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Some integral inequalities related to Wirtinger's result for p-norms

S. S. Dragomir[1,2]

[1] *Mathematics, College of Engineering & Science, Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia.*
`sever.dragomir@vu.edu.au`

[2] *DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.*

ABSTRACT

In this paper we establish several natural consequences of some Wirtinger type integral inequalities for p-norms. Applications related to the trapezoid unweighted inequalities, of Grüss' type inequalities and reverses of Jensen's inequality are also provided.

RESUMEN

En este artículo establecemos varias consecuencias naturales de algunas desigualdades integrales de tipo Wirtinger para p-normas. También se entregan aplicaciones relacionadas a desigualdades trapezoidales sin peso, desigualdades de tipo Grüss y reversos de la desigualdad de Jensen.

Keywords and Phrases: Wirtinger's inequality, trapezoid inequality, Grüss' inequality, Jensen's inequality.

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1 Introduction

The following Wirtinger type inequalities are well known

$$\int_a^b u^2(t)dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

provided $u \in C^1([a,b],\mathbb{R})$ and $u(a) = u(b) = 0$ with equality holding if and only if $u(t) = K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant K, and, similarly, if $u \in C^1([a,b],\mathbb{R})$ satisfies $u(a) = 0$, then

$$\int_a^b u^2(t)dt \leq \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt. \tag{1.2}$$

The equality holds in (1.2) if and only if $u(t) = K\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant K.

For $p > 1$ put $\pi_p := \frac{2\pi}{p}\sin\left(\frac{\pi}{p}\right)$. In [11], J. Jaroš obtained, as a particular case of a more general inequality, the following generalization of (1.1)

$$\int_a^b |u(t)|^p\,dt \leq \frac{(b-a)^p}{(p-1)\pi_p^p}\int_a^b |u'(t)|^p\,dt \tag{1.3}$$

provided $u \in C^1([a,b],\mathbb{R})$ and $u(a) = u(b) = 0$, with equality if and only if $u(t) = K\sin_p\left[\frac{\pi_p(t-a)}{b-a}\right]$ for some $K \in \mathbb{R}$, where $\sin_p$ is the $2\pi_p$-periodic generalized sine function, see [18] or [5].

If $u(a) = 0$ and $u \in C^1([a,b],\mathbb{R})$, then

$$\int_a^b |u(t)|^p\,dt \leq \frac{[2(b-a)]^p}{(p-1)\pi_p^p}\int_a^b |u'(t)|^p\,dt \tag{1.4}$$

with equality iff $u(t) = K\sin_p\left[\frac{\pi_p(t-a)}{2(b-a)}\right]$ for some $K \in \mathbb{R}$.

The inequalities (1.3) and (1.4) were obtained for $a = 0$, $b = 1$ and $q = p > 1$ in [17] by the use of an elementary proof.

For some related Wirtinger type integral inequalities see [1, 2, 4, 8, 9, 11, 12] and [15]-[17].

These inequalities are used in various fields of Mathematical Analysis, Approximation Theory, Integral Operator Theory and Analytic Inequalities Theory since they provide connections between the Lebesgue norms of a function and the corresponding Lebesgue norms of the derivative under some natural assumptions at the endpoints.

Motivated by the above results, in this paper we establish some natural consequences of the Wirtinger type integral inequalities for p-norms (1.3) and (1.4). Applications related to the trapezoid unweighted inequalities, of Grüss' type inequalities and reverses of Jensen's inequality are also provided.

2 Some applications for trapezoid inequality

We have:

Proposition 2.1. *Let $g \in C^1([a,b],\mathbb{R})$. Then for $p>1$ we have the trapezoid inequality*

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{b-a}{2(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b |g'(t)-g'(a+b-t)|^p\,dt\right)^{1/p}. \quad (2.1)$$

In particular, for $p=2$, we have [7]

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{b-a}{2\pi}\left(\frac{1}{b-a}\int_a^b |g'(t)-g'(a+b-t)|^2\,dt\right)^{1/2}. \quad (2.2)$$

Proof. If $g \in C^1([a,b],\mathbb{R})$, then by taking

$$u(t) := \frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2},\ t\in[a,b]$$

we have $u(a)=u(b)=0$ and by (1.3) we have

$$\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p dt \leq \frac{(b-a)^p}{(p-1)\,2^p\pi_p^p}\int_a^b |g'(t)-g'(a+b-t)|^p\,dt, \quad (2.3)$$

namely

$$\left(\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p dt\right)^{1/p} \leq \frac{(b-a)}{2(p-1)^{1/p}\pi_p}\left(\int_a^b |g'(t)-g'(a+b-t)|^p\,dt\right)^{1/p}. \quad (2.4)$$

By Hölder's integral inequality we have for $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned}\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right| dt &\leq \left(\int_a^b dt\right)^{1/q}\left(\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p dt\right)^{1/p}\\ &= (b-a)^{1/q}\left(\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p dt\right)^{1/p}\\ &= (b-a)^{1-1/p}\left(\int_a^b \left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p dt\right)^{1/p}. \quad (2.5)\end{aligned}$$

By making use of the properties of modulus and integral, we also have

$$\int_a^b \left| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right| dt \geq \left| \int_a^b \left[\frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right] dt \right|$$
$$= \left| \int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2}(b-a) \right|. \tag{2.6}$$

By making use of (2.4)-(2.6) we get the desired result (2.1). □

Further, we have:

Proposition 2.2. *Let $g \in C^1([a,b],\mathbb{R})$. Then for $p>1$ we have the trapezoid inequality*

$$\left| \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt \right|$$
$$\leq \frac{b-a}{(p-1)^{1/p}\pi_p} \left(\frac{1}{b-a}\int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p dt \right)^{1/p}. \tag{2.7}$$

In particular, for $p=2$, we have [7]

$$\left| \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt \right| \leq \frac{b-a}{\pi} \left(\frac{1}{b-a}\int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^2 dt \right)^{1/2}. \tag{2.8}$$

Proof. If $g \in C^1([a,b],\mathbb{R})$, then by taking

$$u(t) := g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a},\ t \in [a,b]$$

we have $u(a)=u(b)=0$ and by (1.3) we have

$$\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p dt \leq \frac{(b-a)^p}{(p-1)\pi_p^p} \int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p dt, \tag{2.9}$$

which gives that

$$\left(\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p dt \right)^{1/p}$$
$$\leq \frac{b-a}{(p-1)^{1/p}\pi_p} \left(\int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p dt \right)^{1/p}. \tag{2.10}$$

By Hölder's integral inequality we have for $p,\ q>1$, $\frac{1}{p}+\frac{1}{q}=1$ that

$$\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right| dt$$
$$\leq \left(\int_a^b dt \right)^{1/q} \left(\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p dt \right)^{1/p}$$
$$= (b-a)^{1/q} \left(\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p dt \right)^{1/p}. \tag{2.11}$$

By making use of the properties of modulus and integral, we also have

$$\begin{aligned}\int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right| dt &\geq \left| \int_a^b \left[g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right] dt \right| \\ &= \left| \int_a^b g(t)\, dt - \frac{g(a) + g(b)}{2}(b-a) \right|. \end{aligned} \tag{2.12}$$

By making use of (2.10)-(2.12) we get the desired result (2.7). □

We also have:

Proposition 2.3. *Let $g \in C([a,b], \mathbb{R})$. Then for $p > 1$ we have the inequality*

$$\begin{aligned}&\left| \frac{b+a}{2} \int_a^b g(s)\, ds - \int_a^b t g(t)\, dt \right| \\ &\qquad \leq \frac{(b-a)^2}{(p-1)^{1/p} \pi_p} \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^p dt \right)^{1/p}. \end{aligned} \tag{2.13}$$

In particular, for $p = 2$, we have [7]

$$\begin{aligned}&\left| \frac{b+a}{2} \int_a^b g(s)\, ds - \int_a^b t g(t)\, dt \right| \\ &\qquad \leq \frac{(b-a)^2}{\pi} \left[\frac{1}{b-a} \int_a^b g^2(t)\, dt - \left(\frac{1}{b-a} \int_a^b g(s)\, ds \right)^2 \right]^{1/2}. \end{aligned} \tag{2.14}$$

Proof. Assume that $g : [a,b] \to \mathbb{C}$ is continuous, then by taking

$$u(t) := \int_a^t g(s)\, ds - \frac{t-a}{b-a} \int_a^b g(s)\, ds, \ t \in [a,b]$$

we have $u(a) = u(b) = 0$, $u \in C^1([a,b], \mathbb{C})$ and by (1.3) we get

$$\int_a^b \left| \int_a^t g(s)\, ds - \frac{t-a}{b-a} \int_a^b g(s)\, ds \right|^p dt \leq \frac{(b-a)^p}{(p-1)\pi_p^p} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^p dt.$$

This is equivalent to

$$\begin{aligned}&\left(\int_a^b \left| \int_a^t g(s)\, ds - \frac{t-a}{b-a} \int_a^b g(s)\, ds \right|^p dt \right)^{1/p} \\ &\qquad \leq \frac{b-a}{(p-1)^{1/p} \pi_p} \left(\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^p dt \right)^{1/p}. \end{aligned} \tag{2.15}$$

By Hölder's integral inequality we also have for p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned}(b-a)^{1/q}&\left(\int_a^b\left|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^p dt\right)^{1/p}\\&\geq\int_a^b\left|\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)\right|dt\\&\geq\left|\int_a^b\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)dt\right|.\end{aligned}\tag{2.16}$$

Observe that, integrating by parts, we have

$$\begin{aligned}\int_a^b\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)dt&=\int_a^b\left(\int_a^t g(s)\,ds\right)dt-\frac{b-a}{2}\int_a^b g(s)\,ds\\&=b\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt-\frac{b-a}{2}\int_a^b g(s)\,ds\\&=\frac{b+a}{2}\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt.\end{aligned}\tag{2.17}$$

By making use of (2.15)-(2.17) we get the desired result (2.13). □

3 Inequalities for the Čebyšev functional

For two *Lebesgue integrable* functions f, $g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt.\tag{3.1}$$

In 1935, Grüss [10] showed that

$$|C(f,g)|\leq\frac{1}{4}(M-m)(N-n),\tag{3.2}$$

provided that there exist real numbers m, M, n, N such that

$$m\leq f(t)\leq M\quad\text{and}\quad n\leq g(t)\leq N\quad\text{for a. e. }t\in[a,b].\tag{3.3}$$

The constant $\frac{1}{4}$ is the best possible in (3.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [3], states that

$$|C(f,g)|\leq\frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty,\tag{3.4}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty=\sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (3.4) also holds if $f,\, g : [a, b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and $f',\, g' \in L_\infty [a, b]$ while $\|f'\|_\infty = ess\ sup_{t\in[a,b]} |f'(t)|$.

A mixture between Grüss' result (3.2) and Čebyšev's one (3.4) is the following inequality obtained by Ostrowski in 1970, [14]:

$$|C(f, g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{3.5}$$

provided that f is *Lebesgue integrable* and satisfies (3.3) while g is absolutely continuous and $g' \in L_\infty [a, b]$. The constant $\frac{1}{8}$ is the best possible in (3.5).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [13] in which he proved that

$$|C(f, g)| \le \frac{1}{\pi^2}(b-a)\|f'\|_2 \|g'\|_2, \tag{3.6}$$

provided that $f,\, g$ are absolutely continuous and $f',\, g' \in L_2 [a, b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

We have:

Theorem 3.1. *If $f : [a, b] \to \mathbb{R}$ is continuous, $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $g : [a, b] \to \mathbb{C}$ is absolutely continuous with $g' \in L_q [a, b]$, then*

$$|C(f, g)| \le \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p} \times \left(\int_a^b |g\prime(t)|^q\,dt\right)^{1/q}. \tag{3.7}$$

In particular, for $p = q = 2$, we get

$$|C(f, g)| \le \frac{(b-a)^{1/2}}{\pi}\left(\frac{1}{b-a}\int_a^b f^2(t)\,dt - \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2\right)^{1/2} \times \left(\int_a^b |g\prime(t)|^2\,dt\right)^{1/2}. \tag{3.8}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \left(\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(s)\,ds\right) g\prime(x)\,dx \\
&= \frac{1}{b-a}\left[\left(\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(s)\,ds\right) g(x)\Big|_a^b - \int_a^b g(x)\left(f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds\right)dx\right] \\
&= -\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{b-a}\int_a^b f(s)\,ds \frac{1}{b-a}\int_a^b g(x)\,dx,
\end{aligned}$$

which gives that

$$C(f,g)=\frac{1}{b-a}\int_a^b\left(\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right)g\prime(x)\,dx. \tag{3.9}$$

Using Hölder's integral inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\begin{aligned}|C(f,g)|&=\left|\frac{1}{b-a}\int_a^b\left(\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right)g\prime(x)\,dx\right|\\&\le\frac{1}{b-a}\int_a^b\left|\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right||g\prime(x)|\,dx\\&\le\frac{1}{b-a}\left(\int_a^b\left|\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right|^p dx\right)^{1/p}\left(\int_a^b|g\prime(x)|^q dx\right)^{1/q}=:I \end{aligned}\tag{3.10}$$

Using (2.15) we have

$$\begin{aligned}I&\le\frac{1}{b-a}\left(\int_a^b\left|\int_a^t f(s)\,ds-\frac{t-a}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^b|g\prime(x)|^q dx\right)^{1/q}\\&\le\frac{1}{b-a}\frac{b-a}{(p-1)^{1/p}\pi_p}\left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^b|g\prime(x)|^q dx\right)^{1/q}\\&=\frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^b|g\prime(x)|^q dx\right)^{1/q}\end{aligned}\tag{3.11}$$

for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, which proves (3.7).

Now, if we put $p=q=2$ and take into account that

$$\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2 dt=\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2,$$

then by (3.7) we derive (3.8). □

This results can be used to obtain various inequalities by taking particular examples of functions f and g as follows.

We have the following trapezoid type inequality:

Proposition 3.2. *Assume that $g:[a,b]\to\mathbb{C}$ has an absolutely continuous derivative with $g''\in L_q[a,b]$, where p, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right|\le\frac{(b-a)^{1+1/p}}{2(p-1)^{1/p}(p+1)^{1/p}\pi_p}\left(\int_a^b|g''(t)|^q dt\right)^{1/q}. \tag{3.12}$$

Proof. We use the following identity that can be proved integrating by parts

$$\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)g'(t)\,dt=C\left(\ell-\frac{a+b}{2},g'\right),$$

where $\ell(t) = t,\ t \in [a,b]$.

Using (3.7) we have

$$
\begin{aligned}
&\left| C\left(\ell - \frac{a+b}{2}, g'\right)\right| \\
&\leq \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b \left| t - \frac{a+b}{2} - \frac{1}{b-a}\int_a^b \left(s - \frac{a+b}{2}\right) ds\right|^p dt\right)^{1/p}\left(\int_a^b |g''(x)|^q\,dx\right)^{1/q} \\
&\qquad = \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b \left| t - \frac{a+b}{2}\right|^p dt\right)^{1/p}\left(\int_a^b |g''(x)|^q\,dx\right)^{1/q} \\
&\qquad = \frac{(b-a)^{1+1/p}}{2(p-1)^{1/p}(p+1)^{1/p}\pi_p}\left(\int_a^b |g''(x)|^q\,dx\right)^{1/q},
\end{aligned}
$$

which proves the desired inequality (3.12). □

Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M) and $f : [a,b] \to [m,M]$ be absolutely continuous so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L[a,b]$. If $f' \in L_\infty[a,b]$, then we have the Jensen's reverse inequality [6]

$$
\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\
&\leq \frac{1}{b-a}\int_a^b (\Phi' \circ f)(t) f(t)\,dt - \frac{1}{b-a}\int_a^b \Phi' \circ f(t)\,dt \frac{1}{b-a}\int_a^b f(t)\,dt = C(\Phi' \circ f, f). \quad (3.13)
\end{aligned}
$$

We have the following reverse of Jensen's inequality:

Proposition 3.3. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M) and $f : [a,b] \to [m,M]$ be absolutely continuous so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L[a,b]$.*

(i) If $f' \in L_q[a,b]$, $\Phi' \circ f \in L_p[a,b]$ with $p,\ q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\
&\leq \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b \left|(\Phi' \circ f)(t) - \frac{1}{b-a}\int_a^b (\Phi' \circ f)(s)\,ds\right|^p dt\right)^{1/p} \\
&\qquad \times \left(\int_a^b |f'(t)|^q\,dt\right)^{1/q}. \quad (3.14)
\end{aligned}
$$

(ii) If Φ is twice differentiable and $(\Phi'' \circ f) f' \in L_q[a,b]$ with $p,\ q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\begin{aligned}
0 &\le \frac{1}{b-a}\int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\
&\le \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p} \\
&\quad \times \left(\int_a^b \left|(\Phi'' \circ f)(t) f'(t)\right|^q dt\right)^{1/q}. \qquad (3.15)
\end{aligned}
$$

The proof follows by Theorem 3.1 for $C(\Phi' \circ f, f)$ and the inequality (3.13).

We have the following mid-point type inequalities:

Corollary 3.4. *Let $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (a,b).*

(i) If $\Phi' \in L_p[a,b]$ with $p > 1$, then

$$
\begin{aligned}
0 &\le \frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right) \\
&\le \frac{b-a}{(p-1)^{1/p}\pi_p}\left(\frac{1}{b-a}\int_a^b \left|\Phi'(t) - \frac{\Phi(b)-\Phi(a)}{b-a}\right|^p dt\right)^{1/p}. \qquad (3.16)
\end{aligned}
$$

(ii) If Φ is twice differentiable and $\Phi'' \in L_q[a,b]$ with $p,\ q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
0 \le \frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right) \le \frac{(b-a)^{1+1/p}}{2(p-1)^{1/p}(p+1)^{1/p}\pi_p}\left(\int_a^b |\Phi''(t)|^q dt\right)^{1/q}. \qquad (3.17)
$$

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References

[1] M. W. Alomari, "On Beesack–Wirtinger Inequality", Results Math., vol. 73, no. 2, pp. 1213–1225, 2017.

[2] P. R. Beesack, "Extensions of Wirtinger's inequality", Trans. R. Soc. Can., vol. 53, pp. 21–30, 1959.

[3] P. L. Chebyshev, "Sur les expressions approximatives des intégrales dèfinis par les autres prises entre les même limites", Proc. Math. Soc. Charkov, vol. 2, pp. 93–98, 1882.

[4] J. B. Diaz and F. T. Metcalf, "Variations on Wirtinger's inequality", in *Inequalities*, New York: Academic Press, 1967, pp. 79–103.

[5] P. Drábek and R. Manásevich, "On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian", Differential Integral Equations, vol. 12, no. 6, pp. 773–788, 1999.

[6] S. S. Dragomir, "A Grüss type inequality for isotonic linear functionals and applications", Demonstratio Math., vol. 36, no. 3, pp. 551–562, 2003.

[7] S. S. Dragomir, "Some integral inequalities related to Wirtinger's result", Preprint, RGMIA Res. Rep. Coll., vol. 21, art. 60, 2018.

[8] R. Giova, "An estimate for the best constant in the L^p-Wirtinger inequality with weights", J. Func. Spaces Appl., vol. 6, no. 1, pp. 1–16, 2008.

[9] R. Giova and T. Ricciardi, "A sharp weighted Wirtinger inequality and some related functional spaces", Bull. Belg. Math. Soc. Simon Stevin, vol. 17, no. 2, pp. 209–218, 2010.

[10] G. Grüss, "Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$", Math. Z., vol. 39, no. 1, pp. 215–226, 1935.

[11] J. Jaroš, "On an integral inequality of the Wirtinger type", Appl. Math. Letters, vol. 24, no. 8, pp. 1389–1392, 2011.

[12] C. F. Lee, C. C. Yeh, C. H. Hong and R. P. Agarwal, "Lyapunov and Wirtinger inequalities", Appl. Math. Lett., vol. 17, no. 7, pp. 847–853, 2004.

[13] A. Lupaş, "The best constant in an integral inequality", Mathematica (Cluj), vol. 15, no. 38, pp. 219–222, 1973.

[14] A. M. Ostrowski, "On an integral inequality", Aequationes Math., vol. 4, pp. 358–373, 1970.

[15] T. Ricciardi, "A sharp weighted Wirtinger inequality", Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8), vol. 8, no. 1, pp. 259–267, 2005.

[16] C. A. Swanson, “Wirtinger’s inequality”, SIAM J. Math. Anal., vol. 9, no. 3, pp. 484–491, 1978.

[17] S.-E. Takahasi, T. Miura and T. Hayata, “On Wirtinger’s inequality and its elementary proof”, Math. Inequal. Appl., vol. 10, no. 2, pp. 311–319, 2007.

[18] S. Takeuchi, “Generalized elliptic functions and their application to a nonlinear eigenvalue problem with p-Laplacian”, J. Math. Anal. Appl., vol. 385 , no. 1, pp. 24–35, 2012.