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PODOLSKY'S ELECTRODYNAMICS UNDER A CHIRAL APPROACH

ELECTRODINÁMICA DE PODOLSKY BAJO UN ENFOQUE QUIRAL

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RESUMEN

En este trabajo se muestra que un nuevo esquema conduce a la electrodinámica de Maxwell y a la electrodinámica de Podolsky, partiendo con relaciones constitutivas quirales en lugar de la usual ley de Coulomb.

Palabras clave: Electrodinámica de Podolsky, ecuaciones de Maxwell.

ABSTRACT

In this paper we show that a new approach leads to Maxwell's and Podolsky's electrodynamics, provided we start from chiral constitutive relations instead of the usual Coulomb's law.

Keywords: Podolsky's electrodynamics, Maxwell's equations.

INTRODUCTION

"*On the Question of Obtaining the Magnetic Field, Magnetic Force, and the Maxwell Equations from Coulomb's Law and Special Relativity*", where it can be shown that any attempt to derive Maxwell equations from Coulomb's law of electrostatics and the laws of special relativity ends in failure unless one makes use of additional assumptions. Kobe [1] gave the answer: all one needs to arrive at Maxwell equations is

(i) Coulomb's law;
(ii) the principle of superposition;
(iii) the assumption that electric charge is a conserved scalar (which amounts to assuming the independence of the observed charge of a particle on its speed [2];
(iv) the requirement of form invariance of the electrostatic field equations under Lorentz transformations, i.e. the electrostatic field equations are thought as covariant space-space components of covariant field equations.

Neuenschwander and Turner [3] obtained Maxwell equations by generalizing the laws of magnetostatics, which follow from the Biot-Savart law and magnetostatics, to be consistent with special relativity.

The preceding considerations leads us to the interesting question: what would happen if we followed the same route as Kobe did, using an electrostatic force law other than the usual Coulomb's one? We shall show that if we start from the force law proposed by Podolsky [4], i.e.,

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{QQ^{'}}{4\pi\varepsilon_0}\left(\frac{1-e^{r/a}}{r^2} - \frac{e^{-r/a}}{ra}\right)\frac{\boldsymbol{r}}{r} \qquad (1)$$

where a is a positive parameter with dimension of length, Q and Q' are the charges at $\boldsymbol{r}$ and $\boldsymbol{r} = 0$, respectively, and $\boldsymbol{F}(\boldsymbol{r})$ is the force on the particle with charge Q due to the particle with charge Q' and if we follow the steps previously outlined, we arrive at the outstanding electrodynamics derived by Podolsky in the early 40 s. In other words, we shall show that the same route that leads to Maxwell equations leads also to Podolsky equations. A notable feature of Podolsky's generalized electrodynamics is that it is free of those infinities which are usually associated with a point source. For instance, (1) approaches a finite value $QQ'/8\pi\varepsilon_0 a^2$ as r approaches zero. Thus, unlike Coulomb's law, Podolsky's electrostatic force law is finite in the whole space.

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In Sec. II we derive the equations that make up Podolsky's electrodynamic under the chiral approach [5, 7, 11]. In Sec. III we arrive at Podolsky's field equations by generalizing the equations of Sec. II, so that they are form invariant under Lorentz transformations. For consistency, we show in Sec. IV that (1) is indeed the electrostatic force law related to Podolsky's theory. The conclusions are presented in Sec. V. Natural units $\hbar = c = 1$,are used throughout. As far as the electromagnetic theories are concerned, we will use the Heaviside-Lorentz units with $c = 1$.

CHIRAL FIELD EQUATIONS

To begin with let us establish some conventions and notations to be used from now on. We use the metric tensor

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

with Greek indices running over 0, 1, 2, 3. Roman indices *i, j* etc, - denote only the three spatial components. Repeated indices are summed in all cases. The space-time four vectors (contravariant vectors) are $x^{\mu} = (t, \vec{x})$, and the covariant vectors, as a consequences are $x_{\mu} = (t, -\vec{x})$. The four-velocities are found, according to

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = \gamma(1, \vec{v})$$
$$u_{\mu} = \gamma(1, -\vec{v})$$

where τ is the proper time ($d\tau^2 = dt^2 - dx^2$), and γ denotes $dt / d\tau = (1-v^2)^{-1/2}$. Let us then generalize (6) so that it satisfies the requirement of form invariance under Lorentz transformations. To do that, we write the mentioned equation in terms of the Levi-Civita density ε^{nml}, which equals +1 (–1) if n, m, l is an even(odd) permutation of 1, 2, 3, and vanish if two indices are equal. The curl equation becomes

$$\varepsilon^{jkl}\partial E_l = 0 \qquad (2)$$

It we define the quantities

$$F^{0i} = -F_{0i} = E^i = -E_i \qquad (3)$$

Equation (10) can be rewritten as

$$\varepsilon^{jkl}\partial_k F_{0l} = 0$$

We imagine now the curl law to be the space-space components of a manifestly covariant field equation (invariance under Lorentz transformations). As a result, we get

$$\varepsilon^{\mu\alpha\nu\alpha\beta}\partial_{\nu} F_{\alpha\beta} = 0 \qquad (4)$$

where $F_{\alpha\beta}$ is a completely antisymmetric tensor of rank four with $\varepsilon^{0123} = +1$.

Of course, this generalization introduces the components F_{00}, F_{01}, and F_{lk}, for which at this point we lack a physical interpretation. Note that the F_{0i} are not necessarily static anymore.

On the other hand, as is well-known, the charge density ρ is defined as the charge per unit of volume, which has as a consequence that the charge dq in an element of volume d^3x is $dq = \rho d^3x$. Since dq is a Lorentz scalar [3], ρ transforms as the time-component of a four-vector, namely, the time-component of the charge-current four-vector $u = (\rho, \vec{j})$. The electric charge, in turn, is conserved locally [3], which implies that it obeys a continuity equation

$$\partial_{\mu} j^{\mu} = 0 \qquad (5)$$

Assuming $e^{j\omega t}$ time dependence, Maxwell's time-harmonic equations [1] for isotropic, homogeneous, linear media are

$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B} \qquad \nabla \bullet \boldsymbol{B} = 0 \qquad (6)$$

$$\nabla \times \boldsymbol{H} = -j\omega \boldsymbol{D} + \boldsymbol{J} \qquad \nabla \bullet \boldsymbol{D} = \rho \qquad (7)$$

Chirality is introduced into the theory by defining the following constitutive relations to describe the isotropic chiral medium [5, 7]

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} + \varepsilon T \nabla \times \boldsymbol{E} \qquad (8)$$

$$\boldsymbol{B} = \mu \boldsymbol{H} + \mu T \nabla \times \boldsymbol{H} \qquad (9)$$

Where the chirality admittance T indicates the degree of chirality of the medium, and the ε y μ are permittivity and permeability of the chiral medium, respectively. In natural units $\varepsilon = 1$, $\mu = 1$ (the factor 1/4 is absorbed in the current value). Since $\boldsymbol{D}$ and $\boldsymbol{E}$ are polar vectors and $\boldsymbol{B}$ and $\boldsymbol{H}$ are axial vectors, it follows that ε and μ are true scalars and T is a pseudoscalar. This means that when the axes of a right-handed Cartesian coordinate system are reversed to form a left-handed Cartesian coordinate system, T changes in sign whereas ε and μ remain unchanged.

Since $\nabla \bullet \boldsymbol{B} = 0$ always, this conditions will hold identically if B is expressed as the curl of a vector potential A since the divergence of the curl of a vector is identically zero. Thus by rearranging equation (7) we have

$$(1 - k_o^2 T^2)\nabla \times \boldsymbol{B} = j\omega \boldsymbol{E} + (\boldsymbol{J} + T\nabla \times \boldsymbol{J}) \qquad (10)$$

In terms of (3) can now be rewritten as

$$(1 + T^2 \partial_t{}^2)\nabla \times \boldsymbol{B} = j\omega \boldsymbol{E} + (\boldsymbol{J} + T\nabla \times \boldsymbol{J})$$

or

$$(1 + T^2 \partial_t{}^2)(\nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t}) = \frac{\partial}{\partial t}(-T^2 \frac{\partial^2}{\partial t^2}\boldsymbol{E} + 2T\nabla \times \boldsymbol{E}) + (\boldsymbol{J} + T\nabla \times \boldsymbol{J}) \qquad (11)$$

In relativistic form we have

$$(1 + T^2 \partial_i \partial^i)\partial_j E^j = j^0 \qquad (12)$$

where $j^0 = j^0_{chiral} + j^0_{ordinary}$ is given by a chiral current plus a ordinary current of electrons and protons, the chiral current is given in ref [1], $\partial_i = \partial / \partial x^i$ and $\partial^i = \partial / \partial x_i$. Note that $\partial_i = -\partial^i$ Using (11), yields $(1 + T^2 \partial_i \partial^i)\partial_j F^{0j} = j^0$

In order that the left-hand side of the preceding equation transforms as the time-component of a four-vector, we must write it as

$$(1 + T^2 \Box)\partial_j F^{0j} = j^0$$

where

$$\Box = \partial_i \partial^i = \eta^{\mu\nu}\partial_\mu \partial_\nu = \partial^2 / \partial t^2 - \nabla^2 \qquad (13)$$

The requirement of form invariance of this equation under Lorentz transformations leads then to the following result

$$(1 + T^2 \Box)\partial_\nu F^{\mu\nu} = j^\mu \qquad (14)$$

Now if $j^0 = j^0_{chiral} + j^0_{ordinary} \approx j^0_{ordinary}$, we can imagine now a particle of mass m and charge Q at rest in a lab frame where there is an electrostatic field $\boldsymbol{E}$. Newton's second law allows us to write

$$\frac{d\boldsymbol{p}}{dt} = Q\boldsymbol{E} \qquad (15)$$

In terms of the proper time this becomes

$$\frac{d\boldsymbol{p}}{d\tau} = Q\gamma \boldsymbol{E} = Qu^0 \boldsymbol{E}$$

where u^0 is the time part of the velocity four-vector u^μ. For the component along de x^i direction, we have

$$\frac{dp^i}{d\tau} = Qu^0 F^{0i}$$

In order that the right-hand side of this equation transforms like a space-component of a four-vector, it must be rewritten as

$$\frac{dp^i}{d\tau} = Qu_\nu F^{\nu i}$$

whose covariant generalization is

$$\frac{dp^\mu}{d\tau} = Qu_\nu F^{\nu\mu} \qquad (16)$$

If (16) is multiplied by $p_\mu = mu\mu$, where m is the rest mass, the result is

$$\frac{1}{2}\frac{d}{d\tau}(p_\mu p^\mu) = Qmu_\mu u^\mu F^{\nu\mu}$$

However,

$$p_\mu p^\mu = m^2 \gamma^2 (1 - \vec{v}^{\,2}) = m^2 \gamma^2 \gamma^{-2} = m^2$$

Therefore, we come to the conclusion that

$$u_\mu u^\mu F^{\nu\mu} = 0$$

Using this result Kobe [1] and Neuenschwander and Turner [3] showed that $F^{\nu\mu}$ is an antisymmetric tensor

($F^{\nu\mu} = -F^{\mu\nu}$). Since $F^{\nu\mu}$ is an antisymmetric tensor of second rank, it has only six independent components, three of which have already been specified. We name therefore the remaining components

$$B^i = \frac{1}{2}\varepsilon^{ilm} F_{lm} \tag{17}$$

Note that $F^{kl} = -\square^{klj} B_j$. Writing out the components of (17) explicitly,

$$B^1 = F_{23} = F^{23} = -B_1$$
$$B^2 = -F_{13} = -F^{13} = -B_2$$
$$B^3 = F_{12} = F^{12} = -B_3$$

Hence, a clever physicist who were only familiar with Podolsky's electrostatics and special relativity could predict the existence of the magnetic field $\vec{B}$, which naturally still lacks physical interpretation.

The content of (12) and (14) can now be seen. For $\mu = 0$, (12) gives

$$\nabla \cdot \boldsymbol{B} = 0 \tag{18}$$

showing that there are no magnetic monopoles in Podolsky's electrodynamics, while for $\mu = i$ we obtain

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{19}$$

which says that time-varying magnetic fields can be produced be $\boldsymbol{B}$ fields with circulation.

The components $\mu = 0$ and $\mu = i$ of (14) give, respectively,

$$(1 + T^2\square)\nabla \cdot \boldsymbol{E} = \rho \tag{20}$$

$$(1 + T^2\square)(\nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{dt}) = \boldsymbol{j} \tag{21}$$

which are nothing but a generalization of Gauss and Ampère- Maxwell laws in this order.

For $\nu = i$, (16) becomes

$$\frac{d\boldsymbol{p}}{dt} = Q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{22}$$

containing the Lorentz force. For $\nu = 0$, (15) assumes the form

$$\frac{dU}{dt} = Q\boldsymbol{v} \cdot \boldsymbol{E} \tag{23}$$

where $U = p^0$ is the particle's energy. Accordingly, our smart physicist, who was able to predict the $\boldsymbol{B}$ field only from its knowledge of electrostatics and special relativity, can now-by making judicious use of (22) and (23) - observe, measure and distinguish the $\boldsymbol{B}$ field from the $\boldsymbol{E}$ field of (15). The new field couples to moving electric charge, does not act on a static charged particle, and, unlike the electrostatic field, is capable only of changing the particle's momentum direction.

Equations (18-21) make up Podolsky's higher-order field equations. Of course, in the limit $T = 0$, all the preceding arguments apply equally well to Maxwell's theory.

Two comments fit in here:

(1) Equation (14) is consistent with the continuity equation (13). In fact, if the divergence of (14) is taken, we obtain

$$(1 + T^2\square)\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\mu j^\mu$$

Since $F^{\mu\nu}$ is an antisymmetric tensor $\partial_\mu \partial_\nu F^{\mu\nu}$ is identically zero. On the other hand, according to (14) $\partial_\mu j^\mu$. Thus, the equation in hand is identically zero;

(2) As was recently shown [8], it is not necessary to introduce a formula for the force density f^μ representing the action of the field on a text particle. We have only to assume that ($-f^\mu$) is the simplest contravariant vector constructed with the current j^μ and a suitable derivative of the field $F^{\mu\nu}$. Applying this simplicity criterion to Podolsky's electrodynamics, we promptly obtain

$$f^\mu = -F^{\mu\nu} j_\nu$$

where, as we have already mentioned, $j^\mu = (\rho, \vec{j})$. Therefore,

$$f^0 = -F^{0i} j_i = \boldsymbol{E} \cdot \boldsymbol{j}$$

and

$$f^k = -F^{k\beta} j_\beta = F^{0k} j_0 + F^{ik} j_i = (\rho \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B})^k$$

Thus, the force density for Podolsky's electrodynamics is the same as that for Maxwell's electrodynamics, namely, the well-known Lorentz force density.

THE FORCE LAW FOR PODOLSKY'S ELECTROSTATICS

We show now that (1) is indeed the force law for Podolsky's electrostatics. It follows

That

$$\boldsymbol{F} = Q\boldsymbol{E} \qquad \boldsymbol{E} = -\nabla V \tag{24}$$

where

$$V(\vec{r}) = \int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')(1-e^{-R/T})}{4\pi R}$$

Eq. (20) can then be rewritten as

$$(1 - T^2\nabla^2)\nabla^2 V(\vec{r}) = -\rho(\vec{r})$$

For a charge Q at the origin of the radius vector this equation reduces to

$$(1 - T^2\nabla^2)\nabla^2 V(\vec{r}) = -Q\delta^3(\vec{r}) \tag{25}$$

We solve this equation using the Fourier transform method. First we define $V(\vec{k})$ as follows:

$$V(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k} e^{-i\vec{k}\cdot\vec{r}}\, \tilde{V}(\vec{k}) \tag{26}$$

$$\tilde{V}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{r} e^{i\vec{k}\cdot\vec{r}} V(\vec{r}) \tag{27}$$

where $d^3\vec{k}$ and $d^3\vec{r}$, respectively, stands for volumes in the three-dimensional k-space and the coordinate space. If we substitute (26) into (25) and take into account that

$$\delta^3(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k} e^{-i\vec{k}\cdot\vec{r}}$$

we obtain

$$\tilde{V}(\vec{k}) = \frac{Q}{(2\pi)^{3/2} T^2 k^2 (k^2 + \frac{1}{T^2})}$$

So,

$$V(\vec{r}) = \frac{Q}{(2\pi)^3 T^2} \int d^3\vec{k} \frac{e^{-i\vec{k}\cdot\vec{r}}}{k^2(k^2 + \frac{1}{T^2})}$$

Integral (28) may be found in any textbook on the theory of functions of a complex variable [8]. As a result,

$$V(\vec{r}) = \frac{Q}{(2\pi)^2 r}\left(\frac{1 - e^{-r/T}}{r}\right)$$

Accordingly, the electric field due to a charge Q at the origin is given by

$$\boldsymbol{E} = -\nabla V(\vec{r}) = Q/4\pi\left(\frac{1 - e^{-r/T}}{r^2} - \frac{e^{-r/T}}{rT}\right)\frac{\boldsymbol{r}}{r} \tag{28}$$

It follows then the force law for Podolsky's electrostatics is

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{QQ'}{4\pi\varepsilon_0}\left(\frac{1 - e^{r/a}}{r^2} - \frac{e^{-r/a}}{ra}\right)\frac{\boldsymbol{r}}{r}$$

which is nothing but the force law for which we were looking (see Eq. (1)).

Recently an algorithm was devised which allows one to obtain the energy and momentum related to a given field in a simple way [8]. Using this prescription we can show that in the framework of Podolsky's electrostatics the energy is given by

$$\varepsilon_{field} = \frac{1}{2}\int d^3\vec{x}\left[\boldsymbol{E}^2 + T^2(\nabla\cdot\boldsymbol{E})^2\right]$$

Making use of the expression for the electrostatic field we have just found, we promptly obtain

$$\varepsilon_{field} = \frac{Q^2}{2T}$$

which tells us that the energy for the field of a point charge has a finite value in the whole space. This is indeed a important feature of Podolsky's generalized electrodynamics.

Equations (28) and $\nabla \times \boldsymbol{E} = 0$ are the fundamental laws of Podolsky's electrostatics. We will slightly analyze an interesting feature of Podolsky's electrostatics by computing the flux of the electrostatic field across a spherical surface of radius R with a charge Q at its center. Using (28) we arrive at the result

$$\oint \boldsymbol{E} \cdot d\boldsymbol{S} = Q(1-(1+R/T)e^{R/T})$$

which tells us that

$$\oint \boldsymbol{E} \cdot d\boldsymbol{S} = 0, \quad R \ll T$$
$$\oint \boldsymbol{E} \cdot d\boldsymbol{S} = Q, \quad R \gg T$$

Therefore, a sphere of radius $R \ll T$, unlike what happens in Maxwell's theory, shields its exterior from the field due to a charge placed at its center. We remark that in Maxwell's electrostatics a closed hollow conductor shields its interior from fields due to charges outside, but does not shield its interior from the field due to charges placed inside it [6]. Note, however, that in order not to conflict with well established results of quantum electrodynamics, the parameter a must be small. Incidentally, it was shown recently that this parameter is of the order of magnitude of the Compton wavelength of the neutral vector boson z, $\lambda \approx 2.15 \times 10^{16}$ *cm*, which mediates the unified and electromagnetic interactions [7].

ABOUT ELECTRON SIZE

In actuality, we don't know how big the electron is. All of our measurements point to the electron having no size, but we haven't measured down far enough. The electron, if it were a black hole, would have to be smaller than $1\text{x}10^{-57}$ meters, quite a bit smaller than we've ever measured!

But, another reason that the electron is not considered a black hole, even assuming that its radius is infinitely small, is that it obeys the laws of quantum field theory. Normally, when one speaks about black holes, one is talking about them in terms of Einstein's theory of general relativity. No one is sure how nature merges Einstein's theory with quantum field theory. So we aren't really sure if the idea of a black hole makes sense on distance scales as small as the (possible) radius of the electron.. Our best idea to unify general relativity with quantum field theory is an idea called string theory, but string theory still appears to be a long way from being put to any experimental tests.

According to general relativity all massive objects possess an event horizon known as the Schwarzschild radius. This is a surface in three-dimensional space surrounding the object. Any light rays emitted from within this radius are unable to escape. If an object exists *entirely* within its Schwarzschild radius then it is referred to as a black hole. This radius grows with the mass of the object according to the formula:

$$R_s = 2mG_N / c^2 = T$$

Notice that for our sun we obtain a radius of 2.95 kilometers. This is well within the interior of the sun so it is not a black hole. For an electron we would obtain $1.35\ \text{x}10^{-51}$ m. If the electron were a point particle, it seems it would be within even this fantastically small radius and would indeed be a black hole!

However, as a subatomic particle the electron is also a quantum-mechanical object. Recall the wave-particle duality hypothesis of de Broglie. All objects have a wave function which represents the probability of locating that object at a particular point in space. During a collision this wave function momentarily collapses and the particle is truly at one 'point' in space, but it immediately starts to spread out again after the instant of collision. The typical spread of the wave-function of a point particle is given by the Compton wavelength: $\lambda = h / mc = 2\pi\hbar / mc = 4\pi T$, in according with our chiral theory and this can be considered the true quantum-mechanical "size" of the object [7]. Notice that this size gets *smaller* as the mass gets larger. For you or I or the sun this quantum-mechanical size is essentially zero (there's not much uncertainty as to where the sun is!) but for an electron the size is $2.42\ \text{x}10^{-12}$ m. Though still small, this is much, much larger than the Schwarzschild radius. So quantum-mechanically most of the electron is 'outside' its event horizon. That's why it and other subatomic particles are not black.

FINAL REMARKS

Despite the simplicity of its fundamental assumptions, Podolsky's model has been little noticed. Currently some of its aspects have been further studied in the literature [7, 8, 12, 13]. In particular, the classical self-force acting on a point charge in Podolsky's model was evaluated and it was shown that in this model, unlike what happens in Maxwell's electrodynamics, the electromagnetic mass is finite and enters the particle's equation of motion in a form consistent with special relativity.

To conclude we call attention to the fact the same assumptions that lead to Maxwell's equations lead also to Podolsky's equations and our chiral equations, provided we start from a generalization of the Coulomb's law instead of the usual Coulomb's law. Yet, in spite of the great similarity between the three theories, Podolsky's generalized electrodynamics and chiral electrodynamics lead to results that are free of those infinities which are usually associated with a point source.

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