Hyper generalized pseudo $Q$-symmetric semi-Riemannian manifolds

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ABSTRACT

The object of the present paper is to study the properties of a hyper generalized pseudo $Q$-symmetric semi-Riemannian manifold, proving that under certain assumptions, it is a perfect fluid spacetime.

RESUMEN

El objetivo del presente artículo es estudiar las propiedades de una variedad semi-Riemanniana hiper generalizada pseudo $Q$-simétrica, probando que bajo ciertas condiciones, es un espacio-tiempo fluido perfecto.

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1 Introduction

Let \( R, S, L \) and \( r \) denote the curvature tensor, Ricci tensor, Ricci operator and the scalar curvature of a (semi)-Riemannian manifold, respectively. It is Mantica and Suh [5] who have introduced the notion of \( Q \)-curvature tensor. In an \( n \)-dimensional Riemannian or semi-Riemannian manifold \((M^n, g)\) \((n > 2)\), the \( Q \)-curvature tensor is defined as

\[
R(Y, U, V, W) = Q(Y, U, V, W) + \frac{\psi}{n-1}[g(Y, W)g(U, V) - g(Y, V)g(U, W)],
\]

where \( Y, U, V, W \) are arbitrary vector fields on \( M^n \) and \( \psi \) is a scalar function. Semi-Riemannian manifolds with Ricci tensor \( S \) of the form

\[
S(Y, V) = ag(Y, V) + bT(Y)T(V),
\]

for any vector fields \( Y, V \), are often termed as perfect fluid spacetimes, where \( a \) and \( b \) are scalars and the vector field \( \varrho \), metrically equivalent to the 1-form \( T \) (that is, \( g(\varrho, \varrho) = T(Y) \)), is a unit time like vector field (that is, \( g(\varrho, \varrho) = -1 \)).

An \( n \)-dimensional semi-Riemannian manifold is said to be hyper generalized pseudo \( Q \)-symmetric (which will be abbreviated hereafter as \((HGPQS)_n\)) if it satisfies the equation

\[
\]

where

\[
(g \wedge S)(Y, U, V, W) = g(Y, W)S(U, V) + g(U, V)S(Y, W) - g(Y, V)S(U, W) - g(U, W)S(Y, V)
\]

and \( A_1, A_2 \) are non-zero 1-forms whose \( g \)-dual vector fields will be denoted by \( \theta_1 \) and \( \theta_2 \), i.e. \( A_1(X) = g(X, \theta_1) \) and \( A_2(X) = g(X, \theta_2) \).

We organized our paper as follows: section 2 is concerned with preliminaries. After preliminaries, some curvature properties of \((HGPQS)_n\) manifolds are studied in section 3. It is obtained that the \( Q \)-curvature tensor in a \((HGPQS)_n\) manifold satisfies 2nd Bianchi’s identity. It is further obtained that the scalar function \( \psi \) is always constant. In section 4 we investigate properties of divergence-free \((HGPQS)_n\) manifolds and we prove that a divergence-free \((HGPQS)_n\) manifold \((n > 2)\) under the assumption \( A_1(Q(Y, U)V) = 0 \) is a perfect fluid spacetime as well as the integral
curves of the vector field $\varrho$ are geodesics and the vector field $\varrho$ is irrotational, if the associated vector fields $\varrho$ and $\sigma$ corresponding to the 1-forms $T_1$ and $T_2$ are related by $(r - 1) \varrho + n\sigma = 0$.

2 Preliminaries

In this section, some relations useful to the study of $(HGPQS)_n$ manifolds are obtained. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, where $1 \leq i \leq n$.

From (1.1) we can easily verify that the tensor $Q$ satisfies the following properties:

\[(i)\quad Q(Y, U)V + Q(U, Y)V = 0,\]
\[(ii)\quad Q(Y, U)V + Q(U, V)Y + Q(V, Y)U = 0,\]  \hspace{1cm} (2.1)

where $g(Q(X, Y)U, V) = Q(X, Y, U, V)$.

Also from (1.1) we have

\[
\sum_{i=1}^{n} \epsilon_i Q(X, Y, e_i, e_i) = 0 = \sum_{i=1}^{n} \epsilon_i Q(e_i, e_i, W, U) \hspace{1cm} (2.2)
\]

and

\[
\sum_{i=1}^{n} \epsilon_i Q(e_i, Y, V, e_i) = \sum_{i=1}^{n} \epsilon_i Q(Y, e_i, e_i, V) = S(Y, V) - \psi g(Y, V) =: Z(Y, V), \hspace{1cm} (2.3)
\]

where

\[
\epsilon_i = g(e_i, e_i) = \pm 1, \quad S(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(X, e_i) e_i, Y), \quad r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i).
\]

From (1.1) and (2.1) it follows that

\[(i)\quad Q(X, Y, U, V) + Q(X, Y, V, U) = 0,\]
\[(ii)\quad Q(X, Y, U, V) - Q(U, V, X, Y) = 0.\]  \hspace{1cm} (2.4)

3 Some curvature properties of $(HGPQS)_n$ manifolds

In this section we prove that in a $(HGPQS)_n$ manifold, the $Q$-curvature tensor satisfies 2nd Bianchi's identity, that is,

\[
(\nabla_X Q)(Y, U, V, W) + (\nabla_Y Q)(U, X, V, W) + (\nabla_U Q)(X, Y, V, W) = 0. \hspace{1cm} (3.1)
\]
In view of (1.1), (1.2) and (3.1) we get
\[
(\nabla_X Q)(Y, U, V, W) + (\nabla_Y Q)(U, X, V, W) + (\nabla_U Q)(X, Y, V, W) \tag{3.2}
\]
\[
= A_1(V)[Q(Y, U, X, W) + Q(U, X, Y, W) + Q(X, Y, U, W)] \\
+ A_1(W)[Q(Y, U, V, X) + Q(U, X, V, Y) + Q(X, Y, V, U)] \\
+ A_2(V)[(g \wedge S)(Y, U, X, W) + (g \wedge S)(U, X, Y, W)] \\
+ A_2(W)[(g \wedge S)(Y, U, V, X) + (g \wedge S)(U, X, V, Y)] \\
+ (g \wedge S)(U, X, Y, W) + A_2(W)[(g \wedge S)(Y, U, V, X) \\
+ (g \wedge S)(U, X, V, Y)] \\
(3.2)
\]

Using (1.3) and 1st Bianchi’s identity for the $Q$-curvature tensor in (3.2) and then simplifying, we obtain (3.1).

Thus we can state the following:

**Theorem 3.1.** The $Q$-curvature tensor in a $(HGPQS)_n$ manifold satisfies 2nd Bianchi’s identity.

Using (1.1) in (3.1), we have
\[
\]
\[
- \frac{d\psi(X)}{n-1}[g(Y, W)g(U, V) - g(Y, V)g(U, W)] \\
- \frac{d\psi(Y)}{n-1}[g(U, W)g(X, V) - g(U, V)g(X, W)] \\
- \frac{d\psi(U)}{n-1}[g(X, W)g(Y, V) - g(X, V)g(Y, W)] = 0.
\]

By virtue of 2nd Bianchi’s identity for the Riemannian curvature tensor, (3.3) yields
\[
\frac{d\psi(X)}{n-1}[g(Y, W)g(U, V) - g(Y, V)g(U, W)] \tag{3.4}
\]
\[
+ \frac{d\psi(Y)}{n-1}[g(U, W)g(X, V) - g(U, V)g(X, W)] \\
+ \frac{d\psi(U)}{n-1}[g(X, W)g(Y, V) - g(X, V)g(Y, W)] = 0.
\]

Contracting $U$ and $V$ in (3.4), we have
\[
(n-2)[d\psi(X)g(Y, W) - d\psi(Y)g(X, W)] = 0 \tag{3.5}
\]
which yields after further contraction
\[
(n-1)(n-2)d\psi(X) = 0.
\]

This implies that $d\psi(X) = 0$, that is, $\psi$ is constant since $n > 2$ and leads to the following:

**Theorem 3.2.** In a $(HGPQS)_n$ manifold, the scalar function $\psi$ is always constant.
Consequently, one can easily bring out the following:

**Theorem 3.3.** In a \((HGPQS)_n\) manifold, \((\text{div}Q)(X,Y)Z\) and \((\text{div}R)(X,Y)Z\) are equivalent.

In view of (1.1), (1.2) and Theorem 3.2 we have

\[
\]

which yields

\[
(\nabla_X S)(U, V) = [F_1(X) + F_2(X)]S(U, V) + F_2(U)S(X, Y) + F_2(V)S(U, X) + F_3(X) + F_4(X)g(U, V) + F_4(U)(g(X, V) + F_4(V)g(U, X) + A_1(Q(U, V)) - A_1(Q(V, X))U)
\]

after contraction over \(Y\) and \(W\), where

\[
F_1(X) = A_1(X) + (n + 1)A_2(X),
F_2(X) = A_1(X) + (n - 3)A_2(X),
F_3(X) = rA_2(X) - \psi A_1(X) + 3A_2(LX),
F_4(X) = rA_2(X) - \psi A_1(X) - A_2(LX),
\]

where \(L\) is the Ricci operator defined by \(g(LX, Y) = S(X, Y)\).

**Definition 3.4.** An \(n\)-dimensional semi-Riemannian manifold is called almost generalized pseudo Ricci symmetric if the non-flat Ricci curvature tensor satisfies the equation

\[
(\nabla_X S)(U, V) = [A(X) + B(X)]S(U, V) + A(U)S(X, V) + A(V)S(U, X) + C(X) + D(X)g(U, V) + C(U)g(X, V) + C(V)g(U, X),
\]

where \(A, B, C\) and \(D\) are non-zero 1-forms whose \(g\)-dual vector fields will be denoted by \(\gamma_1, \gamma_2, \delta_1\) and \(\delta_2\), i.e. \(A(X) = g(X, \gamma_1), B(X) = g(X, \gamma_2), C(X) = g(X, \delta_1)\) and \(D(X) = g(X, \delta_2)\).

Thus we can state the following:
Theorem 3.5. A \((HGPQS)_n\) manifold \((n > 2)\) under the assumption \(A_1(Q(X,U)V) = A_1(Q(V,X)U)\) is necessarily almost generalized pseudo Ricci symmetric.

Making use of (2.3) in (3.7), we get

\[
(\nabla_X Z)(U,V) = [F_1(X) + F_2(X)]Z(U,V) + F_2(U)Z(X,V) + F_2(V)Z(U,X)
\]

\[
+ [F_3(X) + \psi F_1(X) + F_4(X)]g(U,V)
\]

\[
+ [F_4(U) + \psi F_2(U)]g(X,V) + [F_4(V) + \psi F_2(V)]g(U,X),
\]

where \(Z = S - \psi g\) is the tensor considered in ([4], [6], [7]). This leads to the following:

Theorem 3.6. A \((HGPQS)_n\) manifold \((n > 2)\) under the assumption \(A_1(Q(X,U)V) = A_1(Q(V,X)U)\) is necessarily almost generalized pseudo Z-symmetric.

4 \((HGPQS)_n\) manifolds \((n > 2)\) with \(\text{div}Q = 0\)

Let \((M^n, g)\) be a semi-Riemannian manifold of dimension \(n\) and let \(\{e_i\}\) be an orthonormal basis of the tangent space \(T_pM\) at any point \(p \in M\) and \(\epsilon_i = \pm 1\). Then the divergence of a vector field \(U\) is defined as

\[
\text{div}U = \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} U, e_i),
\]

and the divergence of a tensor field of type \((1,3)\), which is a tensor field of type \((0,3)\), is defined as

\[
(\text{div}K)(X,Y)Z = \sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} K)(X,Y)Z, e_i).
\]

Now

\[
(\text{div}Q)(Y,U)V = \sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} Q)(Y,U)V, e_i)
\]

\[
= \sum_{i=1}^{n} \epsilon_i [2A_1(e_i)Q(Y,U,V,e_i) + A_1(Y)Q(e_i,U,V,e_i)
\]

\[
+ A_1(U)Q(Y,e_i,V,e_i) + A_1(V)Q(Y,e_i,U,e_i)
\]

\[
+ A_1(e_i)Q(Y,U,V,e_i) + 2A_2(e_i)(g \wedge S)(Y,U,V,e_i)
\]

\[
+ A_2(Y)(g \wedge S)(e_i,U,V,e_i) + A_2(U)(g \wedge S)(Y,e_i,V,e_i)
\]

\[
+ A_2(V)(g \wedge S)(Y,U,e_i,e_i) + A_2(e_i)(g \wedge S)(Y,U,V,e_i)]
\]
Hence

\[(\text{div} Q)(Y, U)V = 3A_1(Q(Y, U)V + T_1(Y)S(U, V) - T_1(U)S(Y, V)\]  
\[+ T_2(Y)g(U, V) - T_2(U)g(Y, V), \]  

where

\[
\begin{align*}
T_1(Y) & = A_1(Y) + (n + 1)A_2(Y) =: g(Y, \varrho), \text{ for } \varrho = \theta_1 + (n + 1)\theta_2, \\
T_2(Y) & = 3A_2(LY) + rA_2(Y) - \psi A_1(Y) =: g(Y, \sigma), \text{ for } \sigma = 3L\theta_2 + r\theta_2 - \psi\theta_1.
\end{align*}
\]

Assuming \((\text{div} Q)(Y, U)V = 0\) and \(A_1(Q(Y, U)V) = 0\), we get from the above equation

\[
T_1(Y)S(U, V) + T_2(Y)g(U, V) = T_1(U)S(Y, V) + T_2(U)g(Y, V). \]  

Now contracting (4.2) over \(U\) and \(V\) we get

\[
S(Y, \varrho) = rT_1(Y) + (n - 1)T_2(Y). \]  

Again putting \(V = \varrho\) in (4.2) we get

\[
(n - 2)[T_1(Y)T_2(U) - T_1(U)T_2(Y)] = 0, \]  

which under the assumption \(n > 2\) implies \(T_1(Y)T_2(U) = T_1(U)T_2(Y)\).

Now putting \(U = \varrho\) in (4.2) and using (4.3) and (4.4) we get

\[
T_1(\varrho)S(Y, V) + T_2(\varrho)g(Y, V) = T_1(Y)[rT_1(V) + nT_2(V)] \]  

and we can state:
Theorem 4.1. A divergence-free \((HGPQS)_n\) manifold \((n > 2)\) under the assumption \(A_1(Q(Y,U)V) = 0\) is a perfect fluid spacetime with unit timelike vector field \(\varrho\), provided the associated vector fields \(\varrho\) and \(\sigma\) corresponding to the 1-forms \(T_1\) and \(T_2\) are related by \((r-1)\varrho + n\sigma = 0\).

In this case, (4.5) becomes

\[ S(Y,V) = ag(Y,V) - T_1(Y)T_1(V), \quad (4.6) \]

where \(a =: T_2(\varrho)\).

Again, \((divQ)(Y,U)V = 0\) gives

\[ (\nabla_Y S)(U,V) - (\nabla_U S)(Y,V) = 0. \quad (4.7) \]

Now using (4.6) in (4.7) we find

\[
\begin{align*}
da(Y)g(U,V) - da(U)g(Y,V) \\
- [T_1(V)(\nabla_Y T_1)(U) + T_1(U)(\nabla_Y T_1)(V)] \\
+ [T_1(V)(\nabla_U T_1)(Y) + T_1(Y)(\nabla_U T_1)(V)] \\
= 0.
\end{align*}
\]

Taking a frame field and contracting \(Y\) and \(V\) we get

\[ (n-1)da(U) + [T_1(U)(\delta T_1) + (\nabla_\varrho T_1)(U)] = 0, \quad (4.9) \]

where

\[ \delta T_1 = \sum_{i=1}^{n} \epsilon_i(\nabla e_i T_1)(e_i). \]

Setting \(V = Y = \varrho\) in (4.8) we find

\[ (\nabla_\varrho T_1)(U) = -da(U) - da(\varrho) T_1(U). \quad (4.10) \]

Substituting (4.10) in (4.9) we get

\[ (n-2)da(U) + T_1(U)(\delta T_1) - da(\varrho) T_1(U) = 0 \quad (4.11) \]

which yields

\[ \delta T_1 = (n-1)da(\varrho) \quad (4.12) \]

for \(U = \varrho\).

Using (4.12) in (4.11) we obtain

\[ da(U) = -T_1(U)da(\varrho), \quad (4.13) \]

provided \(n > 2\).
Putting $V = \varrho$ in (4.8) and using (4.13) we get

$$(\nabla_Y T_1)(U) - (\nabla_U T_1)(Y) = 0.$$  

This means that the 1-form $T_1$ is closed, that is,

$$dT_1(Y, U) = 0.$$  

Hence

$$g(\nabla_U \varrho, Y) = g(\nabla_Y \varrho, U)$$  

for all $U, Y$, (4.14)

which yields

$$g(\nabla_\varrho \varrho, Y) = g(\nabla_Y \varrho, \varrho),$$  

(4.15)

for $U = \varrho$. Since $g(\nabla_Y \varrho, \varrho) = 0$, from (4.15) it follows that $g(\nabla_\varrho \varrho, Y) = 0$ for all $Y$. Hence $\nabla_\varrho \varrho = 0$. This implies that the integral curves of the vector field $\varrho$ are geodesics. Therefore we can state the following:

**Theorem 4.2.** In a divergence-free $(HGPQS)_n$ manifold $(n > 2)$ under the assumption $A_1(Q(Y, U)V) = 0$, the integral curves of the unit timelike vector field $\varrho$ are geodesics, provided the associated vector fields $\varrho$ and $\sigma$ corresponding to the 1-forms $T_1$ and $T_2$ are related by $(r - 1)\varrho + n\sigma = 0$.

Taking into account that the divergence of the conformal curvature tensor of a Riemannian manifold $(M^n, g)$ is ([3], [6]):

$$(\text{div}C)(X, Y)Z = \frac{n - 3}{n - 2}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)]$$  

(4.16)

$$= \frac{n - 3}{n - 2}(\text{div}Q)(X, Y)Z,$$

for any vector fields $X, Y, Z$ on $M^n$, from the Lemma 2.1 of [2] we infer

**Theorem 4.3.** Let $(M, g)$ be a $(HGPQS)_n$ perfect fluid spacetime $(n > 2)$. If $(\text{div}Q)(X, Y)Z = 0$, for any vector fields $X, Y, Z$ on $M$, then the unit timelike vector field $\varrho$ is irrotational.

Also, in [2] was proved the following result:

**Theorem 4.4.** [2] Let $(M, g)$ be a $(HGPQS)_n$ perfect fluid spacetime $(n > 2)$. If $(\text{div}Q)(X, Y)Z = 0$, for any vector fields $X, Y, Z$ on $M$, then $(M, g)$ is a GRW spacetime whose fiber is Einstein.

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