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ON STRUCTURE AND COMMUTATIVITY OF NEAR - RINGS

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Abstract

The aim of this paper is to generalize the results in [1] *and* [13]. *Here, we are interested in two problems concerning certain classes of near rings satisfying one of the following polynomials identities :*

$(*)$ *For each* x, y *in a near-ring* N, *there exist positive integers* $t = (x, y) \geq 1$ *and* $s = s\,(x, y) > 1$ *such that* $xy = \pm y^s x^t$.

$(**)$ *For each* x, y *in a near-ring* N, *there exist positive integers* $t = t\,(x, y) \geq 1$ *and* $s = s\,(x, y) > 1$ *such that* $xy = \pm x^t y^s$.

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The first problem is to prove the decomposition for near-rings satisfying either of the properties $(*)$ or $(**)$ and the second problem is to prove the commutativity of distributively generated near - ring satisfying either $(*)$ or $(**)$. As an application we show that if N is strongly distributively generated near - ring satisfying either $(*)$ or $(**)$, then R is commutative. This generalizes a result by A. Frolich which asserts that a distributively generated near - ring N is distributive if and only if N^2 is distributive or if $\langle N, +\rangle$ is commutative.

A celebrated theorem of I. Herstein [8] asserts that a periodic ring is commutative if its nilpotent elements are central. In order to get the analog of this result in near - rings H. Bell [2] proved that if N is distributively generated (d.g.) near - ring with its nilpotent elements laying in the center, then the set A of all nilpotent elements of N forms an ideal of N and if N/A is periodic, then N must be commutative. Recently, M. Quadri, Ashraf and A. Ali [13] proved that a d. g. near - ring N satisfying any one of the following conditions :

(i) For each $x, y \in N$, there exist positive integers $m = m(x, y), n = n(x, y)$ at least one of them greater than one such that $xy = y^m x^n$,

or

(ii) For each $x, y \in N$, there exist positive integers $m = m(x, y), n = n(x, y)$ at least one of them greater than one such that $xy = x^n y^m$, then R is commutative.

The main purpose of this paper is to generalize the above result. In view of this observation, we want to study the structure and commutativity of near - ring N satisfying one of following conditions :

$(*)$ For each x, y in a near - ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm y^s x^t$.

$(**)$ For each x, y in a near - ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s(x, y) > 1$ such that $xy = \pm x^t y^s$.

The present paper is organized as follows : In the first section, we put together some elementary material for the sake of completeness. In particular, the connection of the direct sum decomposition of rings and analogous of near - rings. In section 2, we establish a rather general theorem which asserts that if a near - ring N satisfying either $(*)$ or $(**)$, then $N = A \oplus B$, that is N is an orthogonal sum of subnear - rings A and B, where A is the set of the nilpotent elements of N and $B = \left\{x \in N \mid x^{n(x)} = x, n(x) \in \mathbf{Z}\right\}$ with $(B, +)$ is abelian.

Section 3 is devoted to the problem that if N satisfies $(*)$ or $(**)$, then under appropriate additional hypothesis a distributively generated near - ring must be a commutative ring. As an application we prove that if strongly distributively generated near ring N satisfying either $(*)$ or $(**)$, then N must be a commutative ring.

1. Preliminaries

All near - rings in this paper are left near - rings. The multiplicative center of a near - ring N will be denoted by $Z(N)$, the set of nilpotent elements of N denoted by A and the set of idempotent elements of N is denoted by B. An element x of a near - ring N is called distributive if $(a+b)\,x = ax + bx$ and anti-distributive if $(a+b)\,x = bx + ax$ for all $a, b \in N$. If all the elements of a near - ring N are distributive, then N is said to be distributive near - ring. A near - ring N is called distributively generated (d. g.) if it contains a multiplicative subsemi - group of distributive elements which generates additive group N^+. A near - ring N will be called strongly distributively generated (s.d.g.) if if contains a set of distributive elements whose square generate N^+ (see [12]).

A near - ring N is called zero - symmetric if $0x = 0$ for all $x \in N$, that is left distributive gives $x0 = 0$. A near ring N is called zero - commutative if $xy = 0$ implies that $yx = 0$ for $x, y \in N$.

An ideal of a near - ring N is defined to be a normal subgroup I of N^+ such that

(i) $NI \subseteq I$ and

(ii) $(x+i)\,y - xy \in I$ for all $x, y \in N$ and $i \in I$. If N is a d. g. near - ring, then (ii) may be replaced by $(ii)'$ $IN \subseteq I$.

A near - ring N is called periodic if for each $x \in N$, there exist distinct positive integers m and n such that $x^m = x^n$.

A near - ring N is an orthogonal sum of subnear - ring X and Y denoted by $N = X \oplus Y$, if $XY = YX = (0)$ and each element of N has a unique representation if the form $x + y$ such that $x \in X$ and $y \in Y$.

2. Decomposition Theorems for near rings

In 1989, Ligh and Luh proved that a ring R satisfying the property $xy = (xy)^{n(x,y)}\,; x, y \in R$, for a positive integer $n(x, y) > 1$ is a direct sum of a J-ring and zero ring (A J-ring is a ring satisfying Jacobson property $x = x^{n(x)},\ x \in R$, for a positive integer $n(x) > 1$). Recently, Bell and ligh [4] established that the direct sum decomposition for rings satisfying the properties $xy = (xy)^2\, p(xy)$ and $xy = (yx)^2\, p(yx)$, where $p(X) \in \mathbf{Z}[X]$, the polynomial ring over $\mathbf{Z}$. Furthermore, in [4] they remarked that in case of near rings the analogous hypothesis do not quite yield such decomposition. Further, they introduced a weaker notion of orthogonal sum and obtained orthogonal sum decomposition of a near - ring N satisfying the property $xy = (yx)^{n(x,y)},\ x, y \in N$, for a positive integer $n(x, y) > 1$.

Motivated by this observation, we shall prove the following results in this section.

Theorem 2.1. : Suppose that N is a near - ring satisfying $(*)$ and the idemoptent elements of N are multiplicative central. Then the set A of all nilpotent elements of N is a subnear - ring with trivial multiplication, and the set B of all idemoptent elements of N is a subnear - ring with $(B, +)$ is abelian. Furthermore, $N = A \oplus B$.

Remark : If a near - ring N satisfies $(**)$, then we may not even get orthogonal sum decomposition of N which is evident from the following :

Example : Let $N = \{0, x, y, z\}$ with addition and multiplication tables, defined as follows :

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\cdot$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

One can easily see that N is a near - ring satisfying the property $(**)$. But the set $B = \{0, x, z\}$ is not a subnear - ring of N.

It is natural to ask a question : What additional conditions are needed to force the orthogonal sum decomposition of a near - ring N satisfies $(**)$? In this direction, we prove.

Theorem 2.2. **:** Let N be a zero - commutative near - ring satisfying $(**)$ and the idempotent elements of N are multiplicative central. Then the set A of all nilpotent elements of N is a subnear-ring with trivial multiplication, and the set B of all idempotent elements of N is a subnear - ring with $(B, +)$ is abelian. Furthermore, $N = A \oplus B$.

Now, we state the following results (see [1], [4] and [8]).

Lemma 2.1. **:** Let N be a near ring with idempotent elements are multiplicative central, and let e and f be any idempotent element of N. Then there exists an idempotent element g such that $ge = e$ and $gf = f$.

Lemma 2.2. **:** Let N be a zero-symmetric near ring. Then the set A of all nilpotent elements in N is an ideal if and only if A is a subgroup of the additive group N^+ of N.

Lemma 2.3. **:** Let N be a near ring having zero commutative. Then the annihilator of any non-empty subset of N is an ideal.

Lemma 2.4. **:** Let N be a zero-symmetric near ring satisfying the following conditions :

(a) For each x in N, there exists an integer $n = n\,(x) > 1$ such that $x^n = x$.

(b) Every non-trivial homomorphic image of N contains a non-zero central idempotent.

Then $(N, +)$ is abelian.

Proof of Theorem 2.1. : We break the proof in the following steps.

Step 1. If N satisfies $(*)$, then it is easy to check that N is zero-symmetric as well as zero-commutative.

Step2. The set A of all nilpotent elements in N form an ideal. To see this, we let $a \in A$ and $x \in N$. The there exist integers $s_1 = s(x,a) \geq 1$, $t_1 = t(x,a) > 1$ such that $ax = \pm x^{s_1} a^{t_1}$. Now, choose $s_2 = s_1(x^{s_1}, a^{t_1}) \geq 1$ and $t_2 = t_1(x^{s_1}, a^{t_1}) > 1$ such that $x^{s_1} a^{t_1} = \pm a^{t_1 t_2} x^{s_1 s_2}, \pm x^{s_1} a^{t_1} = (\pm)^2 a^{t_1 t_2} x^{s_1 s_2}$, and $ax = (\pm)^2 a^{t_1 t_2} x^{s_1 s_2}$. Hence, it is clear that for arbitrary k, we have integers $s_1 = s(x,a) \geq 1$, $s_2 = s_1(x^{s_1}, a^{t_1}) \geq 1, ..., s_k = s_{k-1}(x^{s_{k-1}}, a^{t_{k-1}}) \geq 1$, $t_1 = t(x,a) > 1$, $t_2 = t_1(x^{s_1}, a^{t_1}) > 1, ..., t_k = t_{k-1}(x^{s_{k-1}}, a^{t_{k-1}}) > 1$ such that

$$ax = (\pm)^k a^{t_1 t_2 \ldots t_k} x^{s_1 s_2 \ldots s_k}.$$

Thus $a \in A$, $a^{t_1 t_2 \ldots t_k} = 0$ for sufficiently large k. Hence $ax = 0$ for all $x \in N$ and $a \in A$. By step 1, N is zero-commutative, and hence the nilpotent elements of N annihilate N on both sides. Thus

$$AN = NA = \{0\}.$$

Hence $A^2 = \{0\}$ and $A \subseteq Z(N)$. Let $a, b \in A$ such that $a^p = 0$ and $a^q = 0$ where p and q are positive integers. Then $(a-b)^{p+q} = 0$, that is $a - b \in A$. Hence A is an additive subgroup of N^+. Therefore A is an ideal of N by Lemma 2.2.

Step 3. Let N satsfying $(*)$ and let $n \in N$. Suppose that $s' \geq 1$, $t' > 1$ be integers such that $n^2 = \pm n^{s'+t'}$. Then $n\left(n \mp n^{s'+t'-1}\right) = 0$. By step 1, N is zero-commutative. Hence $\left(n \mp n^{s'+t'-1}\right) n = 0$ and $\left(n \mp n^{s'+t'-1}\right) n^{s'+t'-1} = 0$. This implies that $\left(n \mp n^{s'+t'-1}\right)^2 = 0$ and $n \mp n^{s'+t'-1} \in A$, that is $n - n^{s'+t'-1} \in A$ and $n + n^{s'+t'-1} \in A$. So we can write $n = n - n^{s'+t'-1} + n^{s'+t'-1}$. Now,

$$\begin{aligned} \left(n^{s'+t'-1}\right)^{s'+t'-1} &= n^{(s'+t'-1)(s'+t'-1)} \\ &= n^{(s'+t'-2)(s'+t')+1} \\ &= \left(n^{s'+t'}\right)^{s'+t'-2} \cdot n \\ &= \left(n^2\right)^{s'+t'-2} \cdot n \\ &= \left(n^{s'+t'-2}\right)^2 \cdot n. \end{aligned}$$

Since $n^{s'+t'-2}$ is idempotent by $(*)$, $\left(n^{s'+t'-1}\right)^{s'+t'-1} = n^{s'+t'-1} = n^{s'+t'-1}$ for $s'+t'-2 > 1$ and $n^{s'+t'-1} \in B$. This shows that $N = A+B$.

Step 4. Let N satisfying $(*)$ and let B be a subnear-ring with $(B,+)$ abelian. If $x, y \in B$, then there exist integers $k = k(x) > 1$ and $l = l(y)$ such that $x^k = x$ and $y^l = y$. Let $p = (k-1)\,l-(k+2) = (l-1)\,k-(1-2) > 1$. Then $x^p = x$ and $y^p = y$. Note that $e_1 = x^{p-1}$ and $e_2 = y^{p-1}$ are idempotent elements in N with $e_1 x = x$ and $e_2 y = y$. Thus $xy = \pm (e_2 y)^q (e_1 x)^r$ for some integers $q = q(xy, e_1 e_2) \geq 1$ and $r = r(xy, e_1 e_2) > 1$. But, we have

$$xy = e_1 x e_2 y = e_1 e_2 xy = xy e_1 e_2 = \pm (e_1 e_2)^q (xy)^r .$$

This implies that $xy = e_1 e_2 (xy)^r$. Hence $xy = (xy)^r$ and so xy is idempotent that is $xy \in B$. Moreover, since N/A has $x^k = x$ property, we have an integer $j > 1$ such that

$$(x-y)^j = x - y + a,\ a \in A.$$

Since e_1, e_2 are central idempotent in N, in view of Lemma 2.1. choose an idempotent g for which $ge_1 = e_1$ and $ge_2 = e_2$. Hence $gx = x$ and $gy = y$. Multiplying (1) by g gives $(x-y)^j = x - y$. This shows that $x - y \in B$. Hence B is a subnear-ring. By step 1, N is zero-symmetric and by Lemma 2.4, we get $(B,+)$ is abelian.

Step 5. We want to see that each element in N has at most one representation of the form $a+b$, where $a \in A$, and $b \in B$. Moreover, $M = A \oplus B$. We have by step 2, N is an ideal. Let $a_1, a_2 \in A$, and $b_1, b_2 \in B$ such that $a_1 + b_1 = a_2 + b_2$. Then $a_1 - a_2 = b_2 - b_1 \in A \cap B = \{0\}$ which gives $a_1 = a_2$ and $b_1 = b_2$. Hence $N = A \oplus B$.

Proof of Theorem 2.2. Using the same techniques applied in the proof of Theorem 2.1, we can prove this result.

3. Certain near - rings are rings.

In [9], Ligh prove that distributively generated Boolean near - rings are rings and indicated that some of the more complicated polynomial identities implying commutativity in rings may turn out d. g. near - rings into rings. The aim of this section is to prove that under certain additional conditions such as d. g. near - rings turn out to be commutative ring. In this direction, we prove the following results.

Theorem 3.1. Let N be a d. g. near - ring satisfying $(*)$. Then N is commutative.

Theorem 3.2. Let N be a d. g. near - ring satisfying $(**)$. Then N is commutative.

Besides providing a simpler and attractive proof of a result due to Bell [2], our results generalize the theorems proved in [3] and [13].

We begin with the following known results .

Lemma 3.1. If N is a zero - commutative near - ring, then $ab = 0$, impels that $arb = 0$ for all $r \in N$ and $a, b \in N$.

Lemma 3.2. A d. g. near - ring N is always zero symmetric.

Lemma 3.3 [**7**]**.** A d. g. near - ring N is distributive if and only if N^2 is additively commutative.

Lemma 3.4 [**7**]**.** A d. g. near - ring N with unity 1 is a ring if N is distributive or if N^+ is commutative.

Lemma 3.5 [**7**]**.** If I is a two sided ideal in a d. g. near - ring, then the elements of the quotient group $N^+ - I$ form a d. g. near - ring which will representated by N/I.

Before, we prove our theorem, we first establish the following results.

Lemma 3.6. Let N be a d. g. near - ring satisfying $(*)$. Let A be the set of all nilpotent elements in N. Then $A \subseteq Z(N)$.

Proof. Let $a \in A$ and $x \in N$. Then using the same technique of step 2 in the proof of Theorem 2.1, we get that the nilpotent elements of N annihilate N on both sides and therefore, are central. Thus $A \subseteq Z(N)$.

Lemma 3.7. Let N be a d. g. near - ring satisfying $(**)$. Let A be the set of all nilpotent elements in N. Then $A \subseteq Z(N)$.

Proof. Let $A \in A$ and $x \in N$. Then there exist integers $s_1 = s(x,a) > 1$ and $t_1 = t(x,a) > 1$ such that $xa = \pm x^{s_1} a^{t_1} = (\pm)^2 x^{s_1 s_2} a^{t_1 t_2}$ and $xa = (\pm)^2 x^{s_1 s_2} a^{t_1 t_2}$. Hence, we find positive integers $s_1 > 1$, $s_2 > 1, ..., s_k > 1$ and $t_1 > 1$, $t_2 > 1, ..., t_k > 1$ satisfying

$$xa = (\pm)^k x^{s_1 s_2 \ldots s_k} a^{t_1 t_2 \ldots t_k}.$$

Since $a \in A$, $a^{t_1 t_2 \ldots t_k} = 0$ for sufficiently large k. Hence $xa = 0$ for all $x \in N$.

Using the same arguments we also find that $ax = 0$ for all $x \in N$. Hence nilpotent elements of N annihilate N on both sides. Hence $A \subseteq Z(N)$.

Now, we are in a position to prove our main theorems.

Proof of Theorem 3.1. By Lemma 2.1 and Lemma 2.2 together with Lemma 3.6, we get A is a two sided ideal which in turn together with the main theorem of Bell [2], we get the desired result.

Proof of Theorem 3.2. Using Lemma 3.7, and the argument in the proof of Theorem 3.1, we get our result.

4. Application.

The following results are corollaries of Theorem 3.1 and Theorem 3.2 and applications of Lemma 3.3 and Lemma 3.4.

Theorem 4.1. Let N be a d. g. near - ring satisfying $(*)$ or $(**)$. Further, if $N^2 = N$, then N is a commutative ring.

Proof. By Theorem 3.1, and Theorem 3.2, a d. g. near - ring satisfying $(*)$ or $(**)$ is commutative. For any $a, b, c \in N$, we have $(b + c)\, a = a\, (b + c)$. This shows that N is distributive and by application of Lemma 3.3, N^2 is additively commutative. Further, $N^2 = N$ yields that N is also additively commutative. Hence N is commutative.

Theorem 4.2. Let N be a d. g. near - ring with unity 1 satisfying $(*)$ or $(**)$. Then N is a commutative ring.

Proof. Application of Lemma 3.4 together with Theorem 3.1 and Theorem 3.2 and Theorem 3.2 gives the required result.

Theorem 4.3. Let N be strongly distributively generated near - ring satisfying either $(*)$ or $(**)$. Then N is a commutative ring.

Proof. In view of Theorem 3.1 and Theorem 3.2, a strongly distributively generated near - ring satisfying either $(*)$ or $(**)$ is commutative. Hence N is s. d. g. near - ring in which every element is distributive. By application of Lemma 3.3, N^2 is additively commutative.

Thus the additive group N^+ of the s.d.g. near - ring is also commutrative and hence N is a commutative ring.

References

1. H. E. Bell, *Certain near rings are rings.* J. London Math. Soc., 4, pp. 264 – 270, (1971).

2. H. E. Bell, *A commutativity theorem for near rings*, Canad. Math. Bull., 20, pp. 25 – 28, (1977).

3. H. E. Bell, *A commutativity for rings*, Canad. J. Math., 28, pp. 986 – 989, (1976).

4. H. E. Bell and S. Ligh, *Some decomposition theorems for periodic rings and near rings*, Math. J. Okayama Univ., 31, pp. 93 – 99, (1989).

5. J. R. Clay, *The near rings on groups of low order*, Math. Z., 104, pp. 364 – 371, (1968).

6. J. R. Clay, *Near rings, geneses and applications*, Oxford Univ. Press, U. K. (1992).

7. A. Frohlich, *Distributively generated near rings*, Proc. London Math. Soc., 8, pp. 76 – 94, (1958)

8. I. N. Herstein, *A note on rings with central nilpotent elements*, Proc. Amer. Math. Soc., 16, pp. 239 – 243, (1989).

9. S. Ligh, *On Boolean near - rings*, Bull. Austral. Math. Soc., 1, pp. 375 – 379, (1969).

10. S. Ligh and J. Luh, *Some commutativity theorems for rings and near - rings*, Acta Math. Acad. Sci. Hungar., 28, pp. 19 – 23, (1976).

11. S. Ligh and H. Luh, *Direct sum of J. rings and zero rings*, Amer. Math. Monthly, 96, pp. 40 – 41, (1989).

12. G. Pilz, *Near - rings*, (Second Edition), North - Holland, Mathematics Studies 23, Amsterdam, (1983).

13. M. A. Quadri, M. Ashraf and A. Ali, *Certain conditions under which near - rings are rings*, Bull. Austral. Math. Soc., 42, pp. 91 – 94, (1990).

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