1 \in \mathcal{A}_1$ and $e_2 = u_2 + v_2 \in \mathcal{A}_2$ are semiprincipal idempotent elements, the ordered pair $(e_1, e_2) = (u_1, u_2) + (v_1, v_2) \in \mathcal{A}_1 \vee \mathcal{A}_2$ because $\gamma_1(e_1) = \mathrm{m} + \mathsf{f} = \gamma_2(e_2)$. This element satisfy

$(e_1, e_2)^2 = (e_1^2, e_2^2) = (e_1, e_2)$ and $(u_1, u_2)^2 = (u_1^2, u_2^2) = (0, 0)$, $(v_1, v_2)^2 = (v_1^2, v_2^2) = (0, 0)$. Finally, we have that $(u_1, u_2)(v_1, v_2) = (u_1v_1, u_2v_2) = (u_1, u_2)/2 + (v_1, v_2)/2$. Therefore (e_1, e_2) is a semiprincipal idempotent in $\mathcal{A}_1 \vee \mathcal{A}_2$. ∎

The above result can be generalized in the following sense: if $(\mathcal{A}, \gamma)$ has a semiprincipal idempotent $e = u+v$ and this algebra is isomorphic to the join of a finite family $\{(\mathcal{A}_i, \gamma_i)\}_{i=1}^n$ of dibaric subalgebras, then each algebra $\mathcal{A}_i$ has a semiprincipal idempotent $e_i = u_i + v_i$. To prove it, we use induction over n and the associativity of the join of dibaric algebras. Therefore, we have the corollary below.

Corollary 4.2. : If $(A, \gamma) \cong (\bigvee_{i=1}^n \mathcal{A}_i, \bigvee_{i=1}^n \gamma_i)$, then $\mathcal{A}$ has a semiprincipal idempotent if and only if $\mathcal{A}_i$ have semiprincipal idempotents, for all i.

Lemma 4.4. : Let $(\mathcal{A}, \gamma)$ be a dibaric algebra with $e = u + v$ as semiprincipal idempotent element such that $(\mathcal{A}, \gamma) \cong (\bigvee_{i=1}^n \mathcal{A}_i, \bigvee_{i=1}^n \gamma_i)$. Then for every i, there exists a dibaric subalgebra of $\mathcal{A}$ isomorphic to (A_i, γ_i) with $e = u + v$ as semiprincipal idempotent.

Proof. We will suppose that $(\mathcal{A}, \gamma) = (\bigvee_{i=1}^n \mathcal{A}_i, \bigvee_{i=1}^n \gamma_i)$. If $e = u + v$ is a semiprincipal idempotent element in $\bigvee_{i=1}^n \mathcal{A}_i$, with

$$e = (e_1, \ldots, e_n), \quad u = (u_1, \ldots, u_n) = u \quad \text{and} \quad v = (v_1, \ldots, v_n),$$

then $e_i = u_i + v_i$ is a semiprincipal idempotent in $\mathcal{A}_i$. On the other hand $\bigvee_{i=1}^n \mathcal{A}_i$ can be written as

$$\bigvee_{i=1}^n \mathcal{A}_i = Fu \oplus Fv \oplus \mathrm{N}_1 \oplus \cdots \oplus \mathrm{N}_n,$$

where

$$\mathrm{N}_j = \{(0, \ldots, x_j, \ldots, 0) \in \bigvee_{i=1}^n \mathcal{A}_i \mid x_j \in \ker(\gamma_j)\}$$

is an barideal of $\bigvee_{i=1}^n \mathcal{A}_i$. So, since $e = u + v$ is a semiprincipal idempotent element and N_j is a barideal of $\mathcal{A}$, for $j = 1, 2, \ldots, n$, it follows that $Fu \oplus Fv \oplus \mathrm{N}_j$ is a dibaric subalgebra of $\bigvee_{i=1}^{n} \mathcal{A}_i$. Finally, the linear mapping $f_j : Fu \oplus Fv \oplus \mathrm{N}_j \longrightarrow \mathcal{A}_j$ defined by $f_j(\alpha u + \beta v + x) = \alpha u_j + \beta v_j + x$ for all $\alpha, \beta \in F$ e $x \in \mathrm{N}_j$ is a dibaric isomorphism. ∎

Lemma 4.5. : If a dibaric algebra $(\mathcal{A}, \gamma)$ with semiprincipal idempotent $e = u + v$ satisfying the descendent chain condition, then there exists a finite number of indecomposable dibaric subalgebras $\{(\mathcal{A}_i, \gamma_i)\}_{i=1}^{n}$ of $(\mathcal{A}, \gamma)$, such that $(\mathcal{A}, \gamma) \cong (\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n, \gamma_1 \vee \cdots \vee \gamma_n)$.

Proof. Since $(\mathcal{A}, \gamma)$ satisfy the descendent chain condition, it follows that N satisfies d.c.c. as $\mathrm{M}(\mathcal{A})$ module. So, there exist indecomposable $\mathrm{M}(\mathcal{A})$ submodules $\mathrm{N}_1, \ldots, \mathrm{N}$ of N such that $\mathrm{N} = \mathrm{N}_1 \oplus \mathrm{N}_2 \oplus \cdots \oplus \mathrm{N}_m$. Therefore, for each j, $\mathcal{A}_j = Fu \oplus Fv \oplus \mathrm{N}_j$ is a dibaric subalgebra of $\mathcal{A}$ such that $(\mathcal{A}, \gamma) \cong (\bigvee_{i=1}^{n} \mathcal{A}_i, \bigvee_{i=1}^{n} \gamma_i)$. Finally, we will show that $\mathcal{A}_j$ is indecomposable, for each j. We observe that a dibaric ideal I of $\mathcal{A}_j$ is a dibaric ideal of $\mathcal{A}$, because $I\mathcal{A} = I(Fu \oplus Fv \oplus \mathrm{N}_1 \oplus \cdots \oplus \mathrm{N}_n) = I\mathcal{A}_j \subseteq I$. Analogously $\mathcal{A}I \subseteq I$. So, the $\mathrm{M}(\mathcal{A})$submodules of N_j are equal to the ;$\mathrm{M}(\mathcal{A})$ submodules of N_j. Since N_j is indecomposable as $\mathrm{M}(\mathcal{A})$-module, it follows that N_j is indecomposable as $\mathrm{M}(\mathcal{A}_j)$-module. So, $\mathcal{A}_j$ is indecomposable. ∎

Theorem 4.1. : (Krull-Schmidt) Let $(\mathcal{A}, \gamma)$ be a dibaric algebra with semiprincipal idempotent element $e = u + v$ that satisfies d.c.c. and a.c.c.. If

$$(\mathcal{A}, \gamma) = (\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n, \gamma_1 \vee \cdots \vee \gamma_n), \; (\mathcal{A}, \gamma) = (B_1 \vee \cdots \vee B_m, \chi_1 \vee \cdots \vee \chi_m).$$

where each $(\mathcal{A}_i, \gamma_i)$ and (B_j, χ_j) are indecomposable dibaric subalgebras of $(\mathcal{A}, \gamma)$, then $n = m$ and reindexing, we have that $(\mathcal{A}_i, \gamma_i) \cong (B_i, \chi_i)$, for each $i \in \{1, \ldots, n\}$.

Proof. Since we have two decompositions of $\mathcal{A}$ in indecomposable dibaric subalgebras, then $\mathrm{N} = \ker(\gamma)$ is decomposed in indecomposable M submodules as follows

$$\mathrm{N} = \mathrm{N}_1 \oplus \cdots \oplus \mathrm{N}_n, \quad \mathrm{N} = \mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_m.$$

where $\mathrm{N}_i = \ker(\gamma_i)$ and $\mathcal{P}_j = \ker(\chi_j)$. According to the Krull-Schmidt's Theorem for $\mathcal{A}$ modules, we have that $m = n$ and with a reindexation $\mathrm{N}_i \cong \mathcal{P}_i$ as $\mathcal{A}$-modules. By the same Theorem, we can write

$$\mathrm{N} = \mathrm{N}_1 \oplus \mathrm{N}_2 \oplus \cdots \oplus \mathrm{N}_k \oplus \mathcal{P}_{k+1} \oplus \cdots \oplus \mathcal{P}_n, \tag{4.1}$$

for $0 \leq k \leq n$. Let j be a fixed index. First we will show that $\mathrm{N}_j \cong \mathcal{P}_j$ as algebras. We consider the two decomposition of (4.1)

$$\mathrm{N}_1 \oplus \mathrm{N}_2 \oplus \cdots \oplus \mathrm{N}_{j-1} \oplus \mathrm{N}_j \oplus \mathcal{P}_{j+1} \oplus \cdots \oplus \mathcal{P}_n,$$

$$\mathrm{N}_1 \oplus \mathrm{N}_2 \oplus \cdots \oplus \mathrm{N}_{j-1} \oplus \mathcal{P}_j \oplus \mathcal{P}_{j+1} \oplus \cdots \oplus \mathcal{P}_n.$$

So, if $x \in \mathrm{N}$, then x can be written in two different ways

$$x = x_1 + \cdots + x_n, \qquad x = x'_1 + \cdots + x'_n$$

where $x_j \in \mathrm{N}_j$; $x'_j \in \mathcal{P}_j$; $x_r, x'_r \in \mathrm{N}_r$ for $1 \leq r \leq j-1$, and $x_s, x'_s \in \mathcal{P}_s$ for $j+1 \leq s \leq n$. Let τ_j be the injection of N_j in N and $\pi_j : \mathrm{N} \longrightarrow \mathcal{P}_j$ the projection define via $\pi_j(x) := x'_j$ for all $x \in \mathrm{N}$. Then the composition $p_j := \pi_j \circ \tau_j$ of N_j in $\mathcal{P}_j$ is an isomorphism of algebras. For $x, y \in \mathrm{N}_j$ we have that $p_j(xy) = (xy)'_j = \{(x'_1 + \cdots + x'_n)(y'_1 + \cdots + y'_n)\}'_j = \{x'_1y'_1 + \cdots + x'_ny'_n\}'_j = x'_jy'_j = p_j(x)p_j(y)$ and hence p_j is a homomorphism of algebras. By to prove that p_j is injective, we consider $x \in \ker(p_j)$. Then $0 = p_j(x) = x'_j$ and so, $x \in \mathrm{N} \cap (\mathrm{N}_1 \oplus \mathrm{N}_2 \oplus \cdots \oplus \mathrm{N}_{j-1} \oplus \mathcal{P}_{j+1} \oplus \cdots \oplus \mathcal{P}_n) = \{0\}$. Therefore $x = 0$. Next, by to prove that p_j is onto, we take $y \in \mathcal{P}_j$. Then, $y = y'_j = \pi_j(y) = \pi_j(y_1 + \cdots + y_n) = \pi_j(y_1) + \cdots + \pi_j(y_n) = 0 + \cdots 0 + \pi_j(y_j) + 0 + \cdots + 0 = \pi_j(y_j)$ where, according to above decomposition, $y_j \in \mathrm{N}_j$.

Finally, we will define a dibaric isomorphism between the algebras $\mathcal{A}_j$ and B_j. According to Lemma 4.4 we can assume, without lost of generality, that $(\mathcal{A}_j, \gamma_j)$ and (B_j, χ_j) have the same semiprincipal idempotent element denoted by $e = u + v$. Then, we define the application $f_j : \mathcal{A}_j \longrightarrow \mathrm{B}_j$, by

$$f_j(\alpha u + \beta v + x) = \alpha u + \beta v + p_j(x)$$

where $\alpha, \beta \in F$ e $x \in \mathrm{N}_j$. It is clear that f_j is a linear isomorphism and also that $\gamma_j = \chi_j \circ f_j$. Therefore, only rest to show that f_j is a homomorphism of algebras. Notice that if $w \in \langle u, v\rangle$ and $x \in \mathrm{N}_j$, then

$$p_j(wx) = wp_j(x), \qquad p_j(xw) = p_j(x)w,$$

because $p_j(wx) = \{w(x'_1 + \cdots + x'_n)\}'_j = \{wx'_1 + \cdots + wx'_n\}'_j = wx'_j = wp_j(x)$. Analogously, we have the other equality. So, if $a = \alpha u+\beta v+x$, $b = \eta u + \mu v + y$ are in $\mathcal{A}_j$, then

$$\begin{aligned} f_j(ab) &= f_j\left(\tfrac{1}{2}(\alpha\mu + \beta\eta)(u + v) + \alpha uy + \beta vy + \eta xu + \mu xv + xy\right) \\ &= \tfrac{1}{2}(\alpha\mu + \beta\eta)(u + v) + \alpha p_j(uy) + \beta p_j(vy) + \eta p_j(xu) + \\ &\qquad \mu p_j(xv) + p_j(xy) \\ &= \tfrac{1}{2}(\alpha\mu + \beta\eta)(u + v) + \alpha u p_j(y) + \beta v p_j(y) + \eta p_j(x)u + \\ &\qquad \mu p_j(x)v + p_j(x)\, p_j(y) \\ &= ((\alpha u + \beta v + p_j(x))\,(\eta u + \mu v + p_j(y)) = f_j(a)f_j(b). \quad \blacksquare \end{aligned}$$

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